

A Novel Capacitance Assignment Procedure for the Design of Sensitivity- and Noise-Optimized SC-Filters

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Abstract—A novel procedure that determines the capacitor values for a given integrator-based SC network with given capacitor ratios is presented. The procedure optimally distributes a limited capacitance area among the individual circuit capacitors by minimizing the overall capacitor spread while simultaneously minimizing either sensitivity or noise. Noise in SC circuits is a function of ideal SC design parameters such as capacitor ratios and capacitance levels and of the technology-dependent parameters describing the switches and amplifiers. In our description of the noise performance, we have found a characteristic point which is only a function of SC design parameters and can thus serve as a measure for the noise performance. For its description a closed-form expression is used, which has the same form as the corresponding sensitivity measure. With these expressions an efficient capacitance assignment optimization procedure is derived, which is implemented in the computer-aided design and optimization program package SCSYN.

I. INTRODUCTION

ASSUMING that the filter to be realized is specified by a given rational transfer function, the design process of a switched-capacitor (SC) filter may be divided into i) finding a suitable SC structure, ii) mapping the given polynomial coefficients onto the various capacitor ratios, which includes dynamic scaling, and iii) distributing a limited capacitance area among the individual circuit capacitors. In this contribution we concentrate on the third step: the capacitance assignment problem. We thereby restrict ourselves to the important class of two-phase SC structures that use integrators as building blocks, such as biquad cascades or ladder structures, and state the problem as follows: *For an SC filter to be realized as a given integrator-based structure with given capacitor ratios, assign the fixed total capacitance to the various capacitors in the filter according to some optimization criterion.* Since

the various integrator stages can be scaled individually, the problem has $(n - 1)$ degrees of freedom for an n th-order filter.

The conventional approach to this problem is to assign absolute values to the individual filter capacitors by choosing the smallest capacitors of every integrator stage to be equal to some unit capacitor. The remaining capacitors are then determined by the given capacitor ratios, cf. [1, p. 346]. This conventional procedure does not, however, take into account the individual behavior of the integrator stages, e.g., their contributions to frequency-response errors and output noise. One recent approach [2] has been to optimize the noise performance of an SC biquad by using different unit capacitors for the two integrator stages. The capacitance spread is not considered. This paper solves the problem of general n th-order filters (i.e., biquad and ladder networks) and shows that individual unit capacitors that are optimally adapted for each integrator stage can significantly improve both sensitivity and noise performance, while maintaining minimum capacitance spread. Thus, for example, an almost 9-dB output noise improvement was obtained in a 10th-order filter design, *without increasing* either the total capacitance area or the overall capacitor spread, both of which are important realizability criteria. Details of this example are discussed later in the text, where it is also shown that the improvements in noise performance directly translate into corresponding improvements in signal-to-noise ratio. It is finally noted that the problem of minimizing noise, assuming a limited capacitance area (as stated above), is *equivalent* to the problem of minimizing the capacitance area for a given, desired noise performance.

The outline of this paper is as follows. In Section II our model for capacitor errors and noise sources, as well as our notation, are introduced. In Section III we investigate capacitance assignments that achieve minimum capacitance spread and show that not a *unique* capacitance assignment, but a whole assignment *region*, can achieve minimum spread. This permits an additional optimization to be carried out while maintaining minimum spread. In Section IV these points are illustrated by means of a second-order example: the frequency-response error as

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well as noise is shown to be dependent on the capacitance assignment. For both performance measures, the minimum value is found to be located inside the assignment region with minimum spread. Motivated by this result, we propose an optimization strategy for general n th-order filters in Section V, which minimizes frequency-response errors, or, alternatively, output noise, and simultaneously realizes the minimum overall capacitance spread. The computer-intensive sensitivity and noise computations required in Section IV reveal the need for a simplified analysis. However, whereas the frequency-response errors are computable in terms of SC design parameters only, the noise computations also involve technology-dependent parameters describing nonideal switches and amplifiers. Based on observations made for the second-order filter example of Section IV, we propose a new measure for the noise performance that depends only on SC design parameters, or in other words, is independent of technology parameters. We present an explicit closed-form expression for this noise measure that closely resembles the corresponding expression for the magnitude-response error. Using these formulas as objective functions, we explicitly formulate the constrained optimization problem and derive an efficient algorithm for its solution in Section VI. This optimization algorithm is implemented as part of our SC-filter design program package SCSYN [3]. In Section VII, we use a 10th-order ladder filter to verify the usefulness of the optimization algorithm, and of our noise measure. We do this by comparing the sensitivity and true noise performance of our optimized design (i.e., also taking into account technology-dependent parameters) with that of the corresponding conventional design.

II. ERROR MODELS AND NOTATION

Error Models for Capacitor Values: Two basically different error sources deteriorate the accuracy of capacitors in MOS integrated circuits; one is *systematic*, affecting all capacitors in the same way, and the other is purely *random*. Because the transfer functions of SC filters depend only on capacitor ratios, *systematic* errors that yield constant relative changes do not effect the filter performance. Since a major systematic error is caused by undercutting of capacitor plates, i.e., during etching, a properly designed capacitor has a constant area/perimeter ratio, in order to obtain constant relative changes. A standard method of achieving this is to break up larger capacitors into parallel combinations of n identical small capacitors, so-called *unit capacitors*. *Random* errors are caused by *global* and *local* oxide and edge variations. According to [4], local variations generally result in errors which are uncorrelated, whereas global effects result in errors that are fully correlated between adjacent capacitors. Thus, if the unit capacitors realizing a specific capacitor ratio are placed adjacent to each other on a chip, the errors due to global variations can be neglected.

According to [5], the remaining local errors result in a combined error for the unit capacitor C_0 described by

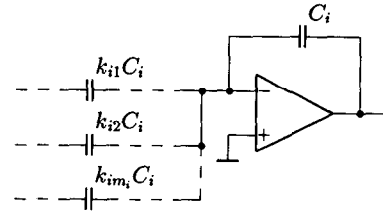


Fig. 1. Notation used for the numbering of the integrator stages and the individual capacitors of a filter.

$\sigma_{c_0} = (K_{le}C_0^{1/2} + K_{lo}C_0)^{1/2}$, where σ_{c_0} denotes the standard deviation of C_0 , K_{le} the local edge effect, and K_{lo} the local oxide effect. For a parallel combination of n unit capacitors $nC_0 \hat{=} C_n$ we obtain, due to uncorrelated error terms, $\sigma_{cn} = (K_{le}n^{1/2}C_n^{1/2} + K_{lo}C_n)^{1/2}$. For a given capacitor C , the resulting σ_c is thus dependent on the chosen size of the unit capacitor C_0 . In [6] it has been shown that for most commonly used technologies the dominant error term is due to local oxide effects. Noting further that $n^{1/4}$ is a slowly increasing function of n , it is reasonable to neglect the local edge effects. This leads to the following simple capacitor error model, which is *independent* of the chosen unit capacitor size.

Each circuit capacitor C_i is assumed to be an independent, Gaussian distributed random variable with mean $C_{i,o}$ being the ideal capacitor-value and variance $E[(C_i - C_{i,o})^2] \hat{=} \sigma_{c,i}^2$. The corresponding standard deviation $\sigma_{c,i}$ is assumed to be $\sigma_{c,i} = c_\alpha \sqrt{C_{i,o}}$ with c_α denoting a technology constant.

Noise Models: In an SC circuit, noise is introduced by switches and amplifiers that are realized by noisy semiconductor devices. We model switches in their ON-state by finite, fixed conductances G in parallel with white-noise current sources of (two-sided) spectral density $2kTG$, where kT denotes the product of Boltzmann's constant and the absolute temperature. In their OFF-state the switches are modeled by open circuits. The amplifiers too are assumed to introduce broadband noise, which, again, is modeled by white noise. The amplifiers are modeled by simple operational transconductance amplifiers (OTA's) consisting of a voltage-controlled current source with transconductance g in parallel with a white-noise current source of spectral density $2kTg\gamma$, with γ denoting the noise factor of the amplifier. This simple amplifier model describes the noise adequately for our purposes at some output terminal of the SC circuit, and serves to develop the ideas leading to our proposed noise optimization.

Notation: The individual capacitors in a filter are numbered as indicated in Fig. 1. The input capacitors to the i th integrator stage with integrating capacitor C_i are denoted by $k_{ij}C_i$. They are connected to the surrounding circuitry by either switches or short circuits. The total capacitance of the i th integrator stage is $(1 + \sum_j k_{ij})C_i$ and is denoted by C_{si} , and the ratio $C_{si}/C_i = (1 + \sum_j k_{ij})$ by β_i . The numbering used within integrator stage i is such that $k_{i1} \leq k_{i2} \leq \dots \leq k_{i,m_i}$ if there are m_i input capacitors in that stage, and the n integrator stages of the filter are

numbered such that stage 1 has the largest stage spread among all stages followed by the one of stage 2, and so on. For SC filters with all capacitance ratios less than unity, this numbering scheme means that $k_{11} \leq k_{21} \leq \dots \leq k_{n1}$.

