Monotonicity Results for Coherent Single-User and Multiple-Access MIMO Rician Channels

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***Abstract*— We introduce a preorder on the line-of-sight (LOS) matrices in coherent multiple-input multiple-output (MIMO) Rician fading channels. We demonstrate that under this preorder, the information rate and the rate-R outage probability corresponding to zero-mean multivariate circularly symmetric Gaussian inputs of arbitrary but fixed covariance matrices are monotonic in the LOS matrix. This result extends previous results obtained by Kim & Lapidoth, *ISIT*, 2003, and Hösli & Lapidoth, *ITG Conference on SCC*, 2004, i.e., our result implies the monotonicity of the information rates corresponding to isotropic Gaussian inputs and of channel capacity in the singular values of the LOS matrix. It is particularly useful in scenarios such as MIMO Rician multiple-access channels, where the achievable rates depend on the LOS matrices of the different users and cannot be determined based on their corresponding singular values alone. We also prove a converse to the main result. That is, given two different LOS matrices, we show that if for all zero-mean multivariate circularly symmetric Gaussian inputs the induced mutual information over one channel is at least as large as over the other channel, then the two LOS matrices can be ordered.**

I. Introduction

It is not difficult to demonstrate that the capacity of a single-input single-output coherent Rician (or Ricean) fading channel is monotonic in the magnitude of the line-of-sight (LOS) component. This follows from the fact that channel capacity is achieved by a zero-mean circularly symmetric Gaussian input and that a non-central chi-square random variable is stochastically monotonic in the non-centrality parameter [1, Lemma 6.2 b)], [2]. This result even extends naturally to the single-input multiple-output scenario.

The extension of this monotonicity result to multiple-input multiple-output (MIMO) Rician channels requires some care, however. The first difficulty one encounters is that to demonstrate monotonicity one has to introduce an ordering on the LOS matrices and it is *a priori* unclear what the natural ordering for the problem at hand is. The second difficulty is that there is no closed-form expression for the capacity-achieving input distribution. While it is straightforward to demonstrate that it is a circularly symmetric multivariate Gaussian, no closed-form solution for the eigenvalues of the optimal covariance matrix are known. Finally, for a given input distribution, the question of which of two LOS matrices gives rise to a larger mutual information for a given realization of the fading matrix depends on the realization. Thus, while one LOS matrix may give rise to a larger information rate for a given realization, it may actually perform worse when one averages over the fading realizations.

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In this paper we shall show that the natural partial ordering for the problem at hand is to define that the $m \times n$ LOS matrix D is "greater or equal" to the $m \times n$ LOS matrix $\tilde{\mathsf{D}}$ if $\mathsf{D}^\dagger\mathsf{D} - \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$ is a positive semidefinite $n \times n$ matrix, i.e., if $\mathsf{D}^\dagger\mathsf{D}$ is greater or equal to $\tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$ in the Loewner sense. (Here and hereafter $(\cdot)^\dagger$ denotes Hermitian conjugation, i.e., the result of transposing the matrix and then applying componentwise complex conjugation.) With this definition we shall show the monotonicity of channel capacity, the monotonicity of the isotropic Gaussian input information rates, and the monotonicity of outage probability.

II. Main Result

The main result from which the monotonicity results will follow can be stated as follows.

Theorem 1: Let $\mathbb{H}$ be a random $m \times n$ matrix whose components are independent, each with a zero-mean unit-variance circularly symmetric complex Gaussian distribution. If two deterministic complex $m \times n$ matrices D, $\tilde{\mathsf{D}}$ are such that

$$\mathsf{D}^\dagger\mathsf{D} \succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$$

then

$$\begin{aligned} &\Pr\left[\log\det\left(\mathsf{I}_m + (\mathbb{H}+\mathsf{D})\mathsf{K}(\mathbb{H}+\mathsf{D})^\dagger\right) \leq t\right] \\ &\leq \Pr\left[\log\det\left(\mathsf{I}_m + (\mathbb{H}+\tilde{\mathsf{D}})\mathsf{K}(\mathbb{H}+\tilde{\mathsf{D}})^\dagger\right) \leq t\right], \qquad (1) \\ &\qquad t \geq 0, \quad \mathsf{K} \in \mathcal{H}^+(n). \end{aligned}$$

