Sending a Bi-Variate Gaussian Source over a Gaussian MAC

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***Abstract*— We consider a problem where a memoryless bi-variate Gaussian source is to be transmitted over an additive white Gaussian multiple-access channel with two transmitting terminals and one receiving terminal. The first transmitter only sees the first source component and the second transmitter only sees the second source component. We are interested in the pair of mean squared-error distortions at which the receiving terminal can reproduce each of the source components.**

It is demonstrated that in the symmetric case, below a certain signal-to-noise ratio (SNR) threshold, which is determined by the source correlation, uncoded communication is optimal. For SNRs above this threshold we present outer and inner bounds on the achievable distortions.

I. Introduction

We consider the situation where a memoryless bi-variate Gaussian source is to be transmitted over an additive white Gaussian multiple-access channel with two transmitting terminals and one receiving terminal. Each of the two source components is fed to a different average-power constrained encoder. Our interest lies in the achievable expected squared-error distortion region. We show that in the symmetric case, where the source components are of the same variance and the transmitting terminals are subjected to the same average power constraint, uncoded transmission is optimal below a threshold signal-to-noise ratio (SNR) that is determined by the correlation between the source components. For SNRs above this threshold we provide outer and inner bounds on the achievable distortions.

The problem at hand can be viewed as the Gaussian version of the problem addressed by Cover, El Gamal and Salehi [1] (see also [2] and [3]). It also appears to be closely related to the quadratic Gaussian CEO problem [6], [7] and the quadratic Gaussian two-terminal source-coding problem [4], [5]. However, it differs in character from the CEO problem and from the two-terminal source coding problem in that no error-free bit-pipes of finite rates can be assumed. This is due to the fact that the source-channel separation theorem does not apply to our situation. Furthermore, the CEO problem focuses on the reconstruction of a single Gaussian random variable, whereas in our case the interest lies in the reconstruction of both source components.

II. Problem Statement

The time-k output $Y_k \in \mathbb{R}$ of the discrete-time two-user additive white Gaussian multiple-access channel is given by

$$Y_k = x_{1,k} + x_{2,k} + Z_k,$$

where $x_{1,k} \in \mathbb{R}$ denotes the time-k symbol transmitted by the first transmitter, $x_{2,k} \in \mathbb{R}$ is the time-k symbol transmitted by the second transmitter, and Z_k denotes the time-k noise term. The noise terms $\{Z_k\}$ are independent identically distributed (IID) zero-mean variance-N Gaussian random variables that are independent of the input sequences $(\{x_{1,k}\}, \{x_{2,k}\})$. We shall consider the case where Transmitter 1 and Transmitter 2 are average-power limited to P_1 and P_2 respectively. See (1) ahead.

At time k the source emits the pair $(S_{1,k}, S_{2,k})$ where the $\{(S_{1,k}, S_{2,k})\}$ are IID zero-mean Gaussians of covariance

$$\mathsf{K}_{SS} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix},$$

with $\rho \in [-1,1]$, and $0 < \sigma_i^2 < \infty$, $i = 1,2$.

The sequence $\{S_{1,k}\}$ is fed to Transmitter 1 and the sequence $\{S_{2,k}\}$ is fed to Transmitter 2. Based on the channel output we wish to reconstruct the source vector. The performance criterion we focus on is the expected squared-error distortions in reconstructing each of the components of the source vector.

Definition 1: Given $\sigma_1, \sigma_2 > 0$, $\rho \in [-1,1]$, and $P_1, P_2 > 0$ we say that the tuple $\left(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2\right)$ is achievable if there exists a sequence of encoder pairs $(f_1^{(n)}, f_2^{(n)})$

$$f_i^{(n)} : \mathbb{R}^n \to \mathbb{R}^n, \quad i = 1,2$$

and a sequence of reconstruction pairs $(\phi_1^{(n)}, \phi_2^{(n)})$

$$\phi_i^{(n)} : \mathbb{R}^n \to \mathbb{R}^n, \quad i = 1,2$$

such that the average power constraints are satisfied

$$\frac{1}{n}\mathsf{E}\left[\|f_i^{(n)}(S_i^n)\|^2\right] \le P_i, \quad i = 1,2 \tag{1}$$

