A class of powerful error-correcting convolutional codes is demonstrated which exhibit catastrophic error-propagation in the sense that any reasonable decoder for a code in this class will commit infinitely many decoding errors for certain channel error patterns with only finitely many errors.

Introduction

The general form of a rate $R = 1/2$ binary systematic convolutional coding system is shown in Fig. 1. $I(D) = i_0 \oplus i_1 D \oplus i_2 D^2 \oplus \ldots$ is the $D$-transform of the sequence of information digits.

$$G(D) = g_0 \oplus g_1 D \oplus \ldots \oplus g_M D^M$$

is the digital transfer function which defines the code, and the code is said to have memory-order $N$, $M < \infty$. $I(D)$ is transmitted directly over the channel as is also the filter output $I(D)G(D)$. Since $N = 2$ transmitted digits and $K = 1$ information digit occur at each time instant, the code is said to have rate $R = K/N = 1/2$. $E_1(D)$ and $E_2(D)$ are the transforms of the sequences of channel noise digits which corrupt the transmitted digits. Letting $||A(D)||$ denote the number of non-zero terms in the transform $A(D)$, we see that $||E_1(D)|| + ||E_2(D)||$ is the total number of errors introduced by the channel into the transmitted data.

At the decoder, the syndrome $S(D)$ is first formed. From Fig. 1, we have

$$S(D) = G(D) [I(D) \oplus E_1(D)] \oplus [I(D) G(D) \oplus E_2(D)]$$

which reduces to

$$S(D) = G(D)E_1(D) \oplus E_2(D). \quad (1)$$

The asterisks in Fig. 1 denote the decoding estimates of the indicated quantities, e.g. $I^*(D) = i_0^* \oplus i_1^* D \oplus i_2^* D^2 \oplus \ldots$ where $i_u^*$ is the decoding estimate of $i_u$. Note that

$$I^*(D) = I(D) \oplus [E_1(D) \oplus E_1^*(D)]$$

so that $E_1(D) \oplus E_1^*(D)$ is the transform of the sequence of decoding mistakes. Note further that the time instant $u$ term in $E_1(D)$ is estimated by applying
the boolean function \( f \) to the digits in \( S(D) \) over time units \( u \) through \( u + M \) as modified by the feedback of the \( M \) preceding decoding estimates. This feedback gives rise to the error-propagation effect\(^1\) whereby erroneous decoding estimates tend to trigger successive decoding mistakes.

**Catastrophic Error-Propagation**

**Definition:** A decoder is said to exhibit catastrophic error-propagation when a finite number of channel errors results in infinitely many decoding mistakes, i.e. when

\[
||E_1(D)|| + ||E_2(D)|| < \infty \text{ but } ||E_1(D) \ominus E_1^R(D)|| = \infty.
\]

We next exhibit the existence of some apparently well-designed convolutional coding systems that exhibit catastrophic error-propagation. The discussion will be facilitated by introducing the notation \( A(D)[u, u+v] \) to denote the portion of \( A(D) \) consisting of the terms with degrees \( u \) through \( u + v \) inclusive.

If the convolutional code has minimum distance \( d \), then it is well known that decoding functions \( f \) exist such that \( E_1^R(D) = E_1(D) \) whenever

\[
||E_1(D)[u, u+N]\|| + ||E_2(D)[u, u+N]\|| \leq \epsilon \quad u = 0, 1, 2, \ldots \tag{2}
\]

for any integer \( \epsilon \) such that \( d > 2\epsilon \). Such a decoder is said to be \( \epsilon \)-error-correcting. It is further known\(^2\) that for any rate \( R = K/N \), there exist codes such that \( H(d/n) \geq 1-R \) where \( n = N(N+1) \) is the decoding constraint length and \( H(x) = -x \log_2 x - (1-x) \log_2 (1-x) \) is the entropy function.

Now let \( G'(D) \) be any code of memory order \( M' \) with minimum distance \( d' \), \( d' \geq 5 \). Consider the new code of memory order \( M = 2M' + 1 \) defined by

\[
G(D) = (1 \oplus D^{M'+1}) \oplus G'(D) \tag{3}
\]

This code has minimum distance \( d \geq d' \) since from (3) we see that \( G(D) \) and \( G'(D) \) coincide over the original constraint length. Note also that \( n = 2(M+1) = 2\times(2M'+1) = 2n' \). We now show that any \( \epsilon \)-error-correcting decoder for this code, \( \epsilon = 2 \), exhibits catastrophic error-propagation.

To show this, consider first the error patterns \( E_1(D) = (1 \oplus D^{M'+1})^{-1} \) and \( E_2(D) = 0 \). From (1) and (3), we find \( S(D) = G'(D) \). We further note that

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\[ ||E_1(D)_{[u,u+M]}|| + ||E_2(D)_{[u,u+M]}|| = \]
\[ ||(1 \oplus D^{M'+1})_{[u,u+M]}|| \leq 2 \]
\[ u = 0, 1, 2, \ldots \]
so we may use (2) to conclude that \( E_1^*(D) = E_1(D) = (1 \oplus D^{M'+1})^{-1}. \)

Consider next that \( E_1(D) = 0 \) and \( E_2(D) = G'(D). \) Again, from (4) we find \( S(D) = G'(D) \) so the decoder must again estimate \( E_1^*(D) = (1 \oplus D^{M'+1})^{-1}. \) But note in our second case
\[ ||E_1(D)|| + ||E_2(D)|| = ||G'(D)|| < \infty \]
but
\[ ||E_1(D) \oplus E_1^*(D)|| = ||(1 \oplus D^{M'+1})^{-1}|| = \infty \]
so that the decoder exhibits catastrophic error propagation as claimed. We note further that \( d' \) might have been the maximum distance attainable with memory order \( M' \) in which case
\[ H(d/n) \geq H \left( \frac{d'}{2n} \right) \geq \frac{1}{2} H \left( \frac{d'}{n} \right) \geq \frac{1}{2}(1-R) \]
where the middle inequality follows from the convexity of \( H(x). \)

The above arguments can be readily generalized to the following:

**Theorem:** For any code rate \( R = K/N, \) there exist systematic binary convolutional codes for every constraint length \( n \) such that their minimum distance satisfies
\[ H \left( \frac{d}{n} \right) \geq \frac{1}{2}(1-R) \]
and such that any e-error-correcting decoder, \( e \geq 2, \) for such codes exhibits catastrophic error-propagation.

There are other disturbing facts about the codes considered above. For instance, if \( E_1(D) = F_1(D) \) and \( E_2(D) = F_2(D) \oplus G'(D) \) where
\[ ||F_1(D)_{[u,u+M]}|| + ||F_2(D)_{[u,u+M]}|| \leq e - 2 \]
\[ u = 0, 1, 2, \ldots, \]
then for any e-error correcting decoder

\[ E_1(D) \circ E_1^*(D) = (1 \circ D^{M+1})^{-1} \]

i.e. the same decoding mistakes are made as were made previously. The import of this is that the decoder is not easily "jarred" from its catastrophic error-propagation condition by further channel errors beyond those initially triggering the catastrophe.

A final note of caution is in order concerning the interpretation of this paper. We have not shown that feedback decoders for convolutional codes are generally undesirable—in fact, our opinion is wholly to the contrary. What we have shown is that injudicious choice of the code can render feedback decoding a perilous venture.

References


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FIG. 1 A General $R = \frac{1}{2}$ Binary Convolutional Coding System