Proof: The proof of the theorem is based on a theorem by T. W. Anderson [3] and can be found in [4]. ∎

In this theorem and throughout the relation $\mathsf{A} \succeq \mathsf{B}$ indicates that $\mathsf{A} - \mathsf{B}$ is positive semidefinite; $\mathcal{H}^+(n)$ denotes the set of $n \times n$ positive semidefinite Hermitian matrices; $\mathcal{U}(n)$ denotes the set of unitary $n \times n$ matrices; I_m denotes the $m \times m$ identity matrix; and $\log(\cdot)$ denotes the natural logarithmic function. All vectors will be column vectors.

As we shall see, Theorem 1 has applications in both single-user and multiple-access communication over coherent Rician fading channels. In single-user communication it can be used to prove that channel capacity as well as the information rates corresponding to isotropic Gaussian inputs are both monotonic in the singular values of the matrix D, to which we shall refer as the "mean matrix".

In multiple-access communications it is used to establish monotonicity results both for the capacity region and for the information rates corresponding to the case where each sender uses isotropic Gaussian inputs of the user's allowed power. Such monotonicity results cannot be obtained using the techniques of [5] and [6] because the problems of computing these regions do not diagonalize. That is, these regions cannot be determined from the singular values of the mean matrices of the different users alone. In addition to the singular values of each mean matrix, one also needs to know how these matrices relate to each other.

Before we discuss the aforementioned applications we introduce two functions that will simplify the notation in the following. In the notation of Theorem 1, we define for any $t \geq 0$, $\mathsf{K} \in \mathcal{H}^+(n)$, and $\mathsf{D} \in \mathbb{C}^{m \times n}$

$$F(t; \mathsf{K}, \mathsf{D}) \triangleq \Pr\left[\log\det\left(\mathsf{I}_m + (\mathbb{H} + \mathsf{D})\mathsf{K}(\mathbb{H} + \mathsf{D})^\dagger\right) \leq t\right]$$

and

$$\mathcal{I}(\mathsf{K}, \mathsf{D}) \triangleq \mathsf{E}\left[\log\det\left(\mathsf{I}_m + (\mathbb{H} + \mathsf{D})\mathsf{K}(\mathbb{H} + \mathsf{D})^\dagger\right)\right]. \quad (2)$$

Note the rotational symmetry in $F(t; \mathsf{K}, \mathsf{D})$ and $\mathcal{I}(\mathsf{K}, \mathsf{D})$. First observe that the law of $\mathbb{H}$ is invariant under left and right rotations, i.e., for any $\mathsf{U} \in \mathcal{U}(m)$ and $\mathsf{V} \in \mathcal{U}(n)$,

$$\mathsf{U}\mathbb{H}\mathsf{V}^\dagger \stackrel{\mathscr{L}}{=} \mathbb{H}$$

where $\stackrel{\mathscr{L}}{=}$ denotes *equality in law*. Consequently, we have for such U and V

$$\begin{aligned}
&F(t; \mathsf{K}, \mathsf{U}\mathsf{D}\mathsf{V}^\dagger) \\
&= \Pr\left[\log\det\left(\mathsf{I}_m + (\mathbb{H} + \mathsf{U}\mathsf{D}\mathsf{V}^\dagger)\mathsf{K}(\mathbb{H} + \mathsf{U}\mathsf{D}\mathsf{V}^\dagger)^\dagger\right) \leq t\right] \\
&= \Pr\left[\log\det\left(\mathsf{U}\left(\mathsf{I}_m + (\mathbb{H} + \mathsf{D})\mathsf{V}^\dagger\mathsf{K}\mathsf{V}(\mathbb{H} + \mathsf{D})^\dagger\right)\mathsf{U}^\dagger\right) \leq t\right] \\
&= \Pr\left[\log\det\left(\mathsf{I}_m + (\mathbb{H} + \mathsf{D})\mathsf{V}^\dagger\mathsf{K}\mathsf{V}(\mathbb{H} + \mathsf{D})^\dagger\right) \leq t\right] \\
&= F(t; \mathsf{V}^\dagger\mathsf{K}\mathsf{V}, \mathsf{D}). \quad (3)
\end{aligned}$$