and

$$\varlimsup_{n\to\infty} \frac{1}{n}\mathsf{E}\left[\Big\|(S_{i,1}, \ldots, S_{i,n}) - \phi_i^{(n)}\left(f_1^{(n)}(S_1^n) + f_2^{(n)}(S_2^n) + (Z_1, \ldots, Z_n)\right)\Big\|^2\right] \le D_i, \quad i = 1,2, \tag{2}$$

whenever $\{(S_{1,k}, S_{2,k})\}$ are IID zero-mean bi-variate Gaussian vectors of covariance matrix K_{SS} as above and $\{Z_k\}$ are IID zero-mean variance-N random variables that are independent of $\{(S_{1,k}, S_{2,k})\}$. Here we used the shorthand notation where S_1^n denotes $(S_{1,1}, \ldots, S_{1,n})$ and similarly for S_2^n.

The problem we address here is, for given $\sigma_1^2, \sigma_2^2, \rho, P_1, P_2$, to find the set of pairs (D_1, D_2) such that $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2)$ is achievable.

By the symmetric version of this problem we shall refer to the case where $\sigma_1^2 = \sigma_2^2$, where $P_1 = P_2$, and where we seek the set of pairs (D, D) that are achievable. That is, if we set $\sigma^2 = \sigma_1^2 = \sigma_2^2$ and $P = P_1 = P_2$ then we are interested in

$$D^*(P, N, \sigma^2, \rho) \triangleq \sup\{D : (D, D, \sigma^2, \sigma^2, \rho, P, P) \text{ is achievable}\}. \quad (3)$$

III. Preliminary Remarks

Before discussing our results, we make three remarks regarding the general nature of the problem. The firs two remarks show that there is no loss in generality by assuming that the correlation coefficient is non-negative and that the source components are of equal variance. As a consequence we shall assume for the remainder that $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and that $\rho \in [0, 1]$. The third remark addresses a convexification issue of the distortion regions.

1) The optimal distortion region depends on the correlation coefficient only via its absolute value $|\rho|$. That is, the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2)$ is achievable if, and only if, the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, -\rho, P_1, P_2)$ is achievable. To see this note that if $(f_1^{(n)}, f_2^{(n)}, \phi_1^{(n)}, \phi_2^{(n)})$ achieves the distortion (D_1, D_2) for the source of correlation coefficient ρ, then $(\tilde{f}_1^{(n)}, f_2^{(n)}, \tilde{\phi}_1^{(n)}, \phi_2^{(n)})$ where

$$\tilde{f}_1^{(n)}(S_1^n) = f_1^{(n)}(-S_1^n)$$

and

$$\tilde{\phi}_1^{(n)}(Y_1, \ldots, Y_n) = -\phi_1^{(n)}(Y_1, \ldots, Y_n)$$

achieves (D_1, D_2) on the source with correlation coefficient $-\rho$.

2) The optimal distortions scale linearly with the source variances. That is, if α_1, α_2 are positive then $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2)$ is achievable if, and only if, $(\alpha_1^2 D_1, \alpha_2^2 D_2, \alpha_1^2\sigma_1^2, \alpha_2^2\sigma_2^2, \rho, P_1, P_2)$ is achievable. Consequently, there is a simple linear transformation from the set of tuples (D_1, D_2) for which $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2)$ is achievable and the set of tuples $(\tilde{D}_1, \tilde{D}_2)$ for which $(\tilde{D}_1, \tilde{D}_2, \alpha_1^2\sigma_1^2, \alpha_2^2\sigma_2^2, \rho, P_1, P_2)$ is achievable.
To see this note that if $(f_1^{(n)}, f_2^{(n)}, \phi_1^{(n)}, \phi_2^{(n)})$ demonstrate the achievability of $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2)$ then the encoders

$$\tilde{f}_i^{(n)}(S_i^n) = f_i^{(n)}(S_i^n/\alpha_i) \quad i = 1, 2$$

and the reconstructions

$$\tilde{\phi}_i^{(n)}(Y_1, \ldots, Y_n) = \alpha_i \cdot \phi_i^{(n)}(Y_1, \ldots, Y_n), \quad i = 1, 2$$

demonstrate the achievability of the tuple $(\alpha_1^2 D_1, \alpha_2^2 D_2, \alpha_1^2\sigma_1^2, \alpha_2^2\sigma_2^2, \rho, P_1, P_2)$.
Applying the same argument in the other direction with scalings by $1/\alpha_1$ and $1/\alpha_2$ concludes the proof.

