On the Gaussian MAC with Imperfect Feedback

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***Abstract*— New achievable regions are derived for the two-user additive white Gaussian multiple-access channel (MAC) with noisy feedback. We treat the general scenario as well as the symmetric setting and the partial feedback setting.**

Unlike previously-known achievable regions, the new regions yield sum-rates that are strictly larger than the no-feedback sum-rate capacity, irrespective of the (positive & finite) Gaussian feedback-noise variance.

In the symmetric setting, our proposed coding scheme achieves sum-rates that converge to Ozarow's noiseless-feedback sum-rate capacity as the feedback-noise variance tends to zero.

In the partial-feedback setting, where one of the transmitters has a perfect feedback link and the other has no feedback at all, we show that the Cover-Leung region (which was originally proposed for perfect-feedback channels but which was later shown to be achievable also with partial feedback) is not tight. This answers in the negative the question posed by van der Meulen as to whether the Cover-Leung region is tight for the Gaussian multiple-access channel with partial feedback.

We also propose a coding scheme for the case where the receiver is cognizant of the realization of the noise on the feedback-link.

I. Introduction

We consider the two-user additive white Gaussian multiple-access channel (MAC) with causal *noisy* feedback from the channel output to the transmitters. We address the general setting as well as two special cases, which are then studied in greater detail: the "symmetric case", where the transmitters have equal average power constraints and where the variances of the noises on the feedback links are equal; and the "partial feedback" case, where one of the feedback links is noise-free whereas the other is non-existent. Additionally, we also address the case where the receiver (but not transmitters) is cognizant of the realizations of the noise on the feedback links.

For these set-ups we propose coding schemes and analyze the rates that they achieve in the Shannon sense, i.e., the rates of communication that are possible using these schemes when the allowed probability of error is fixed but arbitrarily small.

In all cases, the set of rates that are achievable with the proposed scheme is strictly larger than when no feedback is available. Moreover, in the symmetric case, the proposed scheme achieves sum-rates that converge to Ozarow's perfect-feedback sum-rate capacity as the variance of the feedback-noise tends to zero.

In the partial feedback case, the proposed scheme yields achievable rates that include the Cover-Leung region [3], a region that was originally derived for the perfect feedback case and that was later shown by Carleial [1] and (for the discrete memoryless case) by Willems & van der Meulen [9] to be achievable also for partial feedback. For some parameters, the inclusion is strict, thus answering in the negative the question posed by van der Meulen in [8] as to whether the Cover-Leung region equals the capacity region of the Gaussian MAC with partial feedback.

The capacity region of the two-user additive white Gaussian MAC with noisy feedback remains, however, still unknown. It is only known in the two extreme cases: when there is no feedback (which can be viewed as the limiting case of infinite-variance feedback noise) [2], [11] and when the feedback links are noise-free [6].

An outer bound on the capacity region of the MAC with imperfect feedback was recently presented by Gastpar and Kramer [5]. For Gaussian channels this outer bound does not meet any of the achievable regions which are known to date, except for the case of perfect feedback or no feedback.

Achievable regions for the noisy feedback case were given by Carleial [1] who generalized the Cover-Leung coding scheme [3] to the imperfect-feedback case and by Gastpar [4] who extended Ozarow's scheme. Our region, however, satisfies the following two properties:

- for all finite feedback-noise variances, the scheme achieves sum-rates that strictly exceed the sum-rate capacity without feedback, and
- the proposed scheme achieves sum-rates that converge to Ozarow's perfect feedback sum-rate capacity when the feedback-noise variance tends to zero.

The work of Michèle A. Wigger has been partly supported by the Swiss National Science Foundation under Grant 200021-111863 /1

II. Channel Model and Previous Results

This paper focuses on the white Gaussian multiple-access channel with two transmitters, Transmitter 1 and Transmitter 2, wishing to transmit messages M_1 and M_2 to a single receiver. The two messages are assumed to be independent of each other and uniformly distributed over the sets $\mathcal{M}_1 = \{1, \ldots, \lfloor e^{nR_1} \rfloor\}$ and $\mathcal{M}_2 = \{1, \ldots, \lfloor e^{nR_2} \rfloor\}$, where n denotes the blocklength, and R_1 and R_2 are the corresponding rates of transmission in nats per channel-use.

To describe the channel model we introduce the sequence $\{Z_k\}$ of independent and identically distributed (IID) zero-mean variance-N Gaussian random variables that will be used to model the additive noise at the receiver terminal. We shall refer to N as the channel-noise variance. Using this sequence we can describe the time-k channel output Y_k corresponding to the time-k channel inputs $x_{1,k}$ and $x_{2,k}$ by

$$Y_k = x_{1,k} + x_{2,k} + Z_k. \tag{1}$$

The sequence $\{Z_k\}$ is assumed to be independent of the messages (M_1, M_2).