Observing further that

$$\mathcal{I}(\mathsf{K}, \mathsf{D}) = \int_0^\infty \left(1 - F(t; \mathsf{K}, \mathsf{D})\right) \mathrm{d}t \quad (4)$$

we thus also have

$$\mathcal{I}(\mathsf{K}, \mathsf{U}\mathsf{D}\mathsf{V}^\dagger) = \mathcal{I}(\mathsf{V}^\dagger\mathsf{K}\mathsf{V}, \mathsf{D}). \quad (5)$$

With (4) we also obtain the following corollary of Theorem 1.

Corollary 2: If $\mathsf{D}^\dagger\mathsf{D} \succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$, then

$$\mathcal{I}(\mathsf{K}, \mathsf{D}) \geq \mathcal{I}(\mathsf{K}, \tilde{\mathsf{D}}), \quad \forall \mathsf{K} \in \mathcal{H}^+(n).$$

That the partial ordering on the mean matrices is "natural" is supported by the following proposition, which shows that the reverse of the corollary is also true:

Proposition 3: If $\mathcal{I}(\mathsf{K}, \mathsf{D}) \geq \mathcal{I}(\mathsf{K}, \tilde{\mathsf{D}})$ for all $\mathsf{K} \in \mathcal{H}^+(n)$, then

$$\mathsf{D}^\dagger\mathsf{D} \succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}.$$

Proof: Instead of proving Proposition 3 directly, we will prove the equivalent statement

$$\mathsf{D}^\dagger\mathsf{D} \not\succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}} \quad \Rightarrow \quad \mathcal{I}(\mathsf{K}, \mathsf{D}) < \mathcal{I}(\mathsf{K}, \tilde{\mathsf{D}}) \text{ for some } \mathsf{K} \in \mathcal{H}^+(n).$$

We first note that $\mathsf{D}^\dagger\mathsf{D} \not\succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$ means that there exists a vector $\mathbf{a} \in \mathbb{C}^n$ such that

$$\mathbf{a}^\dagger\mathsf{D}^\dagger\mathsf{D}\mathbf{a} < \mathbf{a}^\dagger\tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}\mathbf{a}. \quad (6)$$

For such a vector let $\mathsf{K}_0 = \mathbf{a}\mathbf{a}^\dagger \in \mathcal{H}^+(n)$. We will show that for K_0 the strict inequality $\mathcal{I}(\mathsf{K}_0, \mathsf{D}) < \mathcal{I}(\mathsf{K}_0, \tilde{\mathsf{D}})$ holds.

By (4) it suffices to show that $F(t; \mathsf{K}_0, \mathsf{D}) > F(t; \mathsf{K}_0, \tilde{\mathsf{D}})$ for all $t > 0$. Indeed, for any such t

$$\begin{aligned}
F(t; \mathsf{K}_0, \mathsf{D}) &= \Pr\left[\log\det\left(\mathsf{I}_m + (\mathbb{H} + \mathsf{D})\mathbf{a}\mathbf{a}^\dagger(\mathbb{H} + \mathsf{D})^\dagger\right) \leq t\right] \\
&= \Pr\left[\log\left(1 + \mathbf{a}^\dagger(\mathbb{H} + \mathsf{D})^\dagger(\mathbb{H} + \mathsf{D})\mathbf{a}\right) \leq t\right] \quad (7) \\
&= \Pr\left[\log\left(1 + (\mathbf{G} + \mathbf{b})^\dagger(\mathbf{G} + \mathbf{b})\right) \leq t\right] \quad (8) \\
&> \Pr\left[\log\left(1 + (\mathbf{G} + \tilde{\mathbf{b}})^\dagger(\mathbf{G} + \tilde{\mathbf{b}})\right) \leq t\right] \quad (9) \\
&= \Pr\left[\log\left(1 + \mathbf{a}^\dagger(\mathbb{H} + \tilde{\mathsf{D}})^\dagger(\mathbb{H} + \tilde{\mathsf{D}})\mathbf{a}\right) \leq t\right] \\
&= \Pr\left[\log\det\left(\mathsf{I}_m + (\mathbb{H} + \tilde{\mathsf{D}})\mathbf{a}\mathbf{a}^\dagger(\mathbb{H} + \tilde{\mathsf{D}})^\dagger\right) \leq t\right] \\
&= F(t; \mathsf{K}_0, \tilde{\mathsf{D}})
\end{aligned}$$