3) The achievable distortion is a convex function of the power constraints (P_1, P_2). That is, if $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2)$ and $(\tilde{D}_1, \tilde{D}_2, \sigma_1^2, \sigma_2^2, \rho, \tilde{P}_1, \tilde{P}_2)$ are achievable then

$$\left(\lambda D_1 + \bar{\lambda}\tilde{D}_1, \lambda D_2 + \bar{\lambda}\tilde{D}_2, \sigma_1^2, \sigma_2^2, \rho, \lambda P_1 + \bar{\lambda}\tilde{P}_1, \lambda P_2 + \bar{\lambda}\tilde{P}_2\right)$$

is achievable for any $\lambda \in [0, 1]$, where $\bar{\lambda} = (1 - \lambda)$.
This follows by a simple time-sharing argument

IV. Main Results

We present necessary conditions as well as sufficient conditions for achievability. In certain cases they agree. The proofs of those conditions will be discussed in the next section.

Our first result is a necessary condition for the achievability of $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2)$.

Theorem 1: A necessary condition for the achievability of $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2)$ is that

$$\frac{1}{2}\log\left(1 + \frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2}}{N}\right) \geq R(D_1, D_2),$$

where the expression for $R(D_1, D_2)$ varies, depending on the values of (D_1, D_2). There are three cases. If (D_1, D_2) are in the set

$$\left\{D_1 \leq \sigma^2(1 - \rho), D_2 \leq (\sigma^2(1 - \rho^2) - D_1)\frac{\sigma^2}{\sigma^2 - D_1}\right\},$$

then

$$R(D_1, D_2) = \frac{1}{2}\log_2\left(\frac{\sigma^4(1 - \rho^2)}{D_1 D_2}\right).$$

If (D_1, D_2) are in the set

$$\Big\{0 \leq D_1 \leq \sigma^2,$$
$$(\sigma^2(1 - \rho^2) - D_1)\frac{\sigma^2}{\sigma^2 - D_1} \leq D_2 \leq \sigma^2(1 - \rho^2) + \rho^2 D_1\Big\},$$

then

$$R(D_1, D_2) =$$
$$\frac{1}{2}\log_2\left(\frac{\sigma^4(1 - \rho^2)}{D_1 D_2 - \left(\rho\sigma^2 - \sqrt{(\sigma^2 - D_1)(\sigma^2 - D_2)}\right)^2}\right),$$

and if (D_1, D_2) are in the set

$$\left\{0 \leq D_1 \leq \sigma^2, D_2 > \sigma^2(1 - \rho^2) + \rho^2 D_1\right\}.$$

then

$$R(D_1, D_2) = \frac{1}{2}\log_2\left(\frac{\sigma^2}{D_1}\right).$$

Corollary 1: In the symmetric case where $P_1 = P_2$, we obtain

$$D^*(\sigma^2,\rho,P,N) \geq \begin{cases} \sigma^2 \frac{P(1-\rho^2)+N}{2P(1+\rho)+N} & \text{for } \frac{P}{N} \in \left(0, \frac{\rho}{1-\rho^2}\right] \\ \sigma^2 \sqrt{\frac{(1-\rho^2)N}{2P(1+\rho)+N}} & \text{for } \frac{P}{N} > \frac{\rho}{1-\rho^2}. \end{cases}$$

