A Class of Maximum Distance Separable Codes over GF(p) Encodable Using Only Addition and Subtraction

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ABSTRACT

The m-stage c-box is defined to be the circuit consisting of the cascade of m identical sub-circuits, each of which contains a unit delay whose output is the sub-circuit output and which output is multiplied by the constant c and added to the sub-circuit input to form the input to the delay cell. It is shown that when a polynomial \( b(x) = b_{n-1} x^{n-1} + \ldots + b_1 x + b_0 \) is shifted into an m-stage c-box, the delay cell contents after \( n \) shifts form the polynomial \( a(x) \) given by \( a(x) = b(x+c) \bmod x^m \).

The \((n,k)\) constacyclic code over GF(p) with \( n = p \) and generated by \( g(x) = (x-c)^{n-k} \) is known to be maximum distance separable. The information polynomial \( i(x) \) in a systematic encoding is encoded as \( x^{n-k} i(x) - r(x) \), where \( r(x) \) is the remainder when \( x^{n-k} i(x) \) is divided by \( g(x) \). It is shown that \( r(x) \) can be obtained by first shifting \( x^{n-k} i(x) \) \( n \) times into an \((n-k)\)-stage c-box and then shifting the result \( n-k \) times into an \((n-k)\)-stage \((-c)\)-box. When \( c = 1 \) (or \(-1\)) the encoder thus uses only additions and subtractions in GF(p), i.e., no scalar multiplications are needed in the encoding circuit for these maximum distance separable codes.
in fact, for \( c = 1 \), the contents \((a_0, a_1, \ldots, a_{m-1})\) after \( N \)-shifts will just be the first \( m \) entries in the \( N \)-th row of Pascal's triangle.

The usefulness of the \( k \)-stage \( c \)-box is the fact, as given by (1), that it can be used to translate the indeterminate of a polynomial by the amount \( c \). Note that when \( c = +1 \) or \( c = -1 \), then the \( m \)-stage \( c \)-box uses no scalar multiplications in \( F \), but only additions or subtractions, respectively.

Massey, Costello and Justesen [1] showed that, for every prime \( p \), the \((n,k)\) constacyclic code of length \( n = p \) over \( F = GF(p) \) generated by \( g(x) = (x-c)^{n-k} \) is maximum distance separable (MDS), i.e., its minimum distance satisfies \( d = n - k + 1 \). They gave a simple decoding procedure for these MDS codes, but gave no simple systematic encoding circuit. We now show how these MDS codes can be systematically encoded using an \((n-k)\)-stage \( c \)-box and an \((n-k)\)-stage \((-c)\)-box.

The information polynomial \( i(x) = i_{k-1} x^{k-1} + \ldots + i_1 x + i_0 \) in a systematic encoding must be encoded into \( x^{n-k} i(x) - r(x) \) where \( r(x) \) is the remainder when \( x^{n-k} i(x) \) is divided by \( g(x) = (x-c)^{n-k} \). It remains to find a circuit which forms \( r(x) \). By Euclid's division algorithm,

\[
x^{n-k} i(x) = (x-c)^{n-k} Q(x) + r(x).
\]

From (3), it follows that

\[
(x+c)^{n-k} i(x+c) = x^{n-k} Q(x+c) + r(x+c)
\]

and hence that

\[
(x+c)^{n-k} i(x+c) = r(x+c) \mod x^{n-k}.
\]

From (4), it follows that we can obtain \( r(x+c) \) as the contents of an \((n-k)\)-stage \( c \)-box after \( n \) shifts with the input \( x^{n-k} i(x) \). But then \( r(x) \) itself can be obtained as the contents of an \((n-k)\)-stage \((-c)\)-box in \( n-k \) shifts with the input \( r(x+c) \). When \( c = 1 \) (in which case the constacyclic code is a cyclic code) or
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SUMMARY

Consider the circuit shown in Figure 1 which is a cascade of identical
sub-circuits and which we shall call an m-stage c-box. The constant c may be
any element in an arbitrary field F. When the polynomial \( b(x) = b_{n-1} x^{n-1} + \ldots \)

\[
\begin{align*}
&b_0, b_1, \ldots, b_{n-2}, b_{n-1} \\
&\quad \Downarrow \text{unit delay} \quad \oplus \text{field adder} \quad \rightarrow \text{multiply by c} \\
&\quad a_0 \quad a_1 \quad a_{m-1} \\
&\quad \text{c-box} \\
&\text{Fig. 1: The m-stage c-box.}
\end{align*}
\]

+ \( b_1 x + b_0 \) with coefficients in F is read into the m-stage c-box, with higher
order coefficients leading as shown in Figure 1, then the contents \( a(x) = a_{m-1} x^{m-1} + \ldots + a_1 x + a_0 \) after n shifts (i.e., just after \( b_0 \) has been shifted in from the
input) is given by

\[
a(x) \equiv b(x + c) \mod x^m. \quad (1)
\]

Equation (1) can be verified by observing that when the input sequence is
1,0,0,... then, after N shifts,

\[
a_1 = \binom{N-1}{i} c^{N-1-i}; \quad (2)
\]
c = -1 (in which case the constacyclic code is a negacyclic code), we see that only addition and subtraction, without scalar multiplication, is required in the encoding circuit. These codes look particularly interesting when p is a prime of the form $p = 2^S - 1$ since then the addition and subtraction in GF(p) are just the usual one's complement operations.

REFERENCE