Proc. 4th Princeton Conf. on Info. Sc. + Systems 1970

DERIVATIVE CONTROLLABILITY*

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The discrete-time system

$$x_{i+1} = Ax_i + Bu_i, \quad x_0 = 0, \; i = 0,1,2,\ldots \tag{1}$$

is called k-step controllable by Cohn[1] if given any n-vector v there exist choices for the first k input vectors such that $x_k = v$. The necessary and sufficient condition is

$$\text{rank}[A^{k-1}B \; A^{k-2}B \; \ldots \; AB \; B] = n. \tag{2}$$

The question then arises as to a corresponding concept for the continuous-time system

$$\dot{x} = Ax + Bu, \; x(0) = 0, \; t \geqq 0. \tag{3}$$

To answer this question, we introduce the input space

$$P_k = \{u(t) = u_0 + u_1 t + u_2 t^2/2! + \ldots + u_{k-1} t^{k-1}/(k-1)!\}$$

of polynomials of degree less than k in time t. Letting $x^{(i)}$ denote the i-th derivative of x, we obtain by direct solution of (3):

<u>Lemma:</u> For u in P_k,

$$x^{(k)}(t) = e^{At}[A^{k-1}B \; A^{k-2}B \; \ldots \; AB \; B] \, [u_0' \; u_1' \; \ldots \; u_{k-1}']'. \tag{4}$$

This lemma then motivates the following:

<u>Definition.</u> The system (3) is k-th derivative state controllable by input P_j [denoted kDSC(P_j)] if there exists $t_1 > 0$ such that for any n-vector v there is an input u in P_j giving $x^{(k)}(t_1) = v$.

<u>Theorem 1:</u> The system (3) is kDSC(P_k) if and only if (2) holds. Moreover, the controllability is for every $t_1 > 0$.

Let K be the least k such that (3) is kDSC(P_k). Clearly, $K \leqq n$ or K is not defined. Inquiry as to the smallest polynomial class needed to control the k-th derivative leads to the following:

<u>Theorem 2:</u> For $k \geqq K$, the system (3) is kDSC(P_i) if and only if $i \geqq K$ for non-singular A or $i \geqq k$ for singular A. Moreover, the controllability is for all $t_1 > 0$.

In the discrete-time case, theorem 2 corresponds to controllability of x_k using only the first K inputs nonzero when A is non-singular.

Finally, we remark that kDSC(P_k) implies iDSC(P_k) for $0 \leqq i < k$ with the controllability being for all except at most finitely many t_1 in any finite positive time interval. We remark further that k-step observability of the system (1) with output has the trivial interpretation for system (3) with output of being able to recover x(0) from the output and its first k-1 derivatives at any fixed time $t_1 > 0$.

[1]. Cohn, M., "Controllability in Linear Sequential Networks", IRE Trans., CT-9, pp 74-78, March 1962.

*This work was supported by the National Science Foundation, Grant GK-13618.