To describe the noisy feedback links we introduce the IID sequence of bi-variate zero-mean Gaussians $\{(W_{1,k}, W_{2,k})\}$ of covariance matrix

$$\begin{aligned} \mathsf{K}_{W_1W_2} &\triangleq \begin{pmatrix} \mathsf{E}\left[W_{1,k}^2\right] & \mathsf{E}[W_{1,k}W_{2,k}] \\ \mathsf{E}[W_{1,k}W_{2,k}] & \mathsf{E}\left[W_{2,k}^2\right] \end{pmatrix} \\ &= \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\varrho \\ \sigma_1\sigma_2\varrho & \sigma_2^2 \end{pmatrix}. \end{aligned}$$

This sequence is assumed to be independent of $(M_1, M_2, \{Z_k\})$. The time-k output $V_{\nu,k}$ of the feedback link from the channel output to Encoder ν is

$$V_{\nu,k} = Y_k + W_{\nu,k}, \quad \nu \in \{1,2\}, \;\; k \in \{1, \ldots, n\}. \tag{2}$$

We are now ready to describe the channel inputs. The time-k signal $X_{\nu,k}$ transmitted by Transmitter ν ($\nu \in \{1,2\}$) is allowed to depend on the message M_ν and on the previous outputs $V_{\nu,1}, \ldots, V_{\nu,k-1}$ of the noisy feedback link to Encoder ν. Thus, for blocklength n codes for User $\nu \in \{1,2\}$ and time $k \in \{1, \ldots, n\}$

$$X_{\nu,k} = f_{\nu,k}^{(n)}\left(M_\nu, V_{\nu,1}, \ldots, V_{\nu,k-1}\right) \tag{3}$$

where

$$f_{\nu,k}^{(n)} : \mathcal{M}_\nu \times \mathbb{R}^{k-1} \to \mathbb{R}. \tag{4}$$

Additionally, we impose average power constraints on the two transmitters and only allow encoding functions fulfilling the power constraints

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[\left(f_{\nu,k}^{(n)}\left(M_\nu, V_{\nu,1}, \ldots, V_{\nu,k-1}\right)\right)^2\right] \leq \mathsf{P}_\nu. \tag{5}$$

We say that a rate pair (R_1, R_2) is achievable if for every blocklength n there exist encoding functions $\left(\{f_{1,k}^{(n)}\}_{k\leq n}, \{f_{2,k}^{(n)}\}_{k\leq n}\right)$ as in (4) that fulfill the power constraints (5), and a decoding function

$$\varphi^{(n)} : \mathbb{R}^n \to \mathcal{M}_1 \times \mathcal{M}_2$$

such that the average probability of error $P_e^{(n)}$ in decoding the messages

$$P_e^{(n)} \triangleq \Pr\left[\varphi^{(n)}\left(Y_1, \ldots, Y_n\right) \neq (M_1, M_2)\right]$$

tends to 0 when the blocklength $n \to \infty$.

The closure of the convex hull of all achievable rate pairs for this setting is called the capacity region and will be denoted by $\mathsf{C}_{\text{FB}}(\mathsf{P}_1, \mathsf{P}_2, N, \mathsf{K}_{W_1W_2})$. The capacity region in the "perfect feedback" case where the feedback links are noiseless, i.e., when $\mathsf{K}_{W_1W_2}$ is the all-zero matrix, will be denoted by $\mathsf{C}_{\text{PerfectFB}}(\mathsf{P}_1, \mathsf{P}_2, N)$. The capacity region in the absence of feedback will be denoted by $\mathsf{C}(\mathsf{P}_1, \mathsf{P}_2, N)$. Formally, this corresponds to $\mathsf{K}_{W_1W_2}$ being a diagonal matrix with the diagonal entries being formally defined as being infinite. The capacity region of the MAC with "partial feedback", i.e., when Transmitter 2 has a noiseless feedback link whereas Transmitter 1 has no feedback at all will be denoted by $\mathsf{C}_{\text{PartialFB}}(\mathsf{P}_1, \mathsf{P}_2, N)$. Formally,

$$\mathsf{K}_{W_1W_2} = \begin{pmatrix} \infty & 0 \\ 0 & 0 \end{pmatrix} \Longleftrightarrow \text{“partial feedback”}.$$

We shall denote the sum-rate capacities by $C_{\text{FB, sum}}(\mathsf{P}_1, \mathsf{P}_2, N, \mathsf{K}_{W_1W_2})$, $C_{\text{PerfectFB, sum}}(\mathsf{P}_1, \mathsf{P}_2, N)$, $C_{\text{sum}}(\mathsf{P}_1, \mathsf{P}_2, N)$, and $C_{\text{PartialFB, sum}}(\mathsf{P}_1, \mathsf{P}_2, N)$ respectively.