where in (7) we have used that $\det(\mathsf{I}_m + \mathsf{A}\mathsf{B}) = \det(\mathsf{I}_n + \mathsf{B}\mathsf{A})$ for any $\mathsf{A} \in \mathbb{C}^{m \times n}$ and $\mathsf{B} \in \mathbb{C}^{n \times m}$; in (8) we have defined $\mathbf{G} = \mathbb{H} \cdot \mathbf{a}$, which is $\mathcal{N}_\mathbb{C}\left(\mathbf{0}, \mathbf{a}^\dagger\mathbf{a} \cdot \mathsf{I}_m\right)$ distributed, and $\mathbf{b} = \mathsf{D} \cdot \mathbf{a}$; and to get (9) we have used the fact that $(\mathbf{G} + \mathbf{b})^\dagger(\mathbf{G} + \mathbf{b})$ has a scaled non-central chi-square distribution with (scaled) non-centrality parameter $\mathbf{b}^\dagger\mathbf{b}$. Defining $\tilde{\mathbf{b}} = \tilde{\mathsf{D}} \cdot \mathbf{a}$ we note that $(\mathbf{G} + \tilde{\mathbf{b}})^\dagger(\mathbf{G} + \tilde{\mathbf{b}})$ in (9) is also a scaled non-central chi-square random variable, which, by (6), has a strictly larger non-centrality parameter $\tilde{\mathbf{b}}^\dagger\tilde{\mathbf{b}} > \mathbf{b}^\dagger\mathbf{b}$. Hence, $(\mathbf{G} + \tilde{\mathbf{b}})^\dagger(\mathbf{G} + \tilde{\mathbf{b}})$ is stochastically strictly larger than $(\mathbf{G} + \mathbf{b})^\dagger(\mathbf{G} + \mathbf{b})$, so that the strict inequality in (9) is justified for any $t > 0$. ■

Theorem 1 and Corollary 2 can be restated as monotonicity results in the mean matrices by introducing a preorder on the set of complex $m \times n$ matrices. Simply define the matrix $\mathsf{D} \in \mathbb{C}^{m \times n}$ to be "larger than or equal to" the matrix $\tilde{\mathsf{D}} \in \mathbb{C}^{m \times n}$ if

$$\mathsf{D}^\dagger\mathsf{D} \succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}. \quad (10)$$

Note that this preorder is not a partial order since the condition $\mathsf{D}^\dagger\mathsf{D} = \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$ only implies that $\tilde{\mathsf{D}} = \mathsf{U}\mathsf{D}$ for some $\mathsf{U} \in \mathcal{U}(m)$; it does *not* imply $\tilde{\mathsf{D}} = \mathsf{D}$ [7], [8].