III. CAPACITANCE ASSIGNMENT AND MINIMAL SPREAD

For an integrator-based SC-filter the individual input capacitances $C_{ij} = k_{ij}C_i$ and the integrating capacitors C_i are related to the total capacitance per stage, C_{si} , by¹ $C_{ij} = (k_{ij}/\beta_i)C_{si}$. Since the k_{ij} are fixed, the stage capacitances C_{si} *uniquely* determine the individual capacitances. We call the vector of stage capacitances $\mathbf{c}_s \triangleq (C_{s1}, C_{s2}, \dots, C_{sn})^T$ the *capacitance assignment*. Denoting the available capacitance area for the whole filter by C_{tot} , we have the obvious constraint $\sum_i C_{si} \leq C_{\text{tot}}$. We may relate \mathbf{c}_s to C_{tot} by

$$\mathbf{c}_s \triangleq \boldsymbol{\lambda} C_{\text{tot}} \quad (1)$$

where the parameter vector $\boldsymbol{\lambda} \triangleq (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ is an alternative formulation of the capacitance assignment.

Note that the constraint $\sum_i C_{si} \leq C_{\text{tot}}$ has no influence on the capacitor spread of the filter realization. As will be shown in Section VI, the constraint always holds with equality for minimum magnitude-response errors and minimum noise. Thus the total available capacitance should be used, i.e., $\sum_i C_{si} = C_{\text{tot}}$, or

$$\sum_{i=1}^n \lambda_i = 1. \quad (2)$$

Using the abbreviations² $C_{i,\max} \triangleq \max_j C_{ij}$ and $k_{i,\max} \triangleq \max_j k_{ij}$, and correspondingly $C_{i,\min}$ and $k_{i,\min}$, the spread of stage i is given by $\rho_i \triangleq C_{i,\max}/C_{i,\min} = k_{i,\max}/k_{i,\min}$. Note that, in contrast to the overall circuit spread ρ , the stage spreads ρ_i are independent of the assignment parameters λ_i and are given by the specified capacitor ratios k_{ij} . With our numbering scheme, stage 1 has the largest stage spread, namely $k_{1,\max}/k_{11}$. Since this spread is fixed by the capacitor ratios k_{1j} to be realized by stage 1, the overall spread in the circuit cannot be less than this stage spread, which then represents the minimum achievable circuit spread. The circuit spread, however, can become larger, depending on the capacitance assignment. This is the case when considering the ratio of maximum and minimum capacitors that belong to *different* integrator stages. Thus we obtain the conditions for minimum achievable circuit spread:

$$C_{i,\max} \leq C_{1,\max}, \quad C_{i,\min} \geq C_{1,\min}, \quad \text{for } 2 \leq i \leq n.$$

¹The notation introduced in the previous section has been generalized in the sense that each capacitor is described by two indices, with the integrating capacitors denoted by C_{i0} . In a similar way the capacitor ratios have been generalized by introducing $k_{i0} \triangleq 1$.

²Note that the maximum (or minimum) is taken over all capacitors in the stage, *including* the integrating capacitor.

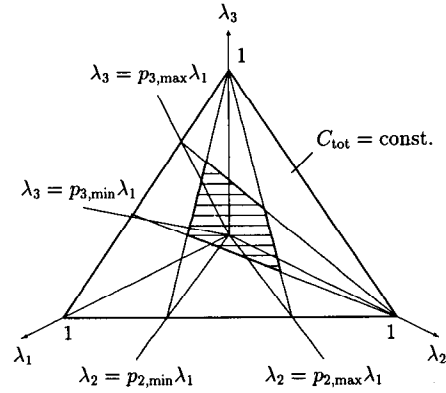


Fig. 2. Minimum spread region for a third-order filter.

With

$$C_{i,\max} = (k_{i,\max} \lambda_i / \beta_i) C_{\text{tot}} \quad \text{and} \quad C_{i,\min} = (k_{i,\min} \lambda_i / \beta_i) C_{\text{tot}}$$

we obtain the minimum spread conditions for the assignment parameters:

$$p_{i,\min} \lambda_1 \leq \lambda_i \leq p_{i,\max} \lambda_1, \quad \text{for } 2 \leq i \leq n \quad (3)$$

where we have defined $p_{i,\max} \triangleq \beta_i k_{1,\max} / (\beta_1 k_{i,\max})$, and $p_{i,\min} \triangleq \beta_i k_{1,\min} / (\beta_1 k_{i,\min})$.

Equation (3) is illustrated for a third-order example in Fig. 2. The region with constant total capacitance C_{tot} , or equivalently $\sum_i \lambda_i = 1$, is given in a three-dimensional $\boldsymbol{\lambda}$ -space. Equation (3) defines a polyhedron achieving minimum capacitor spread, which, with our notation, is given by the spread of stage 1. Outside this region, the spread increases “ $1/\lambda$ -like” [7]; see also the illustrative example in Section IV.

The important point to note is that the minimum spread ρ_{\min} is not restricted to a *single* value of $\boldsymbol{\lambda}$ (corresponding to a *single* capacitance assignment), but to a *range* of $\boldsymbol{\lambda}$ values. This permits an additional optimization such as minimum sensitivity or minimum noise to be carried out, in which case the corresponding optimum $\boldsymbol{\lambda}$ -value is to be found within the minimum spread range. The resulting optimized circuit is then guaranteed to have the minimum capacitance spread ρ_{\min} . This is illustrated by the example in the following section and discussed in detail in Section V.

IV. AN ILLUSTRATIVE EXAMPLE

As an illustrative example we use the biquad circuit shown in Fig. 3. With $k_{11} = k_{12} = 0.0314$, and $k_{13} = k_{21} = 0.314$, this bandpass filter has a nominal center frequency of 100 kHz at a clock rate of 2 MHz and a Q-value of 10. For the minimum achievable circuit spread we find $\rho_{\min} = k_{1,\max}/k_{1,\min} = k_{10}/k_{11} = 1/k_{11} \approx 31.85$. Using (2) we replace λ_1 by λ and λ_2 by $(1-\lambda)$. With (3), the minimum spread is found in the interval $\lambda_l \leq \lambda \leq \lambda_u$ with $\lambda_l \approx 0.512$, and $\lambda_u \approx 0.913$. In Fig. 4, the overall spread ρ as a function of the assignment parameter λ is shown. Outside the minimum spread range, it increases as $1/\lambda$ and $1/(1-\lambda)$ as shown in [7].

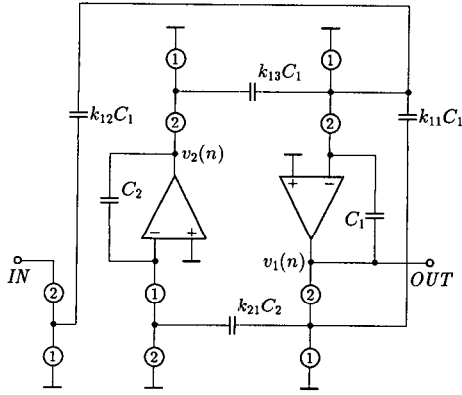


Fig. 3. Circuit diagram of second-order bandpass filter example.

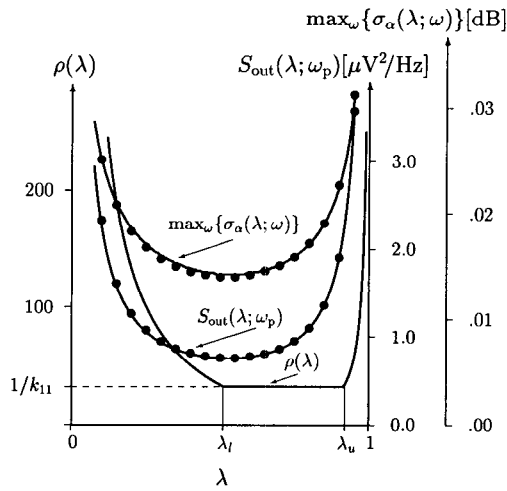


Fig. 4. Spread $\rho(\lambda)$, magnitude-response error $\max_{\omega}\{\sigma_{\alpha}(\lambda; \omega)\}$ and noise $S_{out}(\lambda; \omega_p)$ as functions of the assignment λ . The dots represent numerical values computed by SCANAL and WATSCAD.

Magnitude-Response Errors: Capacitor errors introduce errors both in the magnitude- and phase-response of a given SC circuit. However, here we consider only magnitude-response errors because these are of more practical importance. Five hundred Monte Carlo simulations were carried out with the SC analysis program SCANAL [8] using our capacitor error model with $c_{\alpha} = 10^{-4}\sqrt{\text{pF}}$ and $C_{tot} = 2.69 \text{ pF}$. The maximum value of the standard deviation $\sigma_{\alpha}(\omega)$ of the magnitude response in the frequency range of 80 kHz to 120 kHz is used as a frequency independent measure for the magnitude-response error. As shown in Fig. 4, $\max_{\omega}\{\sigma_{\alpha}(\omega)\}$ is a function of the assignment parameter λ with the minimum located within the minimum capacitance-spread interval $[\lambda_l, \lambda_u]$. Thus the filter can be designed to be optimum with respect to capacitance spread as well as to magnitude-response error.