We refer to the case where $\mathsf{P}_1 = \mathsf{P}_2 = \mathsf{P}$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$ as the "symmetric setting". For such a setting we refer to σ^2 as the feedback-noise variance and define

$$\begin{aligned} \mathsf{C}_{\text{FB}}^{(s)}(\mathsf{P}, N, \sigma^2, \varrho) &\triangleq \mathsf{C}_{\text{FB}}(\mathsf{P}, \mathsf{P}, N, \mathsf{K}), \\ \mathsf{K} &= \begin{pmatrix} \sigma^2 & \sigma^2\varrho \\ \sigma^2\varrho & \sigma^2 \end{pmatrix}. \end{aligned}$$

Similarly, we denote by $\mathsf{C}^{(s)}(\mathsf{P}, N) \triangleq \mathsf{C}(\mathsf{P}, \mathsf{P}, N)$ the symmetric capacity region without feedback and by $\mathsf{C}_{\text{PerfectFB}}^{(s)}(\mathsf{P}, N) \triangleq \mathsf{C}_{\text{PerfectFB}}(\mathsf{P}, \mathsf{P}, N)$ the symmetric capacity region with perfect feedback. Again we denote the corresponding sum-rate capacities by adding a subscript "sum".

In this work we shall also consider a setting where the decoder has perfect side-information about the realizations of the Gaussian noise sequences corrupting the feedback signals $\{W_{1,k}\}$ and $\{W_{2,k}\}$. We denote the capacity region of the MAC with noisy feedback and perfect side-information by $\mathsf{C}_{\text{FBSI}}(\mathsf{P}_1, \mathsf{P}_2, N, \mathsf{K}_{W_1W_2})$ and in the symmetric case by

$$\begin{aligned} \mathsf{C}_{\text{FBSI}}^{(s)}(\mathsf{P}, N, \sigma^2, \varrho) &\triangleq \mathsf{C}_{\text{FBSI}}^{(s)}(\mathsf{P}, \mathsf{P}, N, \mathsf{K}), \\ \mathsf{K} &= \begin{pmatrix} \sigma^2 & \sigma^2\varrho \\ \sigma^2\varrho & \sigma^2 \end{pmatrix}. \end{aligned}$$

Of the settings above, the capacity region is only known when there is no feedback or when the feedback links are noiseless. In the former case the capacity region was independently determined by Cover [2] and Wyner [11]. In the later case the capacity region was determined by Ozarow [6]. The sum-rate capacity with perfect feedback is given by

$$\begin{aligned}&\mathsf{C}_{\text{PerfectFB, sum}}(\mathsf{P}_1, \mathsf{P}_2, N)\\&= \frac{1}{2}\log\left(1+\frac{\mathsf{P}_1+\mathsf{P}_2+2\sqrt{\mathsf{P}_1\mathsf{P}_2}\rho}{N}\right)\end{aligned}$$

when ρ is the solution in the interval $[0, 1]$ of the equation

$$\begin{aligned}&N(N+\mathsf{P}_1+\mathsf{P}_2+2\sqrt{\mathsf{P}_1\mathsf{P}_2}\rho)\\&=(N+\mathsf{P}_1(1-\rho^2))(N+\mathsf{P}_2(1-\rho^2)).\end{aligned} \tag{6}$$

For the MAC with partial feedback no capacity result is known but Carleial [1] showed that the rate region originally proposed by Cover and Leung [3] for the MAC with perfect feedback is also achievable with partial feedback. This achievable rate region, to which we will refer as the Cover-Leung region, is given by the set of non-negative rate pairs (R_1, R_2) that for some $\rho_1, \rho_2 \in [0, 1]$ satisfy

$$\begin{aligned}R_1 &\leq \frac{1}{2}\log\left(1+\frac{\mathsf{P}_1(1-\rho_1)}{N}\right)\\R_2 &\leq \frac{1}{2}\log\left(1+\frac{\mathsf{P}_2(1-\rho_2)}{N}\right)\\R_1+R_2 &\leq \frac{1}{2}\log\left(1+\frac{\mathsf{P}_1+\mathsf{P}_2+2\sqrt{\mathsf{P}_1\mathsf{P}_2\rho_1\rho_2}}{N}\right).\end{aligned} \tag{7}$$

In [8] van der Meulen posed the question whether the Cover-Leung region achieves the capacity region for the Gaussian MAC with partial feedback. We will show that this is not the case.

For this partial feedback setting Willems, van der Meulen, and Schalkwijk proposed an encoding scheme [10] based on a scheme by Schalkwijk & Kailath [7]. Unfortunately, the achievable rate region can only be stated in an implicit form and is difficult to evaluate analytically and to compare to the Cover-Leung region.