Note: Theorem 1 can be easily extended to a much wider class of sources and distortions. Indeed, if the source is any memoryless bi-variate source (not necessarly zero-mean Gaussian) and if the fidelity measures $d_1(s_1,\hat{s}_1), d_2(s_2,\hat{s}_2) \geq 0$ that are used to measure the distortion in reconstructing each of the source components are arbitrary, then the pair (D_1, D_2) is achievable with powers P_1, P_2 only if

$$\min_{P_{\widehat{S}_1,\widehat{S}_2|S_1,S_2}} I(S_1,S_2;\widehat{S}_1,\widehat{S}_2) \tag{4}$$

$$\text{such that } \mathsf{E}\left[(S_1-\widehat{S}_1)^2\right] \leq D_1, \quad \mathsf{E}\left[(S_2-\widehat{S}_2)^2\right] \leq D_2,$$

does not exceed

$$\frac{1}{2}\log\left(1+\frac{P_1+P_2+2\rho_{\text{max}}\sqrt{P_1P_2}}{N}\right),$$

where ρ_{max} is the Hirschfeld-Gebelein-Rényi maximal correlation between S_1 and S_2:

$$\rho_{\text{max}} = \sup \mathsf{E}[g(S_1)h(S_2)] \tag{5}$$

where the supremum is over all functions g, h under which

$$\mathsf{E}[g(S_1)] = \mathsf{E}[h(S_2)] = 0 \qquad \mathsf{E}\left[g^2(S_1)\right] = \mathsf{E}\left[h^2(S_2)\right] = 1 \tag{6}$$

We next present two sufficient conditions for the achievability of $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2)$. The first is obtained by analyzing uncoded transmission.

Theorem 2: For $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2)$ to be achievable it suffices that both of the following conditions hold:

$$D_1 \geq \sigma^2 \frac{(1-\rho^2)P_2+N}{P_1+P_2+2\rho\sqrt{P_1P_2}+N}$$

$$D_2 \geq \sigma^2 \frac{(1-\rho^2)P_1+N}{P_1+P_2+2\rho\sqrt{P_1P_2}+N}.$$

Corollary 2: In the symmetric case

$$D^*(\sigma^2,\rho,P,N) \leq \sigma^2 \frac{P(1-\rho^2)+N}{2P(1+\rho)+N}$$

Combining Corollary 1 and Corollary 2, we obtain:

Corollary 3: For the symmetric case,

$$D^*(\sigma^2,\rho,P,N) = \sigma^2 \frac{P(1-\rho^2)+N}{2P(1+\rho)+N}, \qquad \text{if } \frac{P}{N} < \frac{\rho}{1-\rho^2}$$

i.e., uncoded transmission is optimal for all $P/N < \rho/(1-\rho^2)$.

The second sufficient condition follows from analyzing the scheme where the encoding functions $f_i^{(n)}(s_i^n)$, $i = 1, 2$, are randomly generated independent rate-R_i vector quantizers, i.e. the channel inputs are the rate-R_i vector quantized source sequences.

Theorem 3: The tuple $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2)$ is achievable whenever there exist rates $R_1 > 0$ and $R_2 > 0$ such that all of the following hold:

$$R_1 < \frac{1}{2}\log_2\left(\frac{P_1(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)}\right)$$

$$R_2 < \frac{1}{2}\log_2\left(\frac{P_2(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)}\right)$$

$$R_1+R_2 < \frac{1}{2}\log_2\left(\frac{P_1+P_2+2\tilde{\rho}\sqrt{P_1P_2}+N}{N(1-\tilde{\rho}^2)}\right)$$

$$D_1 > \sigma^2 2^{-2R_1} \cdot \frac{1-\rho^2(1-2^{-2R_2})}{1-\tilde{\rho}^2}$$

$$D_2 > \sigma^2 2^{-2R_2} \cdot \frac{1-\rho^2(1-2^{-2R_1})}{1-\tilde{\rho}^2}.$$

where $\tilde{\rho} = \rho\sqrt{(1-2^{-2R_1})(1-2^{-2R_2})}$.