Output Noise: Using an extended version of the WATSCAD SC-network analysis package [9], [10], the discrete noise at the node "OUT" in our second-order filter example caused by noisy switches and amplifiers has been numerically evaluated, based on the noise models described in Section II. The noise spectral density at the

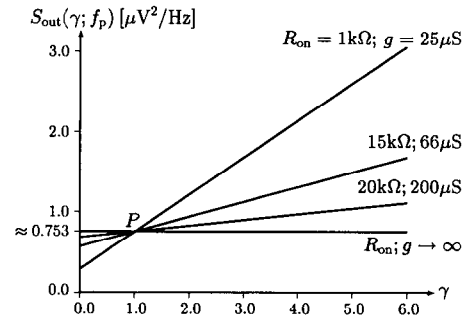


Fig. 5. Noise spectral value at center frequency as a function of the noise factor γ for various combinations of switch ON-resistances and amplifier transconductances. The different pairs $(R_{on} \triangleq 1/G; g)$ are selected to maintain approximately equal charge transfer levels.

filter's center frequency ω_p , $S_{out}(\omega_p)$, is shown in Fig. 5 as a function of the amplifiers noise factor γ for various combinations of parameters $R_{on} \triangleq 1/G$ and g and $\lambda = 0.5$.³ Fig. 5 illustrates the following typical characteristics: The noise $S_{out}(\omega_p)$ is linear in the noise factor γ and there is a point P where all the linear characteristics (which differ according to R_{on} and g) meet. Such a point P has been observed in all other SC-filter examples. Although a more exact analysis reveals that the linear characteristics do *not exactly* meet in a point (except for undamped integrators [11]), but in a small region, it is found in the companion paper [12] that, for practical designs, this region is very small and can be approximated by a *point*. Furthermore, [12] shows that the magnitude of this so-called *crossing point* is dependent only on SC design parameters. It is therefore reasonable to use this magnitude as a *measure* for the true noise performance, which is independent of technology parameters. In Fig. 4, this function, $S_{out}(\lambda; \omega_p)$, is shown for the example at hand as evaluated by WATSCAD. It is seen that the noise measure achieves its minimum value at an assignment λ that lies within the interval $[\lambda_l, \lambda_u]$ where minimum capacitance-spread is achieved. It is therefore possible to realize the filter in our example with a design that is optimum with respect both to spread *and* noise.

As this example also shows, the design optimum with respect to noise is nearly optimum with respect to magnitude-response errors, a behavior which has been found to be typical. Because noise is usually the critical performance measure, it is advisable to optimize a circuit with respect to noise in the knowledge that the resulting magnitude-response error will also be small.

V. OPTIMUM CAPACITANCE ASSIGNMENT

5.1. Strategy for an Optimization

The conventional method for the computation of absolute capacitor values in a given SC-filter circuit is to choose the minimum capacitance of each stage i , $C_{i,min}$, to

³For a fair comparison, the pairs (R_{on}, g) are selected such that the magnitude response errors caused by incomplete charge transfers are in the same order of magnitude for the various combinations and negligible in the present context.

be equal to the unit capacitance C_0 [1]. This results in the stage capacitances $C_{si} = (\beta_i / k_{i,\min})C_0$. Thus a fixed total capacitance area C_{tot} leads to the unit capacitance $C_0 = C_{\text{tot}} / \sum_i \beta_i / k_{i,\min}$. Since $C_{i,\min} = C_0$ for all stages i , we obtain $\lambda_i = p_{i,\min} \lambda_1$, for $2 \leq i \leq n$. In view of (2) this results in the following assignment parameters for the conventional method:

$$\tilde{\lambda}_i = p_{i,\min} / \left(1 + \sum_{j=2}^n p_{j,\min} \right), \quad \text{for } 2 \leq i \leq n.$$

In our second-order example, with λ_1 and λ_2 replaced by λ and $1 - \lambda$, respectively, we obtain $\tilde{\lambda} = \lambda_u$, which is the upper bound of the minimum spread region as shown in Fig. 4. However, this figure shows that a capacitance assignment that minimizes either magnitude-response errors or noise, say λ^* , could considerably improve the corresponding performance, compared to the conventional assignment $\tilde{\lambda} = \lambda_u$. We emphasize that $\tilde{\lambda}$ and λ^* require the same total capacitance C_{tot} , and realize the same minimum capacitor spread $1/k_{11}$.

In general, the optimum assignment vector λ^* may not lie within the minimum-spread region. As mentioned earlier, outside the minimum-spread region the spread grows very fast, which means that for a given C_{tot} , assignments outside this minimum-spread region lead to some capacitors that are extremely small. As is well known, this situation is to be avoided for practical reasons. It is therefore reasonable to limit the capacitance assignments to the minimum-spread region specified in the λ -space by (3). Thus we can formulate the optimization problem as follows: *Find the assignment λ^* that minimizes either magnitude-response error or noise and simultaneously achieves a minimum overall capacitor spread.*

In order to obtain an efficient optimization procedure, we next present explicit closed-form expressions for the magnitude-response error and the noise measure introduced above, which avoid computer-intensive evaluations as performed for the illustrative example in Section IV. The optimization procedure itself is developed in Section VI.

5.2. Magnitude-Response Errors

Seeking an analytical expression for the magnitude-response error that permits efficient computation, we apply a first-order approximation, $\Delta\alpha$, to the amplitude-response error. Thus we use only the linear terms of the corresponding Taylor-series expansion with respect to the capacitance ratio errors Δk_{ij} :

$$\Delta\alpha(\omega) = \sum_{i=1}^n \sum_{j=1}^{m_i} \frac{\partial\alpha_o(\omega)}{\partial k_{ij}} \Delta k_{ij}$$

where $\partial\alpha_o(\omega)/\partial k_{ij}$ denotes the partial derivative of the magnitude response $\alpha(\omega)$ evaluated at the nominal k_{ij} . In the above equation we again use a first-order approximation for Δk_{ij} ; with $k_{ij} = C_{ij}/C_i$, and hence $\Delta k_{ij} = 1/C_i$.

($\Delta C_{ij} - k_{ij}\Delta C_i$), we obtain

$$\Delta\alpha(\omega) = \sum_{i=1}^n \left\{ \sum_{j=1}^{m_i} \frac{\partial\alpha_o(\omega)}{\partial k_{ij}} \Delta C_{ij} - \left(\sum_{j=1}^{m_i} \frac{\partial\alpha_o(\omega)}{\partial k_{ij}} k_{ij} \right) \Delta C_i \right\} \frac{1}{C_i}.$$

Using our capacitance-error model (random, independent capacitance errors ΔC_{ij} with standard deviation $\sigma_{c_{ij}} = c_\alpha \sqrt{C_{ij}}$), we find for the variance of the magnitude-response error $E[(\Delta\alpha)^2] \triangleq \sigma_\alpha^2$:

$$\sigma_\alpha^2(\omega) = \sum_{i=1}^n \left\{ \sum_{j=1}^{m_i} \left(\frac{\partial\alpha_o(\omega)}{\partial k_{ij}} \right)^2 k_{ij} + \left(\sum_{j=1}^{m_i} \frac{\partial\alpha_o(\omega)}{\partial k_{ij}} k_{ij} \right)^2 \right\} \frac{c_\alpha^2}{C_i}. \quad (4)$$

For our second-order example, the evaluation of (4) is plotted in Fig. 4. The curve actually looks like an interpolation of the simulated (dotted) values. This excellent agreement with the Monte-Carlo simulation indicates the correctness of our expression (4) for practical capacitor errors and corresponding capacitor-ratio errors which are usually on the order of less than 1%. Even if larger capacitor errors are assumed (e.g., if the unit capacitors are small, or the process uses quantized values with poor capacitor matching), expression (4) is still a sufficiently good approximation because, in the considered optimization, only the assignment *achieving a minimum* is sought.

5.3. The Noise Measure

In the companion paper [12], a formula for the approximate computation of the noise measure introduced above (i.e., the magnitude of the crossing point noise) has been derived. For an n th-order integrator-based SC-filter this discrete-noise (two-sided) spectral density is given by

$$S_{n,\text{out}}(\omega) = \sum_{i=1}^n \left\{ \sum_{j \in N_i} |H_{ij}(\exp(j\omega T_s))|^2 \frac{1}{k_{ij}} \right\} \frac{2kT}{C_i}. \quad (5)$$

Here, n denotes the number of integrator stages in the filter, the set N_i contains the indices j of the *switched* input capacitors C_{ij} of integrator stage i , and $H_{ij}(z)$ is the transfer function from capacitor C_{ij} to the filter output. Note that, as desired, the above explicit expression for our noise measure depends only on SC design parameters, or in other words, it is independent of technology parameters. For our second-order example this expression is used in Fig. 4 to interpolate the simulated (dotted) values of $S_{\text{out}}(\lambda, \omega_p)$. The excellent agreement with the WATSCAD simulation indicates the correctness of (5). Excellent agreement has also been found for higher order filters as reported in [12].

