DERIVATIVE CONTROLLABILITY

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The discrete-time system

\[ x_{i+1} = Ax_i + Bu_i, \quad x_0 = 0, \quad i = 0,1,2,\ldots \]  \hspace{1cm} (1)

is called k-step controllable by Cohn \cite{1} if given any n-vector \( v \) there exist choices for the first \( k \) input vectors such that \( x_k = v \). The necessary and sufficient condition is

\[ \text{rank}[A^{k-1}B \quad A^{k-2}B \quad \ldots \quad AB] = n. \] \hspace{1cm} (2)

The question then arises as to a corresponding concept for the continuous-time system

\[ \dot{x} = Ax + Bu, \quad x(0) = 0, \quad t \geq 0. \] \hspace{1cm} (3)

To answer this question, we introduce the input space

\[ P_k = \{ u(t) = u_0 + u_1 t + u_2 t^2/2! + \ldots + u_{k-1} t^{k-1}/(k-1)! \} \]

of polynomials of degree less than \( k \) in time \( t \). Letting \( x^{(i)} \) denote the \( i \)-th derivative of \( x \), we obtain by direct solution of (3):

\[ x^{(k)}(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds = v = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds. \]

\[ x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds. \] \hspace{1cm} (4)

This lemma then motivates the following:

**Definition.** The system (3) is \( k \)-th derivative state controllable by input \( P_k \) [denoted \( k\text{DSC}(P_k) \)] if there exists \( t_1 > 0 \) such that for any \( n \)-vector \( v \) there is an input \( u \) in \( P_k \) giving \( x^{(k)}(t_1) = v \).

**Theorem 1.** The system (3) is \( k\text{DSC}(P_k) \) if and only if (2) holds. Moreover, the controllability is for every \( t_1 > 0 \).

Let \( K \) be the least \( k \) such that (3) is \( k\text{DSC}(P_k) \). Clearly, \( K \leq n \) or \( K \) is not defined. Inquire as to the smallest polynomial class needed to control the \( k \)-th derivative leads to the following:

**Theorem 2.** For \( k \leq K \), the system (3) is \( k\text{DSC}(P_k) \) if and only if \( i \leq K \) for non-singular \( A \) or \( i \geq k \) for singular \( A \). Moreover, the controllability is for all \( t_1 > 0 \).

In the discrete-time case, theorem 2 corresponds to controllability of \( x_k \) using only the first \( K \) inputs nonzero when \( A \) is non-singular.

Finally, we remark that \( k\text{DSC}(P_k) \) implies \( 1\text{DSC}(P_k) \) for \( 0 \leq i < k \) with the controllability being for all except at most finitely many \( t_1 \) in any finite positive time interval. We remark further that \( k \)-step observability of the system (1) with output has the trivial interpretation for system (3) with output of being able to recover \( x(0) \) from the output and its first \( k-1 \) derivatives at any fixed time \( t_1 > 0 \).

\[ [1] \text{ Cohn, M., "Controllability in Linear Sequential Networks", IRE Trans., CT-9, pp 74-78, March 1962.} \]

\[ \text{This work was supported by the National Science Foundation, Grant GK-13618.} \]