Corollary 4: In the symmetric case $(D, D, \sigma^2, \sigma^2, \rho, P, P)$ is achievable if there exists some $R > 0$ satisfying

$$R < \frac{1}{4}\log_2\left(\frac{2P(1+\rho(1-2^{-2R}))+N}{N(1-\rho^2(1-2^{-2R})^2)}\right) \tag{7}$$

$$D > \sigma^2 2^{-2R} \cdot \frac{1-\rho^2(1-2^{-2R})}{1-\rho^2(1-2^{-2R})^2}. \tag{8}$$

Here the RHS of (8) is monotonically decreasing in R. Evaluating Corollary 4 and Corollary 1 for $P/N \to \infty$ we get:

Corollary 5: In the symmetric case

$$\lim_{P/N\to\infty} \sqrt{\frac{P}{N}} D^*(\sigma^2,\rho,P,N) = \sigma^2\sqrt{\frac{1-\rho}{2}}.$$

We conclude this section with a note on the superposition of the two discussed coding schemes.

Note: We have analyzed two coding schemes; uncoded transmission and transmission of vector-quantized source sequences. The superposition of those two schemes, analogous to the scheme discussed for the single-user case in [9], seems to yield strict improvements of the above discussed achievable $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2)$. Detailed results are to follow.

V. Notes on the Derivations

In this section we shall try to sketch the ideas behind the proofs of the main results.

The proof of Theorem 1 consists on one hand of upper bounding the mutual information between the the source vectors and the reconstructions, and on the other hand evaluating the rate distortion function for a bi-variate Gaussian source. The key to upper bounding the mutual information between source and reconstructions is to use the average power constraints (1) and the limited correlation between the source components to obtain the upper bound

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{Var}\big(X_{1,k}(S_1^n)+X_{2,k}(S_2^n)\big) \leq P_1+P_2+2\rho\sqrt{P_1P_2} \tag{9}$$

where $X_{1,k}(S_1^n)$ is the k-th component of $f_1^{(n)}(S_1^n)$ and where $X_{2,k}(S_2^n)$ is analogously defined. Once this bound is established for all encoders $f_1^{(n)}$, $f_2^{(n)}$ satisfying the power constraints (1), one can derived necessary conditions for achievability by using the data processing inequality to upper bound the mutual information between the source vectors and their reconstructions by the mutual information between the transmitted waveforms and the received waveform. This latter mutual information is upper bounded by the capacity of the additive Gaussian noise channel subject to the power constraint $P_1 + P_2 + 2\rho\sqrt{P_1 P_2}$.

The rate distortion function is obtained from evaluating (4) under the given distortion constraints and for the given source law P_{S_1,S_2}. From the maximum mutual information theorem it follows that this minimum is achieved if and only if $S_1, S_2, \widehat{S}_1, \widehat{S}_2$ are jointly Gaussian. The minimization problem is then reduced to a minimization over the set of covariance matrices of $S_1, S_2, \widehat{S}_1, \widehat{S}_2$ that satisfy the distortion constraints and where the submatrix in S_1, S_2 is the covariance matrix of the source. The minimizing covariance matrix can be found by noticing that every relevant distortion pair can be achieved, with minimal necessary rate, by combining a scaling of the source with reverse waterfilling. Let $\mathcal{D}(R)$ be the set of all distortion pairs (d_1, d_2) that can be achieved on the source pair (S_1, S_2) with rate R, and let $\mathcal{D}_c(R)$ be the set of (d_1, d_2) that can be achieved with rate R on the scaled source (S_1, cS_2). The region $\mathcal{D}_c(R)$ corresponds to the region $\mathcal{D}(R)$ scaled by a factor c^2 on the S_2-axis. Reverse waterfilling at rate R on the unitarily decorrelated pair (V_1, V_2) of (S_1, cS_2) achieves the point $(d_1^*, d_2^*) \in \mathcal{D}_c(R)$ of minimal sum $d_1 + d_2$. And since R is the minimal rate needed to achieve (d_1^*, d_2^*) on (S_1, cS_2), and