Referring to step ii) of the filter design procedure outlined in Section I, we note that the capacitor ratios are usually obtained by a scaling process which causes the signal levels of all filter nodes to be maximum (scaled for

maximum undistorted output). It is important to note that this scaling process not only maximizes the achievable output swing, but simultaneously minimizes our noise measure (5) for a given capacitance assignment. A detailed discussion of this interesting fact is given in Appendix A which, besides proving the statement, makes clear that the result is not at all obvious. We conclude that *an SC circuit scaled for maximum undistorted output achieves a maximum signal-to-noise ratio if the given capacitance area is distributed such as to minimize our noise measure.*

VI. OPTIMIZATION PROCEDURE

6.1. Problem Formulation

As motivated by the discussion in Section 5.1, we consider only those designs as "good" that achieve the minimum possible capacitance spread. Thus, we restrict ourselves to the region in the λ -space that is given by (3). Within this λ -region, either the magnitude-response error (4), or our noise measure (5), may be minimized. With $C_i = (\lambda_i / \beta_i) C_{\text{tot}}$, the objective function for each case can be written in the form

$$f(\lambda; \omega) = \sum_{i=1}^n \frac{a_i(\omega)}{\lambda_i}.$$

The frequency dependence may be eliminated by use of $F(\lambda) \triangleq \max_{\omega \in \Omega} f(\lambda; \omega)$, which is useful for the magnitude-response error, or by $F(\lambda) \triangleq \int_{\omega \in \Omega} f(\lambda; \omega) d\omega$, which is useful for our noise measure with $F(\lambda)$ corresponding to the noise power in the frequency range Ω . In this way, the frequency-dependent coefficients $a_i(\omega)$ are transformed into *positive* constants b_i , and for λ_i 's satisfying (3), the following constrained minimization problem can be formulated: Minimize $F(\lambda) = \sum_i b_i / \lambda_i$ subject to $\sum_i \lambda_i = 1$ and the inequality constraints $p_{i,\min} \lambda_1 \leq \lambda_i \leq p_{i,\max} \lambda_1$ for $2 \leq i \leq n$ and $\lambda_i > 0$ for all i .

The above formulation inherently proves the fact that the minimum of the objective function is obtained using the entire available capacitance C_{tot} , as claimed in Section III: Assume there exists a minimum of $F(\lambda)$, which uses less capacitance than C_{tot} meaning $\sum_i \lambda_i < 1$. Increasing any parameter λ_i to utilize the entire available capacitance will *decrease* $F(\lambda)$ because the coefficients b_i are positive, as is seen from (4) and (5). Therefore, the minimum demands that $\sum_i \lambda_i = 1$.

To emphasize that the constraints are linear in the above optimization problem we define the $n \times 1$ vector p_{il} as

$$p_{il} \triangleq \left(p_{il}, 0, \dots, \underbrace{(-1)^{l-1}}_{\text{ith place}}, 0, \dots, 0 \right)^T$$

with $p_{il} \triangleq p_{i,\min}$ for $l=0$ and $p_{il} \triangleq -p_{i,\max}$ for $l=1$, and the $n \times 1$ vector $h \triangleq (1, 1, \dots, 1)^T$, and rewrite the opti-

mization problem as

$$\text{minimize } F(\lambda) = \sum_{i=1}^n \frac{b_i}{\lambda_i} \quad (6a)$$

$$\text{subject to } h^T \lambda = 1 \quad (6b)$$

$$\text{and } p_{il}^T \lambda \leq 0, \quad 2 \leq i \leq n, \quad l \in \{0, 1\}. \quad (6c)$$

Note that the physical constraints $\lambda_i > 0$ for all i are not included in (6) because these requirements will automatically be fulfilled by the algorithm to be developed.

6.2. Discussion of the Problem

With a simple and fast algorithm in mind, we next discuss the special properties of our optimization problem (6). First, we note that we are faced with a separable programming problem, i.e., $F(\lambda) = \sum_i F_i(\lambda_i)$. Consequently, the gradient vector of our objective function $F(\lambda)$ is given by $\nabla F(\lambda) = (-b_1/\lambda_1^2, -b_2/\lambda_2^2, \dots, -b_n/\lambda_n^2)^T$ and the Hessian matrix by $\nabla^2 F(\lambda) = 2 \text{diag}\{b_1/\lambda_1^3, b_2/\lambda_2^3, \dots, b_n/\lambda_n^3\}$. Thus $\nabla^2 F(\lambda)$ is *positive definite* for every relevant point λ because $1 > \lambda_i > 0$ and $b_i > 0$ for all i , meaning that $F(\lambda)$ is *convex*. Together with the linear equality and inequality constraints (6b) and (6c) which define a polyhedron, our problem is a *convex programming problem* with the fundamental property that any local minimum is a global minimum. Furthermore, since $\nabla^2 F(\lambda)$ is *strictly* positive definite, the obtained minimum λ^* is also *unique*, cf., [13, p. 257].

Denoting by A the set of those inequality constraints that are active (i.e., that hold with equality) at the minimum λ^* , the first-order necessary conditions for our problem (6), the so-called *Kuhn-Tucker conditions*, cf., [14, pp. 314 ff., p. 327], are found to be

$$\nabla F(\lambda^*) + \sum_{il \in A} \mu_{il} p_{il} + \mu_1 h = 0 \quad (7a)$$

$$h^T \lambda^* = 1 \quad (7b)$$

$$p_{il}^T \lambda^* = 0, \quad il \in A \quad (7c)$$

$$p_{il}^T \lambda^* < 0, \quad il \notin A \quad (7d)$$

$$\mu_i \geq 0, \quad il \in A. \quad (7e)$$

Note that (7a), (7b), and (7c) are the necessary conditions for the equality-constrained problem that result from the objective function (6a) together with the equality constraint (6b), and the *active* constraints of (6c). To verify that A contains the proper set of active constraints, (7d) ensures that λ^* also fulfills the inactive inequality constraints. A point λ^* that fulfills all constraints is called *feasible*. Violating (7d) indicates that further constraints must be made active to guarantee that λ^* is located within the feasible assignment region. Finally, (7e) ensures that removing a constraint from A does not further decrease the objective function

If the set A were known *a priori*, we would merely have to solve an equality constrained problem. Because we do not know A , however, a solution procedure for (6) consists of two components: i) an algorithm for finding the correct set of active inequality constraints, and ii) a vehi-

cle to determine the solution of the corresponding *equality* constrained problem. For i), we use the so-called *active set method*, which is adequate for convex programming problems [14, ch. 11.3]. This is outlined in more detail in Section 6.3. Because ii) has to be solved several times in the course of the active set algorithm, we take advantage of the simple structure of our objective function and the linear constraints, and derive a closed-form solution for the equality-constrained problem; see Section 6.4.

6.3. Active Set Method

The idea of an algorithm based on the active set method is to define a set of active constraints at each step, termed a *working set* W . The working set W , a subset of the given inequality constraints (6c), defines the so-called *working surface* S by the λ -values that fulfill $p_{il}^T \lambda = 0$ for $il \in W$.⁴ Since the constraints are linear, the surface S in our convex program is a *hyperplane*. On this hyperplane, the algorithm searches for the point λ which has a minimum objective function. If this point is *not* feasible, constraints are added to the working set W until a feasible minimum is obtained. If, at this stage, the removal of a constraint from the current working set allows a further decrease of the objective function, the algorithm continues with the search on the hyperplane given by the new reduced working set. Otherwise the algorithm terminates.

Since the number of different working sets W is finite, the algorithm terminates in a finite number of steps *if a decrease of the objective function is guaranteed from step to step*. This decrease of $F(\lambda)$ is a key point of the algorithm since, otherwise, the same working set W could be chosen a second time, causing the algorithm to cycle. As will be shown for the convex program at hand, a proper choice of the working set W will decrease the objective function at each step. This guarantees the successful termination of the algorithm.

To be more precise, assume that step $(k-1)$ has obtained the feasible point $\lambda_{W_{k-1}}$ on the hyperplane S_{k-1} defined by the working set W_{k-1} . Further assume that in step k the new working set W_k is obtained from W_{k-1} by either dropping a constraint or by adding a new constraint.⁵ The search on the hyperplane S_k then corresponds to the following equality-constrained problem:

$$\text{minimize} \quad F(\lambda) = \sum_{i=1}^n \frac{b_i}{\lambda_i} \quad (8a)$$

$$\text{subject to} \quad h^T \lambda = 1 \quad (8b)$$

$$\text{and} \quad p_{il}^T \lambda = 0, \quad il \in W_k \quad (8c)$$

for which the solution is denoted by $\lambda_{W_k}^*$. On the working

hyperplane S_k the obtained minimum may, or may not, be feasible.