Before stating the main results we introduce some further notation. In the following let $\mathbf{A}_1^\ell$ denote the vector $(A_1, A_2, \ldots, A_\ell)^\mathsf{T}$; let $\operatorname{diag}(a_1, \ldots, a_\ell)$ be the diagonal matrix with entries $a_1, \ldots, a_\ell$ on the main diagonal; let I_ℓ be the identity matrix of dimension ℓ, i.e., $\mathsf{I}_\ell = \operatorname{diag}(1, 1, \ldots, 1) \in \mathbb{R}^{\ell\times\ell}$; and let for zero-mean random vectors $\mathbf{S}$ and $\mathbf{T}$ the covariance matrices be defined as $\mathsf{K}_{\mathbf{S},\mathbf{T}} \triangleq \mathsf{E}[\mathbf{S}\mathbf{T}^\mathsf{T}]$ and $\mathsf{K}_{\mathbf{S}} \triangleq \mathsf{E}[\mathbf{S}\mathbf{S}^\mathsf{T}]$.

III. Main Results

The statements of the results of this work are split into four different sections corresponding to the different settings we introduced in Section II: The general (possibly asymmetric) MAC with noisy feedback; the symmetric MAC with noisy feedback; the MAC with partial feedback; and finally the symmetric MAC with noisy feedback and perfect side-information about the feedback-noises at the receiver.

A. Asymmetric MAC with Noisy Feedback

Theorem 1: Irrespective of the channel-noise variance $0 < N < \infty$, of the allowed average transmit powers $0 < \mathsf{P}_1, \mathsf{P}_2 < \infty$, and of the finite covariance matrix of the feedback noises $\mathsf{K}_{W_1W_2}$, the capacity region of the two-user additive white Gaussian MAC in the presence of feedback is strictly larger than in its absence:

$$\mathsf{C}_{\text{FB}}(\mathsf{P}_1, \mathsf{P}_2, N, \mathsf{K}_{W_1W_2}) \supset \mathsf{C}(\mathsf{P}_1, \mathsf{P}_2, N) \tag{8}$$

with the inclusion being strict.

B. Symmetric MAC with Noisy Feedback

Theorem 2: The capacity region of the symmetric two-user white Gaussian multiple-access channel with channel-noise variance N, with transmit powers P, and with memoryless Gaussian feedback-noises of covariance matrix

$$\mathsf{K}_{W_1W_2} = \begin{pmatrix}\sigma^2 & \sigma^2\varrho\\ \sigma^2\varrho & \sigma^2\end{pmatrix}$$

includes all non-negative rate pairs (R_1, R_2) which for some choice of the positive integer η fulfill both

$$R_1, R_2 \leq \frac{1}{2\eta}\sum_{\ell=0}^{\eta-1}\log\left(1+\frac{\mathsf{P}(1-\rho_\ell^2)}{N}\left(\kappa_\ell(1+\lambda_\ell)^2+\mu_\ell\right)\right)$$

and

$$R_1+R_2 \leq \frac{1}{2\eta}\sum_{\ell=0}^{\eta-1}\log\left(1+\frac{2\mathsf{P}(1+\rho_\ell)}{N}\left(\kappa_\ell(1+\lambda_\ell)^2+\mu_\ell\right)\right)$$

where $\kappa_0 = \rho_0 = \lambda_0 = \mu_0 = 0$, and where for $\ell = 1, \ldots, \eta-1$

$$\rho_\ell = (-1)^{\ell-1}\frac{\mathsf{K}^\mathsf{T}_{\mathbf{Y}_1^\ell,U_1}\mathsf{K}^{-1}_{\mathbf{Y}_1^\ell}\mathsf{P}^{-1}\mathsf{L}_\ell\mathsf{K}_{\mathbf{Y}_1^\ell,U_1}}{1-\mathsf{K}^\mathsf{T}_{\mathbf{Y}_1^\ell,U_1}\mathsf{K}^{-1}_{\mathbf{Y}_1^\ell}\mathsf{P}^{-1}\mathsf{K}_{\mathbf{Y}_1^\ell,U_1}}$$

$$\kappa_\ell = \frac{1-\mathsf{K}^\mathsf{T}_{\mathbf{Y}_1^\ell,U_1}\mathsf{K}^{-1}_{\mathbf{Y}_1^\ell}\mathsf{P}^{-1}\mathsf{K}_{\mathbf{Y}_1^\ell,U_1}}{1-\mathsf{K}^\mathsf{T}_{\mathbf{V}_{1,1}^\ell,U_1}\left(\frac{\mathsf{P}^2}{N}\Sigma_\ell+\mathsf{P}\mathsf{K}_{\mathbf{V}_{1,1}^\ell}\right)^{-1}\mathsf{K}_{\mathbf{V}_{1,1}^\ell,U_1}}$$

$$\lambda_\ell = \frac{N}{\mathsf{P}}\mathsf{K}^\mathsf{T}_{\mathbf{V}_{1,1}^\ell,U_1}\left(\Sigma_\ell+\frac{N}{\mathsf{P}}\mathsf{K}_{\mathbf{V}_{1,1}^\ell}\right)^{-1}\mathsf{K}_{\mathbf{W}_\ell,\epsilon_\ell}\mathsf{K}^{-1}_{\epsilon_\ell}$$