III. Applications

A. The Single-User Rician Fading Channel

We first apply the main result to a coherent single-user Rician fading channel. Its output $(\mathbb{H}, \mathbf{Y})$ consists of a random $m \times n$ matrix $\mathbb{H}$ whose $m \cdot n$ components are independent and identically distributed (IID) according to the zero-mean unit-variance circularly symmetric complex Gaussian distribution $\mathcal{N}_\mathbb{C}(0, 1)$ and of a random m-vector $\mathbf{Y} \in \mathbb{C}^m$ which is given by

$$\mathbf{Y} = (\mathbb{H} + \mathsf{D})\mathbf{x} + \mathbf{Z} \quad (11)$$

where $\mathbf{x} \in \mathbb{C}^n$ is the channel input; D is a deterministic $m \times n$ complex matrix called the *line-of-sight (LOS) matrix*; and the random vector $\mathbf{Z}$ takes value in $\mathbb{C}^m$ according to the $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \sigma^2 \mathsf{I}_m)$ law for some $\sigma^2 > 0$. Here $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathsf{K})$ denotes the zero-mean circularly symmetric multivariate complex Gaussian distribution of covariance matrix K. Thus, the m components of $\mathbf{Z}$ are IID $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$. It is assumed that $\mathbb{H}$ and $\mathbf{Z}$ are independent, and that their joint law does not depend on the input $\mathbf{x}$.

Since the law of $\mathbb{H}$ does not depend on the input $\mathbf{x}$, we can express the mutual information between the channel input and output as

$$I\big(\mathbf{X}; (\mathbb{H}, \mathbf{Y})\big) = I\big(\mathbf{X}; \mathbf{Y} \big| \mathbb{H}\big). \tag{12}$$

Of all input distributions of a given covariance matrix, the zero-mean circularly symmetric multivariate complex Gaussian maximizes the conditional mutual information $I(\mathbf{X}; \mathbf{Y} | \mathbb{H} = \mathsf{H})$, irrespective of the realization H. Consequently, it must also maximize the average mutual information $I(\mathbf{X}; \mathbf{Y} | \mathbb{H})$. We shall therefore consider the mutual information induced by $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathsf{K})$ inputs, and we will be primarily interested in the dependence of mutual information on the LOS matrix D when the input distribution, i.e., K, is held fixed. Also, since we can absorb the dependence on σ^2 into K, we assume $\sigma^2 = 1$ throughout without loss of generality.

For a given realization $\mathbb{H} = \mathsf{H}$, we can express the conditional mutual information $I(\mathbf{X}; \mathbf{Y} | \mathbb{H} = \mathsf{H})$ for Gaussian inputs as

$$I(\mathbf{X}; \mathbf{Y} | \mathbb{H} = \mathsf{H}) = \log\det\left(\mathsf{I}_m + (\mathsf{H} + \mathsf{D})\mathsf{K}(\mathsf{H} + \mathsf{D})^\dagger\right), \quad \mathbf{X} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathsf{K}). \tag{13}$$

The average conditional mutual information induced by a $\mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathsf{K})$ input can be thus expressed [9] as an explicit function of K and D as

$$I(\mathbf{X}; \mathbf{Y} | \mathbb{H}) = \mathsf{E}\left[\log\det\left(\mathsf{I}_m + (\mathbb{H} + \mathsf{D})\mathsf{K}(\mathbb{H} + \mathsf{D})^\dagger\right)\right] \tag{14}$$
$$= \mathcal{I}(\mathsf{K}, \mathsf{D}), \quad \mathsf{K} \in \mathcal{H}^+(n),\ \mathsf{D} \in \mathbb{C}^{m \times n}. \tag{15}$$

Thus Corollary 2 can be interpreted as the monotonicity of the average conditional mutual information of the Rician fading channel (11) with fixed input covariance matrix. We can also give a more direct interpretation of Theorem 1 through the notion of *outage probability*. Consider the probability

$$\Pr\left[\log\det\left(\mathsf{I}_m + (\mathbb{H} + \mathsf{D})\mathsf{K}(\mathbb{H} + \mathsf{D})^\dagger\right) \le R\right] = F(R; \mathsf{K}, \mathsf{D}).$$

We can interpret this quantity as the probability that the achievable rate on the Gaussian channel $\mathbf{Y} = (\mathsf{H} + \mathsf{D})\mathbf{x} + \mathbf{Z}$ corresponding to the particular channel realization $\mathbb{H} = \mathsf{H}$ and the input distribution $\mathcal{N}_{\mathbb{C}}(0, \mathsf{K})$ does not exceed R. Under this interpretation, Theorem 1 can be viewed as the monotonicity of the outage probability in the channel LOS matrix.