If $\lambda_{W_k}^*$ is not feasible, meaning that $p_{il}^T \lambda_{W_k}^* > 0$ for one of the free constraints $il \notin W_k$, we proceed by modifying $\lambda_{W_k}^*$ to a feasible point λ_{W_k} that fulfills $F(\lambda_{W_k}) < F(\lambda_{W_{k-1}})$. Since $F(\cdot)$ is strictly convex, this can be done in the same way as is normally done in *quadratic programming*, cf., [14, p. 425]. The idea is to move as far as possible on the line from $\lambda_{W_{k-1}}$ to $\lambda_{W_k}^*$ while maintaining feasibility. The feasible boundary point denoted by λ_{W_k} decreases the objective function: $F(\lambda_{W_k}) < F(\lambda_{W_{k-1}})$. To show this we first note that $\lambda_{W_k}^*$ is different from $\lambda_{W_{k-1}}$ because otherwise $\lambda_{W_k}^*$ would be feasible, violating the present assumption. Since $F(\cdot)$ is strictly convex, it follows that $F(\lambda_{W_k}^*) < F(\lambda_{W_k}) < F(\lambda_{W_{k-1}})$ as long as⁶ $\lambda_{W_k} \neq \lambda_{W_{k-1}}$. With $d_k \triangleq \lambda_{W_k}^* - \lambda_{W_{k-1}}$ we write for the improved point $\lambda_{W_k} = \lambda_{W_{k-1}} + \alpha_k d_k$, with $\alpha \in (0, 1]$ as large as possible while maintaining feasibility: $p_{il}^T \lambda_{W_k} \leq 0$, for $il \notin W_k$. This leads to

$$\alpha_k = \min_{il \in W_k, p_{il}^T d_k > 0} \left\{ 1, \frac{-p_{il}^T \lambda_{W_{k-1}}}{p_{il}^T d_k} \right\}. \quad (9)$$

If $\alpha_k < 1$ is found, a new constraint il becomes active and has to be added to W_k leading to the new working set W_{k+1} . By contrast $\alpha_k = 1$ means that $\lambda_{W_k} = \lambda_{W_k}^*$ which leads to the second situation, meaning that $\lambda_{W_k}^*$ is feasible.

With $\lambda_{W_k}^*$ being feasible, we have to decide whether dropping an active constraint may further decrease the objective function $F(\cdot)$. The solution $\lambda_{W_k}^*$ of the equality constrained problem (8) is given by the following first-order condition

$$\nabla F(\lambda_{W_k}^*) + \sum_{il \in W_k} \mu_i p_{il} + \mu_1 h = 0. \quad (10)$$

If the Lagrange multipliers μ_i in (10) are nonnegative for all $il \in W_k$, then, with $\lambda_{W_k}^*$, our working set W_k fulfills the Kuhn-Tucker conditions (7), and therefore solves the convex program (6). On the other hand, if a specific Lagrange multiplier μ_j with $jl \in W_k$ is negative, $F(\cdot)$ can further be decreased by dropping the corresponding constraint jl . To verify this, we relax constraint jl , obtaining the new working set W_{k+1} , and the corresponding hyperplane S_{k+1} . Since our constraint vectors p_{il} are linearly independent, there exists a $\Delta \lambda_j$ for a move on S_{k+1} , $\bar{\lambda}_{W_{k+1}} \triangleq \lambda_{W_k}^* + \Delta \lambda_j$, such that

$$p_{il}^T \bar{\lambda}_{W_{k+1}} = p_{il}^T \Delta \lambda_j = \begin{cases} c_j, & \text{for } i = j, il \in W_k \\ 0, & \text{for } i \neq j, il \in W_k \end{cases} \quad (11)$$

because $p_{il}^T \lambda_{W_k}^* = 0$ for the active constraints ($il \in W_k$). Note that, except for constraint jl , all other constraints of W_k are still active in W_{k+1} . It can easily be shown (cf., [14, ch. 10.7]) that the move with $\Delta \lambda_j$ on this new

⁴Note that constraints il for $l=0$ and $l=1$ are related to the lower and upper bound of the assignment parameter λ_1 , respectively, and can, therefore, not be active simultaneously. Thus at most one of the indices $i0$ and $i1$ can be in the set W .

⁵Note the feasible point $\lambda_{W_{k-1}}$ belongs to both hyperplanes S_{k-1} and S_k defined by W_{k-1} and W_k , respectively.

⁶The degenerated case $\lambda_{W_k} = \lambda_{W_{k-1}}$ is not interesting because here the new constraint is just added, but no move is performed in the λ -space. This is not considered to be an optimization step, meaning the present step number is still $(k-1)$.

hyperplane S_{k+1} yields $\Delta F(\lambda_{W_k}^*) = -\mu_j c_j$. Therefore, choosing a $\Delta \lambda_j$ that satisfies (11) such that a negative c_j results (as is needed for feasibility) will decrease the objective function $F(\cdot)$ if and only if $\mu_j < 0$. This proves the claim that dropping a constraint decreases $F(\cdot)$.

What we now need for the final formulation of our optimization algorithm is the solution of (8), which consists of closed-form expressions for the minimum point $\lambda_{W_k}^*$ and for the corresponding Lagrange multipliers μ_i of (10).

6.4. Analytic Solution of the Equality Constrained Problem

For notational simplicity, we drop the step index k in the following presentation of the general closed-form solution for the equality constrained problem (8). This solution is derived by first reformulating the original problem by eliminating the equality constraints (8c). This leads to a similar problem with a reduced set of variables, which can be solved in a straightforward manner. A back transformation to the original problem finally produces the desired solution. Leaving the details to Appendix B, we thus find for the components of the minimum point λ_W^* :

$$\lambda_{W,i}^* = \rho_{W,i}^* \left/ \sum_{k=1}^n \rho_{W,k}^* \right., \quad 1 \leq i \leq n \quad (12)$$

where the terms $\rho_{W,i}^*$ are given by

$$\rho_{W,i}^* = \begin{cases} 1, & \text{for } i = 1 \\ \sqrt{b_i} H(W), & \text{for } i \neq 1, il \notin W \\ p_{il}, & \text{for } il \in W \end{cases} \quad (13)$$

and where the auxiliary expression $H(W)$ is defined as

$$H(W) \triangleq \sqrt{\frac{1 + \sum_{il \in W} (-1)^l p_{il}}{b_1 + \sum_{il \in W} (-1)^l \frac{b_i}{p_{il}}}}. \quad (14)$$

The required Lagrange multipliers μ_i , $il \in W$ are also determined in Appendix B. Since only the signs of these Lagrange multipliers are needed to decide whether a constraint il has to be dropped or not, it is sufficient to use the normalized Lagrange multipliers $\tilde{\mu}_i \triangleq \mu_i \lambda_{W,1}^{*2}$, given with (B.10) by

$$\tilde{\mu}_i = (-1)^l \left(\frac{1}{H(W)^2} - \frac{b_i}{p_{il}^2} \right), \quad il \in W. \quad (15)$$

6.5. Optimization Algorithm

With the expressions (9), (12), (13), (14), and (15) we are ready to formulate the optimization algorithm for the convex program (6). This algorithm consists of a main procedure MINIMIZE shown in Fig. 6, and the two auxiliary procedures FEASIBLE and MINIMAL given in Figs. 7 and 8, respectively. Here we use a programming-like language with self-explanatory structure elements; com-

```

PROCEDURE Minimize
BEGIN
  {*Initialize working set by activating all upper bounds.*}
  W0 = {22,32,...n2}
  k = 0
  <compute initial point lambda_W0, cf. Eqs.(12) and (13)>
  REPEAT
    REPEAT
      k = k + 1
      <compute H_k(W_k), cf. Eq.(14)>
    UNTIL feasible
  UNTIL minimal
END Minimize

```

Fig. 6. Procedure MINIMIZE.

```

PROCEDURE feasible: BOOLEAN
BEGIN
  q = phi
  <compute lambda_Wq, cf. Eqs.(12) and (13)>
  d_k = lambda_Wk - lambda_Wq-1
  <find alpha_k and associated constraint q, cf. Eq.(9)>
  lambda_Wk = lambda_Wq-1 + alpha_k d_k
  IF q != phi THEN
    {* Add constraint q *}
    W_{k+1} = W_k + q, feasible = FALSE
  ELSE
    {* No change in working set *}
    W_{k+1} = W_k, feasible = TRUE
  END feasible

```

Fig. 7. Procedure FEASIBLE.

```

PROCEDURE minimal: BOOLEAN
BEGIN
  {*Compute most negative normalized Lagrange multiplier:
  mu_q < mu_i; il in W. *}
  mu_q = 0, q = phi
  FOR il in W_k DO
    <compute mu_i, cf. Eq.(14)>
    IF mu_i < mu_q THEN mu_q = mu_i, q = {il}
  END
  {*Check for minimality.*}
  IF q != phi THEN
    {* Drop constraint q *}
    W_{k+1} = W_k - q, minimal = FALSE
  ELSE
    {* No change in working set *}
    W_{k+1} = W_k, minimal = TRUE
  END minimal

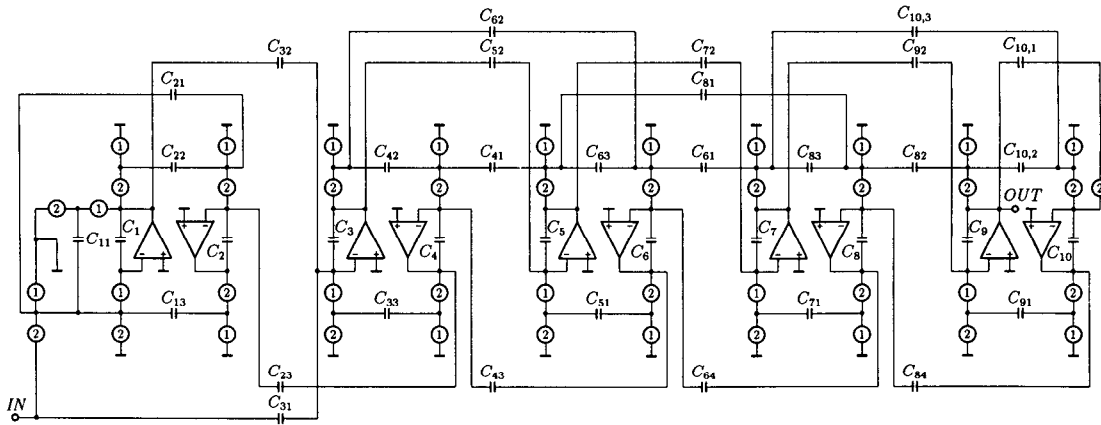
```

Fig. 8. Procedure MINIMAL.

ments are given in {* *} brackets, and "pseudo"-statements use < > brackets. The above algorithm has been implemented in the design program SCSYN [3].