$$\mu_\ell = \frac{N\left(1-\mathsf{K}^\mathsf{T}_{\mathbf{Y}_1^\ell,U_1}\mathsf{K}^{-1}_{\mathbf{Y}_1^\ell}\mathsf{P}^{-1}\mathsf{K}_{\mathbf{Y}_1^\ell,U_1}\right)}{\mathsf{K}_{\epsilon_\ell}\left(\mathsf{K}^\mathsf{T}_{\mathbf{W}_\ell,\epsilon_\ell}\left(\Sigma_\ell+\frac{N}{\mathsf{P}}\mathsf{K}_{\mathbf{V}_{1,1}^\ell}\right)^{-1}\mathsf{K}_{\mathbf{W}_\ell,\epsilon_\ell}\right)^{-1}\mathsf{K}_{\epsilon_\ell}}.$$

In the expressions above U_1 and U_2 are independent zero-mean Gaussian random variables of variance P;

the sequences $\{Z_k\}$, $\{W_{1,k}\}$, $\{W_{2,k}\}$, $\{Y_k\}$, $\{V_{1,k}\}$, and $\{V_{2,k}\}$ are defined in Section II; the channel inputs at time k are given by

$$X_{1,k} = \sqrt{\frac{\mathsf{P}}{\mathsf{D}_{k-1}}} \left(U_1 - \mathbf{c}_{k-1}^{\mathsf{T}} \mathbf{V}_{1,1}^{k-1}\right)$$

$$X_{2,k} = (-1)^{k-1} \sqrt{\frac{\mathsf{P}}{\mathsf{D}_{k-1}}} \left(U_2 - \mathbf{c}_{k-1}^{\mathsf{T}} \mathsf{L}_{k-1} \mathbf{V}_{2,1}^{k-1}\right);$$

and for $k = 1, \ldots, \eta - 1$ the coefficients $\{\mathbf{c}_k\}$ are given in the equation at the top of page 5 and

$$\begin{aligned}
\epsilon_k &= \left(U_1 - \mathsf{K}_{\mathbf{Y}_1^k, U_1}^{\mathsf{T}} \mathsf{K}_{\mathbf{Y}_1^k}^{-1} \mathbf{Y}_1^k\right) \\
&\quad + (-1)^k \left(U_2 - \mathsf{K}_{\mathbf{Y}_1^k, U_2}^{\mathsf{T}} \mathsf{K}_{\mathbf{Y}_1^k}^{-1} \mathbf{Y}_1^k\right) \\
\mathbf{W}_k &= \mathbf{W}_{1,1}^k + (-1)^k \mathsf{L}_k \mathbf{W}_{2,1}^k \\
\Sigma_k &= \mathsf{K}_{\mathbf{W}_k} - \mathsf{K}_{\mathbf{W}_k, \epsilon_k} \mathsf{K}_{\epsilon_k}^{-1} \mathsf{K}_{\mathbf{W}_k, \epsilon_k}^{\mathsf{T}} \\
&\quad - \mathsf{K}_{\mathbf{W}_k, \mathbf{Y}_1^k} \mathsf{K}_{\mathbf{Y}_1^k}^{-1} \mathsf{K}_{\mathbf{W}_k, \mathbf{Y}_1^k}^{\mathsf{T}} \\
\mathsf{D}_k &= \mathsf{P} - 2\mathbf{c}_k^{\mathsf{T}} \mathsf{K}_{\mathbf{V}_{1,1}^k, U_1} + \mathbf{c}_k^{\mathsf{T}} \mathsf{K}_{\mathbf{V}_{1,1}^k} \mathbf{c}_k \\
\mathsf{L}_k &= \operatorname{diag}\left(1, (-1)^1, (-1)^2, \ldots, (-1)^{k-1}\right).
\end{aligned}$$

Note: Choosing in Theorem 2 the parameter η (perhaps sub-optimally) as $\eta = 2$ suffices to establish the symmetric version of Theorem 1.

Remark 1: We can improve the achievable rate region in Theorem 2 by modifying the encoding scheme achieving these rates to apply a different power allocation scheme over the transmitted symbols. In the scheme demonstrating Theorem 2 all the transmitted symbols have the same expected power. Instead, we can allocate different powers to the different transmitted symbols provided, of course, that on the average over a block of η symbols the power constraints are fulfilled. More specifically, for the scheme which achieves the rate region for $\eta = 2$ in Theorem 2 we envision a power allocation scheme over blocks of 2 consecutive channel uses which assigns an enhanced power of $(1+\gamma)\mathsf{P}$ to the first symbol and a reduced power of $(1-\gamma)\mathsf{P}$ to the second symbol in the block, where $\gamma \in [0,1]$. For this power allocation scheme it can be shown that the optimal choice of the parameter γ is given by the unique solution in the interval $[0,1]$ of the cubic equation