$$\min_{\substack{P_{\widehat{S}_1,\widehat{S}_2|S_1,S_2}: \\ \mathsf{E}\left[(S_1-\widehat{S}_1)^2\right]\le d_1 \\ \mathsf{E}\left[(S_2-\widehat{S}_2)^2\right]\le \frac{1}{c^2} d_2}} I(S_1, S_2; \widehat{S}_1, \widehat{S}_2) = \min_{\substack{P_{\widehat{S}_1,\widehat{S}_2|S_1,S_2}: \\ \mathsf{E}\left[(S_1-\widehat{S}_1)^2\right]\le d_1 \\ \mathsf{E}\left[(cS_2-c\widehat{S}_2)^2\right]\le d_2}} I(S_1, cS_2; \widehat{S}_1, c\widehat{S}_2),$$

the rate R is also the minimal rate needed to achieve $(d_1^*, d_2^*/c^2)$ on (S_1, S_2). Hence, by choosing the appropriate scaling c, we can get any relevant point on the boundary of $\mathcal{D}(R)$. The covariance matrix of $(S_1, S_2, \widehat{S}_1, \widehat{S}_2)$ that achieves $(d_1^*, d_2^*/c^2)$ now follows from the covariance matrix of $(V_1, V_2, \widehat{V}_1, \widehat{V}_2)$, where $(\widehat{V}_1, \widehat{V}_2)$ result from reverse waterfilling at rate R on (V_1, V_2).[1]

The proof of Theorem 2 is straightforward. One merely considers the uncoded scheme where

$$f_i^{(n)}(S_i^n) = \frac{\sqrt{P_i}}{\sigma}(S_{i,1}, \ldots S_{i,n}), \quad i = 1, 2$$

and then analyzes the linear minimum mean squared-error estimators of $S_{i,k}$ from Y_k.

[1]We note that this idea generalizes to Gaussian sources with more than two components.

The proof of Theorem 3 involves an analysis of randomly generated independent vector quantizers for the two components. The proposed scheme is conceptually simple, but its analysis gets involved by the included epsilons and deltas. For the sake of clarity and brevity we shall omit these epsilons and deltas here.

The encoder for the i-th, $i = 1, 2$, source component is a rate-R_i Gaussian vector quantizer that scales the quantized sequence to meet the channel input power constraint. Its codebook $\mathcal{C}_i$ consists of 2^{nR_i} codewords that are chosen IID uniformly on the surface of an $\mathbb{R}^n$-sphere of center at the origin and radius $\sqrt{n\sigma^2(1 - 2^{-2R_i})}$. Encoder i chooses the codeword $\mathbf{u}_i^*$ in the codebook $\mathcal{C}_i$ that is closest (in Euclidean distance) to the source sequence $\mathbf{s}_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,n})$, and transmits its scaled version

$$\begin{aligned} \mathbf{x}_i &= \alpha_i \operatorname*{argmin}_{\mathbf{u}\in\mathcal{C}_i} \|\mathbf{s}_i - \mathbf{u}\| \\ &= \alpha_i \operatorname*{argmax}_{\mathbf{u}\in\mathcal{C}_i} \langle \mathbf{s}_i, \mathbf{u} \rangle, \end{aligned}$$

where

$$\alpha_i = \sqrt{\frac{P_i}{\sigma^2(1 - 2^{-2R_i})}},$$

and where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. The distance $\|\mathbf{s}_i - \mathbf{u}_i^*\|$ between the source sequence $\mathbf{s}_i$ and its closest codeword $\mathbf{u}_i^*$ approaches, with high probability, $\sigma^2 \cdot 2^{-2R_i}$ as the blocklength n tends to infinity. It can be shown that, for large n, the correlation coefficient between the chosen codewords $\mathbf{U}_1^*$ and $\mathbf{U}_2^*$ is, with very high probability, close to

$$\tilde{\rho} = \rho\sqrt{(1 - 2^{-2R_1})(1 - 2^{-2R_2})}.$$

This coefficient $\tilde{\rho}$ plays a central role in this coding scheme.