VII. A DESIGN EXAMPLE

As an example of practical relevance we consider the design optimization of a bandpass filter of order 10 with specifications as given in [15]. These specifications define a passband extending from 1.2 kHz to 1.54 kHz, a passband ripple of 0.2 dB, and a maximum pole Q of 48. Using the programs FILSYN [16], and LADNET and SCSYN [3], an SC ladder filter was generated that uses 50 kHz as clock frequency. The SC circuit was derived exactly from the corresponding z-domain transfer function, cf., [17], which, in turn, was obtained by FILSYN by prewarping and bilinearly transforming the corresponding s-domain transfer function. The SC circuit is shown Fig. 9. It is noted that the corresponding capacitor ratios were obtained by scaling the circuit for maximum undistorted output in the passband. In comparing different capacitance assignments, we assume a given total capacitor area of 100 pF. The magnitude response error is assumed to be



$k_{11} = 0.0324$	$k_{21} = 0.0039$	$k_{31} = 0.0028$	$k_{41} = 0.1365$	$k_{51} = 0.1977$	$k_{61} = 0.0321$	$k_{71} = 0.2081$	$k_{81} = 0.0349$	$k_{91} = 0.2848$	$k_{10,1} = 0.0658$
$k_{12} = 0.0547$	$k_{22} = 0.1436$	$k_{32} = 0.1026$	$k_{42} = 0.2433$	$k_{52} = 0.4093$	$k_{62} = 0.0874$	$k_{72} = 0.4118$	$k_{82} = 0.0417$	$k_{92} = 1.2547$	$k_{10,2} = 0.0836$
$k_{13} = 0.2066$	$k_{23} = 0.2679$	$k_{33} = 0.1646$	$k_{43} = 0.6392$		$k_{63} = 0.1803$		$k_{83} = 0.1786$		$k_{10,3} = 0.0893$
					$k_{64} = 0.3789$		$k_{84} = 0.5855$		

Fig. 9. Circuit diagram of 10th-order bandpass ladder filter.

TABLE I
GLOBAL VERSUS CONSTRAINED OPTIMIZATION

Minimum criterion	Spread	MR-error (dB)	Noise (μV^2)
MR-error, global	4236	0.074	333
MR-error, constrained	362	0.082	354
Noise, global	1462	0.081	270
Noise, constrained	362	0.087	293

TABLE II
STANDARD ASSIGNMENTS VERSUS OPTIMUM ASSIGNMENTS.
SPREAD IS 362

Assignment	MR-error (dB)	Noise (μV^2)
Standard	0.226	2263
MR-optimal	0.082	354
Noise-optimal	0.087	293

critical at the upper passband edge, at 1.54 kHz, where also the discrete-time noise spectral density of our noise measure is evaluated. Thus the magnitude-response error σ_{α} , as well as the noise variance $\sigma_{n_{out}}^2$ in a 1-Hz band, both at 1.54 kHz, are taken as performance measures, which are computed by our formulas (4) and (5), respectively.

In Table I we compare unconstrained-optimized (globally optimized) designs with constrained-optimized designs, that is, designs with capacitance assignments achieving the minimum overall capacitor spread, which, in our example, is 362. If the magnitude response error is taken as the optimization criterion, we see that an unconstrained optimization can decrease the error by only about 10% compared to the corresponding constrained optimization, whereas the spread increases by a factor of almost 12. Likewise, we find with the minimum noise criterion that an unconstrained optimization can decrease the output noise by less than 8%, whereas the spread is increased by a factor of 4. This example is typical in that it demonstrates the desirability (and feasibility) of restricting the optimization to designs that result in a minimum capacitance spread.

Next, we compare these minimal spread designs⁷ with the conventional solution discussed in Section III, which assumes the same minimum capacitance value C_{i1} for

each integrator stage. It is seen from Table II that the output noise variance $\sigma_{n_{out}}^2$ of our circuit is improved by almost 9 dB compared with that of the conventional design. Note that such improvements are also observed if the noise is taken in larger frequency bands; thus, for example, from 0.1 kHz to 10 kHz, an improvement of 8.1 dB is obtained. Using the magnitude error criterion similar improvements result. It should be emphasized that all three designs feature the *same overall capacitor spread* of 362, and the *same total capacitor area* of 100 pF. The difference between the two approaches is that in the conventional design the individual capacitors consist of multiples of a unit-capacitor that is equal to the minimum capacitance occurring in the filter (i.e., C_{11}), whereas in the *improved* designs each integrator stage incorporates *optimally adapted* unit-capacitors.

Note that in contrast to our noise measure, which is dependent only on SC design parameters, and is used in the synthesis context (computer-efficient optimization), the true noise performance is also dependent on technology parameters (noise factor γ , amplifier transconductance g , and switch conductance G). To verify our final designs, we compared the *true noise performance* of the

⁷The resulting magnitude-response error-optimal capacitance assignment is $\lambda = (0.0666, 0.1163, 0.1460, 0.1383, 0.1813, 0.1930, 0.0265, 0.0714, 0.0191, 0.0416)^T$, whereas the noise-optimal assignment is $\lambda = (0.0366, 0.0942, 0.1183, 0.1320, 0.1497, 0.1563, 0.0655, 0.1500, 0.0234, 0.0741)^T$.

⁸The transconductance resistance combinations used are selected such that the magnitude-response errors caused by *charge transfer errors* are in the same order of magnitude as the presently treated magnitude-response errors, which are due to inexact capacitances—that is, they are less than about 0.08 dB. The required analysis was carried out using the program SWAP [18].

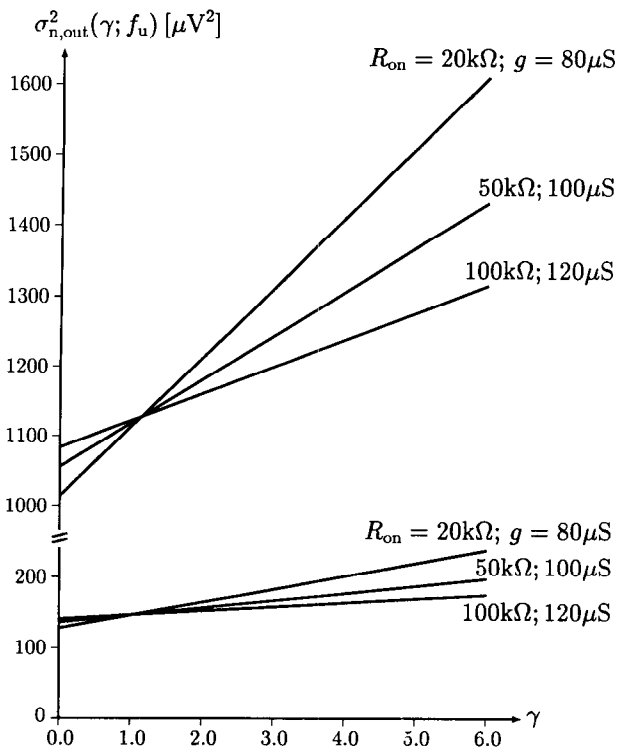


Fig. 10. True noise performance of 10th-order bandpass ladder filter for various combinations of switch ON-resistances and amplifier transconductances. Noise in a 1-Hz band at upper passband edge $f_u = 1.54$ kHz. The upper curves are related to the conventional design, the lower curves to the noise-optimized design.

constrained noise-optimal design with the conventional design. For this comparison we rely on WATSCAD noise computations, which have been shown to agree well with measured results [9]. In Fig. 10, this comparison is illustrated for the noise variance in the 1-Hz band at 1.54 kHz, for various transconductance and switch ON-resistance combinations.⁸ It is seen that not only the crossing point noise (our noise measure) is far superior in the design optimized according to our procedure, but also that the true noise performance is significantly improved compared to the conventional design; the linear noise characteristics are much flatter in the optimized design, which excludes the possibility that the standard design could become superior in true noise performance for a noise factor γ larger than that of the crossing point. Note that these relations are true not only for the illustrated noise variance, but over the whole frequency band. Because similar results have been obtained for numerous other examples, we are confident that the crossing point noise is a very useful measure for the noise performance of SC filters. In a broader context, the above results suggest the following design strategy in order to optimize the crucial noise performance: Using our procedure, choose the capacitance assignment that minimizes the technology-independent crossing point noise. Then compute the true noise performance of the obtained filter design (cf., Fig. 10) to guide a sensible design for the amplifiers and switches.