$$\begin{aligned}
&2\ \gamma \left(4\mathsf{P}(1+\gamma) + 3N + 2\sigma^2 + 2\sigma^2(1-\varrho)\frac{2\mathsf{P}(1-\gamma)}{N}\right) \\
&\cdot \left(\mathsf{P}(1+\gamma) + N + \sigma^2 + \sigma^2(1-\varrho)\frac{2\mathsf{P}(1-\gamma)}{N}\right) \\
&= (1-\gamma^2) 2\mathsf{P}\sigma^2(1-\varrho)\left(1 + \frac{2\mathsf{P}}{N}\right) \\
&+ (1-\gamma^2)\mathsf{P}\left(N + 2\sigma^2 + \sigma^2(1-\varrho)\frac{2\mathsf{P}}{N}\right) \qquad (9)
\end{aligned}$$

and is, in general, not equal to zero, thus, demonstrating that equal power allocation in our scheme is sub-optimal.

Remark 2: Another improvement can be based on a variation on a theme by Carleial [1]. In Carleial's scheme both users apply a rate-splitting strategy. They split their messages into two independent parts and with a fraction of the available power they encode the first part of the message using a scheme similar to the scheme proposed by Cover & Leung [3] whereas with the remaining power they encode the second part of the message ignoring the feedback. Each user then transmits the sum of the two encoded blocks. We propose to modify this scheme by encoding the second parts of the messages using the scheme we propose for Theorem 2 instead of Carleial's feedbackless encoding.

With this modification of the Carleial scheme we obtain a coding scheme for the Gaussian MAC with noisy feedback with achievable rate region including all the achievable rates for this setting known to date.

Theorem 3: The supremum over all achievable sum-rates in Theorem 2 converges to the sum-rate capacity with perfect feedback when the feedback-noise variance σ^2 tends to 0. Thus for the symmetric setting the sum-rate capacity with noisy feedback converges to Ozarow's perfect feedback sum-rate capacity:

$$\lim_{\sigma^2 \downarrow 0} \mathsf{C}_{\text{FB, sum}}^{(s)}(\mathsf{P}, N, \sigma^2, \varrho) = \mathsf{C}_{\text{PerfectFB, sum}}^{(s)}(\mathsf{P}, N). \qquad (10)$$

C. MAC with Partial Feedback

Theorem 4: Consider a two-user Gaussian multiple-access channel with a noiseless feedback link to User 2 and no feedback link to User 1. Let N be the channel-noise variance and P_1 and P_2 be the transmit powers for User 1 and User 2, respectively. Then for non-negative $(R_{1,0}, R_{1,1}, R_{1,2}, R_{2,0}, R_{2,1}, R_{2,2})$ the rate pair $(R_1 = R_{1,0} + R_{1,1} + R_{1,2}, R_2 = R_{2,0} + R_{2,1} + R_{2,2})$ is achievable if for some choice of $\beta_1, \beta_2, \lambda_1, \lambda_2 \in [0,1]$ all the following constraints are satisfied

$$\begin{aligned}
R_{1,0} &\le \frac{1}{4}\log\left(1 + \frac{2\bar{\beta}_1\mathsf{P}_1}{N}\right) \\
R_{2,0} &\le \frac{1}{4}\log\left(1 + \frac{\bar{\beta}_2\mathsf{P}_2\left(2 + \frac{\bar{\beta}_2\mathsf{P}_2}{\beta_1\mathsf{P}_1 + N}\right)}{N}\right) \\
R_{1,0} + R_{2,0} &\le \frac{1}{4}\log\left(1 + \frac{\bar{\beta}_1\mathsf{P}_1 + \bar{\beta}_2\mathsf{P}_2}{N}\right) \\
&+ \frac{1}{4}\log\left(1 + \frac{N_2}{N}\right) \\
R_{1,1} &\le \frac{1}{4}\log\left(1 + \frac{\beta_1\lambda_1\mathsf{P}_1}{\bar{\beta}_1\mathsf{P}_1 + N}\right) \\
R_{1,1} &\le \frac{1}{4}\log\left(1 + \frac{\beta_1\lambda_1\mathsf{P}_1}{N_1 + N}\right) \\
&+ \frac{1}{4}\log\left(1 + \frac{\left(\sqrt{\beta_1\bar{\lambda}_1\mathsf{P}_1} + \sqrt{\beta_2\bar{\lambda}_2\mathsf{P}_2}\right)^2}{N_1 + N + \beta_1\lambda_1\mathsf{P}_1 + \beta_2\lambda_2\mathsf{P}_2}\right) \\
R_{2,1} &\le \frac{1}{4}\log\left(1 + \frac{\beta_2\lambda_2\mathsf{P}_2}{N_1 + N}\right)
\end{aligned}$$