The decoding is performed in two parts. First the transmitted codeword pair is recovered, and then this codeword pair is used to make linear estimates of the source sequences. To recover the transmitted pair $(\mathbf{u}_1^*, \mathbf{u}_2^*)$, the decoder seeks, among all "jointly typical" pairs $(\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{C}_1 \times \mathcal{C}_2$, i.e among all pairs satisfying

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle \approx \tilde{\rho}\, \|\mathbf{u}_1\|\, \|\mathbf{u}_2\|,$$

the codeword pair $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ whose weighted sum $\alpha_1\hat{\mathbf{u}}_1 + \alpha_2\hat{\mathbf{u}}_2$ has the smallest angle to the channel output $\mathbf{y}$, i.e.

$$(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2) = \operatorname*{argmax}_{\substack{(\mathbf{u}_1,\mathbf{u}_2)\in\mathcal{C}_1\times\mathcal{C}_2: \\ \langle \mathbf{u}_1,\mathbf{u}_2\rangle \approx \tilde{\rho}\|\mathbf{u}_1\|\|\mathbf{u}_2\|}} \left\langle \frac{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2}{\|\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2\|}, \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\rangle.$$

The corresponding source estimates are then

$$\begin{aligned} \hat{\mathbf{s}}_1 &= \beta_1\hat{\mathbf{u}}_1 + \gamma_1\hat{\mathbf{u}}_2 \\ \hat{\mathbf{s}}_2 &= \beta_2\hat{\mathbf{u}}_1 + \gamma_2\hat{\mathbf{u}}_2, \end{aligned}$$

where the coefficients β_1, γ_1, β_2, γ_2 are chosen such that $(\hat{\mathbf{s}}_1, \hat{\mathbf{s}}_2)$ would form the minimum mean squared-error estimates of $(\mathbf{s}_1, \mathbf{s}_2)$ if S_1, S_2, U_1^*, U_2^* were zero-mean joint

Gaussians with correlation coefficients

$$\rho(S_1, S_2) = \rho, \qquad \rho(S_1, U_1^*) = \sqrt{1 - 2^{-2R_1}}$$

$$\rho(S_1, U_2^*) = \rho\sqrt{1 - 2^{-2R_2}}, \quad \rho(S_2, U_1^*) = \rho\sqrt{1 - 2^{-2R_1}}$$

$$\rho(S_2, U_2^*) = \sqrt{1 - 2^{-2R_2}}, \qquad \rho(U_1^*, U_2^*) = \tilde{\rho}.$$

The analysis of the three error events $\{\hat{\mathbf{u}}_1 \neq \mathbf{u}_1^*, \hat{\mathbf{u}}_2 = \mathbf{u}_2^*\}$, $\{\hat{\mathbf{u}}_1 = \mathbf{u}_1^*, \hat{\mathbf{u}}_2 \neq \mathbf{u}_2^*\}$, and $\{\hat{\mathbf{u}}_1 \neq \mathbf{u}_1^*, \hat{\mathbf{u}}_2 \neq \mathbf{u}_2^*\}$ gives that reliable transmission of the pair $(\mathbf{u}_1^*, \mathbf{u}_2^*)$ is possible for all rates (R_1, R_2) in the region[2]

$$\mathcal{R} = \Bigg\{ (R_1, R_2) : R_1 < \frac{1}{2}\log_2\left(\frac{P_1(1-\tilde{\rho}^2) + N}{N(1-\tilde{\rho}^2)}\right)$$

$$R_2 < \frac{1}{2}\log_2\left(\frac{P_2(1-\tilde{\rho}^2) + N}{N(1-\tilde{\rho}^2)}\right)$$

$$R_1 + R_2 < \frac{1}{2}\log_2\left(\frac{P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1P_2} + N}{N(1-\tilde{\rho}^2)}\right) \Bigg\}.$$

It can then be shown that for all $(R_1, R_2) \in \mathcal{R}$, the proposed sequence of schemes achieves the distortions[3]

$$D_1 = \sigma^2 2^{-2R_1} \cdot \frac{1 - \rho^2(1 - 2^{-2R_2})}{1 - \tilde{\rho}^2}$$

$$D_2 = \sigma^2 2^{-2R_2} \cdot \frac{1 - \rho^2(1 - 2^{-2R_1})}{1 - \tilde{\rho}^2}.$$

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[2] These rate constraints are similar to Ozarow’s capacity result for the Gaussian multiple-access channel with feedback [8].

[3] These expressions are similar to the single-rate constraints in the quadratic Gaussian two-terminal source coding result [4], [5].