VIII. CONCLUSIONS

A design procedure for the optimum assignment of a given overall capacitance area to the various integrator stages in integrator-based SC filters is proposed. The underlying optimization criterion employed throughout is the requirement for minimum capacitance spread. This is reasonable, because we have found that minimum spread is realized in a whole *assignment region*, whereas unacceptably large spread-values result for assignments slightly outside this optimum region. As a result, the remaining degrees of freedom can be used for an optimization with respect to other criteria. The two criteria investigated are *minimum magnitude-response error* and *minimum noise*.

Closed-form expressions are derived that express the capacitance spread, the magnitude-response error due to capacitor errors, and the output noise as functions of the capacitance assignment. Based on these formulas, an optimization procedure is developed that either minimizes the magnitude-response errors or the noise measure, while maintaining the minimum possible capacitance spread. This procedure is implemented in the SC-filter design and optimization tool SCSYN [3]. This program also permits the structural optimization of biquad cascades where the same criteria for the optimization are used. Due to the relative simplicity of our formulas, it is possible to determine the best in a class of possible realizations by an exhaustive search within reasonable computation time.

While the expressions describing the capacitance spread and the sensitivity of the magnitude response to capacitor errors can readily be formulated in terms of capacitance ratios and assignment parameters, this is not possible for the noise minimization. Here, technology-dependent parameters such as switch ON-resistance and amplifier transconductance also come into the analysis. This is why it was necessary to develop a measure of noise performance that excludes these technology parameters and approximates the true noise in terms only of design parameters (such as capacitor ratios and capacitance levels). The high accuracy of the newly introduced noise measure is illustrated by various designs of a 10th-order ladder filter.

The usefulness of this noise measure, as well as of the optimization algorithm using it, has been demonstrated by comparing the sensitivity and the noise performance of the optimized designs of a 10th-order example filter with the corresponding performance measures resulting from a conventional design. We found that remarkable improvements are achievable with the new designs, e.g., a decrease of almost 9 dB in output noise was obtained for a filter using the same total capacitance and the same capacitance spread as the one based on the conventional design. Note that the decreasing noise directly translates into a corresponding improvement in signal-to-noise ratio, because the optimization does not change the dynamical behavior of the filter (i.e., the optimization permits the capacitor ratios to remain invariant). Finally, we note that noise-optimum designs are preferable to sensitivity-opti-

mum designs, because the critical feature of SC filters is dynamic range and noise, and because, as we have empirically observed, the sensitivity performance of a noise optimum design is not far from its sensitivity optimum.

APPENDIX A SIGNAL-TO-NOISE RATIO AND NODE SCALING

We discuss here the changes in signal-to-noise ratios that are obtained by node scaling. Intuitively, node scaling that causes the signal levels of all filter nodes to be maximum (scaling for maximum undistorted output) *improves* the signal-to-noise ratio. The idea behind this is that the various signals in the circuit (for example the amplifier output voltages) are enlarged by this scaling process, while the noise contribution remains unchanged. The question concerning signal-to-noise ratio improvement by scaling for maximum undistorted output of SC filters is also treated in [1, pp. 339 ff.]. However, the simple noise models assumed there are different from our more accurate noise models in that ideal operational amplifiers and fixed noise sources at the amplifier input terminals are used in [1] to model amplifier noise. Switch noise does not seem to be taken into account at all. If more accurate noise models are assumed, it is not *a priori* clear whether scaling for maximum undistorted output will improve—or even deteriorate—the resulting signal-to-noise ratio. Indeed, the approximate formula describing our noise measure (see Section 5.3) is seen to have equivalent noise sources which are dependent on the various capacitances. These noise sources have a variance of the form $2kT/(k_{ij}C_i)$, where C_i denotes the integrating capacitance of integrator stage i , and k_{ij} are the capacitor ratios of the corresponding switched input capacitors. Because scaling for maximum undistorted output adjusts these ratios, the variances of the noise sources are *also changed* by this operation. This is in contrast to the assumption in [1], and evokes the question whether scaling for maximum undistorted output *improves* the signal-to-noise ratio *under all circumstances*.

Consider the SC-filter section shown in Fig. 11(a) with the accompanying signal-flow graph representation in Fig. 11(b). Node scaling applied to integrator stage i modifies the amplifier output-voltage $V_i(z)$ to $\hat{V}_i(z)$ by simultaneously multiplying the input branches to that stage i by ρ_i and dividing the branches leaving that stage by ρ_i . Scaling the node voltage $V_j(z)$ of the integrator stage j means repeating this process with ρ_i replaced by ρ_j . For the section shown in Fig. 11 these operations are described by

$$\begin{aligned}
 V_i &\mapsto \hat{V}_i = V_i \rho_i \\
 k_{jl} &\mapsto \hat{k}_{jl} = k_{jl} \frac{\rho_j}{\rho_i}.
 \end{aligned} \tag{A.1}$$

Note that the branch connecting the two stages i and j is scaled twice, once as branch leaving stage i by $1/\rho_i$, and once as an input branch to stage j , by ρ_j . The transfer function from the node voltage $V_i(z)$ to the output, $H_j(z) \triangleq V_{\text{out}}(z)/V_i(z)$, is equivalent to the noise transfer func-

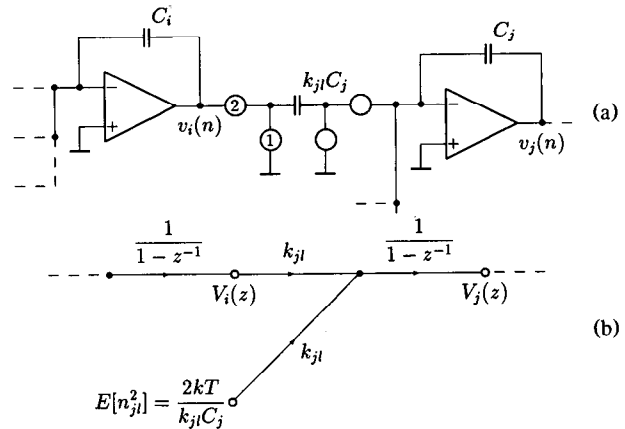


Fig. 11. SC filter section considered for signal-to-noise ratio computations.

tion from the noise source n_{jl} to the output, $H_j(z) \triangleq V_{\text{out}}(z)/n_{jl}(z)$, which is thus scaled to

$$H_j(z) \mapsto \hat{H}_j(z) = H_j(z) \frac{1}{\rho_i}. \tag{A.2}$$

Using (A.1) and (A.2), the contribution of the white noise source n_{jl} with spectral magnitude $E[n_{jl}^2] = 2kT/(k_{jl}C_j)$ to the output noise spectrum, $S_{jl}(\omega)$, is seen to be scaled to

$$\begin{aligned}
 S_{jl}(\omega) &\mapsto \hat{S}_{jl}(\omega) = |\hat{H}_j(\omega)|^2 \frac{2kT}{\hat{k}_{jl}\hat{C}_j} \\
 &= S_{jl}(\omega) \frac{1}{\rho_i \rho_j} \frac{C_j}{\hat{C}_j}
 \end{aligned} \tag{A.3}$$

where \hat{C}_j denotes the integrating capacitance C_j after scaling. Clearly, the scaling operation only adjusts capacitor ratios, but says nothing about absolute capacitance values. To confine a constant total filter capacitance area, we add the requirement that *the total stage capacitances remain unaltered by scaling*. For integrator stage j we have before scaling

$$C_{sj} = \left(1 + \sum_l k_{jl}\right) C_j \tag{A.4}$$

and after scaling

$$\hat{C}_{sj} = \left(1 + \rho_j \sum_l \frac{k_{jl}}{\rho_{m(l)}}\right) \hat{C}_j. \tag{A.5}$$

Note that as an input branch to stage j the signal-flow graph branch with weight k_{jl} becomes $\rho_j k_{jl}$ by scaling of stage j ; as a branch connecting stage $m(l)$ to stage j it becomes $k_{jl}/\rho_{m(l)}$ when scaled by $\rho_{m(l)}$. With unchanged total stage capacitances, i.e., with $C_{sj} = \hat{C}_{sj}$, the ratio C_j/\hat{C}_j required in (A.3) is found using (A.4) and (A.5). The ratio of scaled to unscaled noise contribution of n_{jl}

therefore becomes

$$\frac{\hat{S}_{jl}}{S_{jl}} = \frac{1}{\rho_i \rho_j} \frac{1 + \rho_j \sum_l k_{jl} / \rho_{m(l)}}{1 + \sum_l k_{jl}}. \quad (\text{A.6})$$

In the scaling process discussed here (scaling for maximum undistorted output), we must consider two separate scaling steps, that of *scaling up* and that of *scaling down*. An amplifier output-voltage $V_i(z)$ is scaled up if this output voltage is not as large as distortionless amplifier operation would allow. In this case, ρ_i is larger than unity, but the maximum allowable input signal to the filter does not become larger, since the output voltage of some other amplifier would then reach its limit of distortionless operation. In order to improve the signal-to-noise ratio by scaling up, the contribution of the noise source n_{jl} must decrease; this means that \hat{S}_{jl}/S_{jl} must be less than unity. Assuming that in the present step only scaling up is performed, we have $\rho_{\min} = 1$ from those stages that cannot be scaled up and some $\rho_{\max} > 1$. Writing (A.6) as

$$\frac{\hat{S}_{jl}}{S_{jl}} = \frac{1/(\rho_i \rho_j) + \sum_l k_{jl} / (\rho_i \rho_{m(l)})}{1 + \sum_l k_{jl}} \quad (\text{A.7})$$

the numerator in the above formula is, with $\rho_{\min} = 1$, $\rho_{\max} > 1$, upper bounded by the expression in the denominator, leading to $\hat{S}_{jl}/S_{jl} < 1$ as needed.