$$\mathbf{c}_k = \left(\Sigma_k + \frac{N}{\mathsf{P}}\mathsf{K}_{\mathbf{V}_{1,1}^k}\right)^{-1}\frac{N}{\mathsf{P}}\mathsf{K}_{\mathbf{V}_{1,1}^k,U_1} + N\frac{\left(\Sigma_k + \frac{N}{\mathsf{P}}\mathsf{K}_{\mathbf{V}_{1,1}^k}\right)^{-1}\left(1-\mathsf{K}^{\mathsf{T}}_{\mathbf{V}_{1,1}^k,U_1}\left(\frac{\mathsf{P}^2}{N}\Sigma_k + \mathsf{P}\mathsf{K}_{\mathbf{V}_{1,1}^k}\right)^{-1}\mathsf{K}_{\mathbf{V}_{1,1}^k,U_1}\right)}{\mathsf{K}_{\epsilon_k} + \frac{N}{\mathsf{P}}\mathsf{K}^{\mathsf{T}}_{\mathbf{V}_{1,1}^k,U_1}\left(\Sigma_k + \frac{N}{\mathsf{P}}\mathsf{K}_{\mathbf{V}_{1,1}^k}\right)^{-1}\mathsf{K}_{\mathbf{W}_k,\epsilon_k}}\mathsf{K}_{\mathbf{W}_k,\epsilon_k};$$

$$R_{1,1}+R_{2,1} \le \frac{1}{4}\log\left(1+\frac{\beta_1\mathsf{P}_1+\beta_2\mathsf{P}_2+2\sqrt{\beta_1\beta_2\mathsf{P}_1\mathsf{P}_2\bar{\lambda}_1\bar{\lambda}_2}}{N_1+N}\right)$$

$$R_{1,2} \le \frac{1}{4}\log\left(1+\frac{\beta_1\lambda_1\mathsf{P}_1}{\bar{\beta}_1\mathsf{P}_1\frac{N}{\bar{\beta}_1\mathsf{P}_1+N}+N}\right)$$

$$R_{1,2} \le \frac{1}{4}\log\left(1+\frac{\beta_1\lambda_1\mathsf{P}_1}{N_2+N}\right) + \frac{1}{4}\log\left(1+\frac{\left(\sqrt{\beta_1\bar{\lambda}_1\mathsf{P}_1}+\sqrt{\beta_2\bar{\lambda}_2\mathsf{P}_2}\right)^2}{N_2+N+\beta_1\lambda_1\mathsf{P}_1+\beta_2\lambda_2\mathsf{P}_2}\right)$$

$$R_{2,2} \le \frac{1}{4}\log\left(1+\frac{\beta_2\lambda_2\mathsf{P}_2}{N_2+N}\right)$$

$$R_{1,2}+R_{2,2} \le \frac{1}{4}\log\left(1+\frac{\beta_1\mathsf{P}_1+\beta_2\mathsf{P}_2+2\sqrt{\beta_1\beta_2\mathsf{P}_1\mathsf{P}_2\bar{\lambda}_1\bar{\lambda}_2}}{N_2+N}\right).$$

Here $\bar{x} \triangleq 1-x$; $N_1 \triangleq \bar{\beta}_1\mathsf{P}_1+\bar{\beta}_2\mathsf{P}_2$; $N_2 \triangleq \bar{\beta}_2\mathsf{P}_2 + \bar{\beta}_1\mathsf{P}_1\frac{\bar{\beta}_2\mathsf{P}_2+N}{\bar{\beta}_1\mathsf{P}+\bar{\beta}_2\mathsf{P}_2+N} + 2\sqrt{\mathsf{P}_1\mathsf{P}_2\bar{\beta}_1\bar{\beta}_2\frac{\bar{\beta}_1\mathsf{P}_1}{\bar{\beta}_1\mathsf{P}_1+N}\frac{\bar{\beta}_2\mathsf{P}_2}{\bar{\beta}_1\mathsf{P}_1+\bar{\beta}_2\mathsf{P}_2+N}}$.

The above rate region includes the Cover-Leung region (7), with the inclusion being for some values of $\mathsf{P}_1, \mathsf{P}_2, N$ strict. This proves that, in general, for the two-user Gaussian multiple-access channel with partial feedback the Cover-Leung region is not equal to the capacity region.

Note: The rate region stated in Theorem 4 can be achieved by a modification of the encoding scheme described in Remark 2 to the case of partial feedback.

An example of a set of channel parameters for which the Cover-Leung region is a strict subset of the region in Theorem 4, is given by $\mathsf{P}_1 = 1, \mathsf{P}_2 = 5, N = 5$. The strictness of the inclusion can in fact be shown by (possibly sub-optimally) choosing $\beta_1 = \beta_2 = 0$.