The situation where $V_i(z)$ is scaled down, meaning that ρ_i is less than unity, is applied if stage i limits the overall output level. Scaling down leads therefore to a larger permissible input signal to the filter. Assuming only down scaling in this step, we have $\rho_{\max} = 1$ for those stages not being scaled, and a $\rho_{\min} < 1$ for those stages being scaled down. Noting that $1/\rho_{\min}$ is exactly the factor by which the maximum allowable filter input signal is being increased, the signal-to-noise ratio will be improved if the ratio \hat{S}_{jl}/S_{jl} does not grow as fast as $1/\rho_{\min}^2$, i.e., $\hat{S}_{jl}/S_{jl} < 1/\rho_{\min}^2$. This is easily seen to hold by observing that the numerator of the right hand side in (A.7) is upper bounded by $1/\rho_{\min}^2$ times its denominator.

Note that a circuit with all amplifier output levels as large as distortionless operation permits, allows neither a further scaling up, nor a further scaling down step, and therefore no further improvement of the signal-to-noise ratio. We thus state the following.

Assuming noise described by (5), an integrator-based SC filter with fixed total capacitance area achieves the best obtainable signal-to-noise ratio if, and only if, it is scaled such that all its amplifier output voltages are at their maximum permissible level (scaled for maximum undistorted output).

APPENDIX B SOLUTION OF THE EQUALITY-CONSTRAINED PROBLEM

Here, we present in detail the general closed-form solution for the equality-constrained problem (8). In sub-

stituting the equality constraints (8c) in (8a), we obtain the following objective function:

$$F(\boldsymbol{\lambda}) = \left(b_1 + \sum_{il \in W} (-1)^l \frac{b_i}{p_{il}} \right) \frac{1}{\lambda_1} + \sum_{i \neq 1, il \notin W} \frac{b_i}{\lambda_i}$$

where the number of assignment parameters λ_i has been reduced by the number of equality constraints in (8c) or, equivalently, by the number of active constraints n_c in the working set W . Using this reduced set of variables, we reformulate (8), and obtain

$$\begin{aligned} \text{minimize} \quad & \hat{F}(\hat{\boldsymbol{\lambda}}) = \sum_{il \notin W} \frac{\hat{b}_i}{\hat{\lambda}_i} \\ \text{subject to} \quad & \hat{\mathbf{h}}^T \hat{\boldsymbol{\lambda}} = 1 \end{aligned} \quad (\text{B.1})$$

with

$$\hat{b}_i = \begin{cases} \left(b_1 + \sum_{il \in W} (-1)^l \frac{b_i}{p_{il}} \right) \left(1 + \sum_{il \in W} (-1)^l p_{il} \right), & \text{for } i = 1 \\ b_i, & \text{for } i \neq 1, il \notin W \end{cases}$$

$$\hat{\lambda}_i = \begin{cases} \lambda_1 \left(1 + \sum_{il \in W} (-1)^l p_{il} \right), & \text{for } i = 1 \\ \lambda_i, & \text{for } i \neq 1, il \notin W. \end{cases}$$

Here the vector $\hat{\mathbf{h}}$ has $(n - n_c)$ unity components. The reduced problem (B.1) is again a convex equality-constrained program with a unique local, and therefore also global, minimum. The necessary first-order conditions for the minimum $\boldsymbol{\lambda}_W^*$ are given by

$$\begin{aligned} \nabla \hat{F}(\boldsymbol{\lambda}_W^*) + \mu \hat{\mathbf{h}} &= 0 \\ \hat{\mathbf{h}}^T \boldsymbol{\lambda}_W^* &= 1 \end{aligned}$$

where μ denotes the Lagrange multiplier. For the components of the minimum point $\boldsymbol{\lambda}_W^*$, these conditions yield:

$$\hat{\lambda}_{W,i}^* = \frac{\sqrt{\hat{b}_i}}{\sum_{il \notin W} \sqrt{\hat{b}_j}}, \quad il \notin W. \quad (\text{B.2})$$

Since any realizable assignment parameter λ_i fulfills $\lambda_i > 0$, which also means $\hat{\lambda}_i > 0$, we consider only positive square root values in (B.2).

In transforming (B.2) back to obtain the solution of the original problem (8), we use the relations

$$\frac{\lambda_i}{\lambda_1} = \begin{cases} \frac{\hat{\lambda}_i}{\hat{\lambda}_1} \left(1 + \sum_{il \in W} (-1)^l p_{il} \right), & \text{for } i \neq 1, il \notin W \\ p_{il}, & \text{for } il \in W \end{cases}$$

given by the expressions below (B.1) for $i \neq 1, il \notin W$, and by (8c) for $il \in W$. With (B.2) these normalized λ -values yield for the minimum point $\boldsymbol{\lambda}_W^*$:

$$\frac{\lambda_{W,i}^*}{\lambda_{W,1}^*} = \begin{cases} \sqrt{b_i} H(W), & \text{for } i \neq 1, il \notin W \\ p_{il}, & \text{for } il \in W \end{cases} \quad (\text{B.3})$$

where the auxiliary expression $H(W)$ is defined by

$$H(W) \triangleq \sqrt{\frac{1 + \sum_{il \in W} (-1)^l p_{il}}{b_1 + \sum_{il \in W} (-1)^l \frac{b_i}{p_{il}}}}. \quad (\text{B.4})$$

With the notation $\rho_{W,i}^* \triangleq \lambda_{W,i}^* / \lambda_{W,1}^*$, we finally obtain for the components of the minimum point λ_{W}^* with (8b):

$$\lambda_{W,i}^* = \rho_{W,i}^* \left/ \sum_{k=1}^n \rho_{W,k}^* \right., \quad 1 \leq i \leq n. \quad (\text{B.5})$$

In the proceeding, we calculate the Lagrange multipliers μ_i corresponding to the active constraints (8c). First, we expand (10) in the minimum point λ_{W}^* , and obtain:

$$-\frac{b_i}{\lambda_{W,i}^{*2}} + \sum_{jl \in W} \mu_j p_{jl} + \mu_1 = 0, \quad \text{for } i = 1 \quad (\text{B.6a})$$

$$-\frac{b_i}{\lambda_{W,i}^{*2}} + (-1)^{l-1} \mu_i + \mu_1 = 0, \quad \text{for } il \in W$$

$$-\frac{b_i}{\lambda_{W,i}^{*2}} + \mu_1 = 0, \quad \text{for } i \neq 1, il \notin W. \quad (\text{B.6b})$$

Note that only (B.6a) and (B.6b) are needed to determine the Lagrange multipliers μ_i for $il \in W$. Using (B.6b), we obtain

$$\mu_i = \left(\frac{b_i}{\lambda_{W,i}^{*2}} - \mu_1 \right) (-1)^{l-1}, \quad il \in W. \quad (\text{B.7})$$

Since the active constraints fulfill

$$\lambda_{W,i}^* = (-1)^l p_{il} \lambda_{W,1}^*, \quad il \in W$$

we rewrite (B.7) as

$$\mu_i = \left(\frac{b_i}{(p_{il} \lambda_{W,1}^*)^2} - \mu_1 \right) (-1)^{l-1}, \quad il \in W \quad (\text{B.8})$$

and thus obtain an equation for μ_1 , if we substitute (B.8) in (B.6a):

$$-\frac{b_1}{\lambda_{W,1}^{*2}} + \sum_{jl \in W} \left(\frac{b_j}{(p_{jl} \lambda_{W,1}^*)^2} - \mu_1 \right) (-1)^{l-1} p_{jl} + \mu_1 = 0.$$

Solving the above equation for μ_1 leads, in a first step, to

$$\mu_1 = \frac{b_1 + \sum_{jl \in W} (-1)^l \frac{b_j}{p_{jl}}}{1 + \sum_{jl \in W} (-1)^l p_{jl}} \frac{1}{\lambda_{W,1}^{*2}}$$

and with (B.4) to

$$\mu_1 = \frac{1}{(H(W) \lambda_{W,1}^*)^2}. \quad (\text{B.9})$$

Inserting (B.9) into (B.8) finally yields for the Lagrange multipliers

$$\mu_i = \frac{1}{\lambda_{W,1}^{*2}} (-1)^l \left(\frac{1}{H(W)^2} - \frac{b_i}{p_{il}^2} \right), \quad il \in W. \quad (\text{B.10})$$

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