D. Symmetric MAC with Noisy Feedback and Receiver Side Information

Theorem 5: Consider the channel described in Theorem 2 under the additional assumption that the receiver has perfect side-information about the realizations of the feedback-noise sequences. Then a non-negative rate pair (R_1, R_2) is achievable if for some positive integer η both

$$R_1, R_2 \le \frac{1}{2\eta}\sum_{\ell=0}^{\eta-1}\log\left(1+\frac{\mathsf{P}\left(1-\rho_\ell^2\right)}{N}\frac{\alpha_\ell}{\mathsf{D}_\ell}\right)$$

and

$$R_1 + R_2 \le \frac{1}{2\eta}\sum_{\ell=0}^{\eta-1}\log\left(1+\frac{2\mathsf{P}\left(1+(-1)^\ell\rho_\ell\right)}{N}\frac{\alpha_\ell}{\mathsf{D}_\ell}\right)$$

where we define recursively for $\ell = 0, \ldots, \eta-1$

$$\alpha_{\ell+1} = \alpha_\ell\left(\frac{\frac{\alpha_\ell}{\mathsf{D}_\ell}\mathsf{P}\left(1-\rho_\ell^2\right)+N}{\frac{\alpha_\ell}{\mathsf{D}_\ell}2\mathsf{P}\left(1+(-1)^\ell\rho_\ell\right)+N}\right)$$

$$\rho_{\ell+1} = \frac{\alpha_\ell}{\alpha_{\ell+1}}\left(\rho_\ell - \frac{(-1)^\ell\frac{\alpha_\ell}{\mathsf{D}_\ell}\mathsf{P}\left(1+(-1)^\ell\rho_\ell\right)^2}{\frac{\alpha_\ell}{\mathsf{D}_\ell}2\mathsf{P}\left(1+(-1)^\ell\rho_\ell\right)+N}\right)$$

$$\mathsf{D}_{\ell+1} = \mathsf{D}_\ell\left(\frac{\mathsf{P}(1-\vartheta_\ell^2)+N+\sigma^2}{2\mathsf{P}\left(1+(-1)^\ell\vartheta_\ell\right)+N+\sigma^2}\right)$$

$$\vartheta_{\ell+1} = \frac{\mathsf{D}_\ell}{\mathsf{D}_{\ell+1}}\left(\vartheta_\ell - \frac{(-1)^\ell\mathsf{P}\left(1+(-1)^\ell\vartheta_\ell\right)^2}{2\mathsf{P}\left(1+(-1)^\ell\vartheta_\ell\right)+N+\sigma^2}\cdot\frac{2\mathsf{P}\left(1+(-1)^\ell\vartheta_\ell\right)+N+2\sigma^2-\varrho\sigma^2}{2\mathsf{P}\left(1+(-1)^\ell\vartheta_\ell\right)+N+\sigma^2}\right)$$

and $\alpha_0 = \mathsf{P}, \rho_0 = 0, \mathsf{D}_0 = \mathsf{P}, \vartheta_0 = 0$.

REFERENCES

[1] A. B. Carleial. Multiple-access channels with different generalized feedback signals. *IEEE Trans. Inform. Theory*, 28(6):841–850, Nov. 1982.

[2] T. M. Cover. Some advances in broadcast channels. In A. Viterbi, editor, *Advances in Communication Systems*, volume 4. San Francisco: Academic Press, 1975.

[3] T. M. Cover and C. S. K. Leung. An achievable rate region for the multiple-access channel with feedback. *IEEE Trans. Inform. Theory*, 27(3):292–298, May 1981.

[4] M. Gastpar. On noisy feedback in Gaussian networks. In *Proc. 43rd Allerton Conf. Comm., Contr. and Comp.*, Allerton H., Monticello, Il, 2005.

[5] M. Gastpar and G. Kramer. On cooperation via noisy feedback. In *Proc. IZS*, pages 146–149, Feb. 22–24, 2006.

[6] L. H. Ozarow. The capacity of the white Gaussian multiple-access channel with feedback. *IEEE Trans. Inform. Theory*, 30(4):623–629, July 1985.

[7] J. P. M. Schalkwijk and T. Kailath. A coding scheme for additive noise channels with feedback-i:no bandwidth constraint. *IEEE Trans. Inform. Theory*, 12:172–182, Apr. 1966.

[8] E. C. van der Meulen. Capacity theorems for multiple-access channels with feedback. In *ISICT'87*, Campinas, Brazil, July 27 - August 01 1987.

[9] F. M. J. Willems and E. C. van der Meulen. Partial feedback for the discrete memoryless multiple access channel. *IEEE Trans. Inform. Theory*, 29(2):287–290, Mar. 1983.

[10] F. M. J. Willems, E. C. van der Meulen, and J. P. M. Schalkwijk. A coding scheme for the additive white Gaussian noise multiple access channel with semi-feedback. *Tijdschrft van het Nederlands Elektronica-en Radiogenootschap*, 48(3):103–107, 1983.

[11] A. D. Wyner. Recent results in the Shannon theory. *IEEE Trans. Inform. Theory*, 20:2–10, Jan. 1974.