These monotonicity results can be further refined for the power-$\mathcal{E}$ isotropic Gaussian input information rates

$$I^{\mathrm{IG}}(\mathcal{E}, \mathsf{D}) \triangleq \mathcal{I}\left(\frac{\mathcal{E}}{n}\mathsf{I}_n, \mathsf{D}\right)$$

and for the capacity $C(\mathcal{E}, \mathsf{D})$ of the Rician channel under the average input power constraint $\mathsf{E}\left[\mathbf{X}^\dagger \mathbf{X}\right] \le \mathcal{E}$, which is given by

$$C(\mathcal{E}, \mathsf{D}) \triangleq \max_{\mathsf{K}} \mathcal{I}(\mathsf{K}, \mathsf{D}) \tag{16}$$

where the maximum is taken over the set of all covariance matrices K satisfying the trace constraint

$$\mathrm{tr}\,(\mathsf{K}) \le \mathcal{E}. \tag{17}$$

It follows immediately from Corollary 2 that, if $\mathsf{D}^\dagger\mathsf{D} \succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$, then $I^{\mathrm{IG}}(\mathcal{E}, \mathsf{D}) \ge I^{\mathrm{IG}}(\mathcal{E}, \tilde{\mathsf{D}})$ and $C(\mathcal{E}, \mathsf{D}) \ge C(\mathcal{E}, \tilde{\mathsf{D}})$.

Another refinement of Theorem 1 is obtained for the outage probability corresponding to isotropic Gaussian inputs

$$P_{\mathsf{out}}^{\mathrm{IG}}(R, \mathcal{E}, \mathsf{D}) \triangleq F\left(R, \frac{\mathcal{E}}{n}\mathsf{I}_n, \mathsf{D}\right)$$

and for the optimal power-$\mathcal{E}$ rate-R outage probability $P_{\mathsf{out}}^{*}(R, \mathcal{E}, \mathsf{D})$, which is the smallest outage probability that can be achieved for the rate R and the average power $\mathcal{E}$ and is given by

$$P_{\mathsf{out}}^{*}(R, \mathcal{E}, \mathsf{D}) \triangleq \min_{\mathsf{K}} F(R, \mathsf{K}, \mathsf{D}) \tag{18}$$

where the minimum is over all positive semidefinite matrices K satisfying (17). From Theorem 1 we now obtain that $\mathsf{D}^\dagger\mathsf{D} \succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$ implies that $P_{\mathsf{out}}^{\mathrm{IG}}(R, \mathcal{E}, \mathsf{D}) \le P_{\mathsf{out}}^{\mathrm{IG}}(R, \mathcal{E}, \tilde{\mathsf{D}})$ and $P_{\mathsf{out}}^{*}(R, \mathcal{E}, \mathsf{D}) \le P_{\mathsf{out}}^{*}(R, \mathcal{E}, \tilde{\mathsf{D}})$.

Using the rotational invariance (5), we can strengthen these refinements by stating them in terms of the singular values of the LOS matrices. Indeed, for any unitary matrix V, we have $\mathrm{tr}\left(\mathsf{V}^\dagger\mathsf{K}\mathsf{V}\right) = \mathrm{tr}\,(\mathsf{K})$, and hence it follows from (5) that for any $\mathsf{U} \in \mathcal{U}(m)$ and $\mathsf{V} \in \mathcal{U}(n)$

$$I^{\mathrm{IG}}(\mathcal{E}, \mathsf{U}\mathsf{D}\mathsf{V}^\dagger) = I^{\mathrm{IG}}(\mathcal{E}, \mathsf{D})$$

and

$$C(\mathcal{E}, \mathsf{U}\mathsf{D}\mathsf{V}^\dagger) = C(\mathcal{E}, \mathsf{D})$$

i.e., that the isotropic Gaussian input information rates and capacity depend on the LOS matrix only via its singular values. By a similar argument, it can be verified that, by (3), both the outage probability corresponding to isotropic Gaussian inputs $P_{\mathsf{out}}^{\mathrm{IG}}(R, \mathcal{E}, \mathsf{D})$ and the optimal outage probability $P_{\mathsf{out}}^{*}(R, \mathcal{E}, \mathsf{D})$ depend on the LOS matrix also only via its singular values. Consequently, all these quantities are monotonic in the singular values of the LOS matrix:

Corollary 4: Let $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_{\min\{m,n\}}$ and $\tilde{\sigma}_1 \ge \tilde{\sigma}_2 \ge \cdots \ge \tilde{\sigma}_{\min\{m,n\}}$ be the singular values of the LOS matrices D and $\tilde{\mathsf{D}}$, respectively. Suppose that $\sigma_i \ge \tilde{\sigma}_i$ for all i. Then

$$I^{\mathrm{IG}}(\mathcal{E}, \mathsf{D}) \ge I^{\mathrm{IG}}(\mathcal{E}, \tilde{\mathsf{D}})$$
$$C(\mathcal{E}, \mathsf{D}) \ge C(\mathcal{E}, \tilde{\mathsf{D}})$$
$$P_{\mathsf{out}}^{\mathrm{IG}}(R, \mathcal{E}, \mathsf{D}) \le P_{\mathsf{out}}^{\mathrm{IG}}(R, \mathcal{E}, \tilde{\mathsf{D}})$$

and

$$P_{\mathsf{out}}^{*}(R, \mathcal{E}, \mathsf{D}) \le P_{\mathsf{out}}^{*}(R, \mathcal{E}, \tilde{\mathsf{D}}).$$

$$\mathcal{R}(\mathsf{K}_1,\ldots,\mathsf{K}_k;\mathsf{D}_1,\ldots,\mathsf{D}_k) = \left\{(R_1,\ldots,R_k)\in\mathbb{R}_+^k : \sum_{i\in\mathcal{S}} R_i \le \mathsf{E}\left[\log\det\left(\mathsf{I}_m + \sum_{i\in\mathcal{S}}(\mathbb{H}_i+\mathsf{D}_i)\mathsf{K}_i(\mathbb{H}_i+\mathsf{D}_i)^\dagger\right)\right], \forall\mathcal{S}\subseteq\{1,\ldots,k\}\right\},$$
$$\mathsf{K}_i\in\mathcal{H}^+(n_i),\quad \mathsf{D}_i\in\mathbb{C}^{m\times n_i}\quad (i=1,\ldots,k). \qquad (20)$$

B. *The Rician Multiple-Access Channel*

The application of the main result becomes more interesting in a multiple-access scenario where the LOS matrices cannot be assumed to be jointly diagonalizable, not even in the case of (unequal power) isotropic inputs or in the calculation of the capacity region.

The coherent MIMO Rician multiple-access channel (MAC) with k senders is modeled as follows. The channel output consists of k independent random matrices $\mathbb{H}_1,\ldots,\mathbb{H}_k$, where $\mathbb{H}_i$ is a random $m\times n_i$ matrix whose components are IID $\mathcal{N}_{\mathbb{C}}(0,1)$, and of a random m-vector $\mathbf{Y}\in\mathbb{C}^m$ of the form

$$\mathbf{Y}=\sum_{i=1}^{k}(\mathbb{H}_i+\mathsf{D}_i)\mathbf{x}_i+\mathbf{Z} \qquad (19)$$

where $\mathbf{x}_i\in\mathbb{C}^{n_i}$ is the i-th transmitter's input vector, D_i is a deterministic $m\times n_i$ complex matrix corresponding to the LOS matrix of the i-th user, and $\mathbf{Z}\sim\mathcal{N}_{\mathbb{C}}(\mathbf{0},\sigma^2\mathsf{I}_m)$ corresponds to the additive noise vector. It is assumed that all fading matrices are independent of $\mathbf{Z}$ and that the joint law of $(\mathbb{H}_1,\ldots,\mathbb{H}_k,\mathbf{Z})$ does not depend on the inputs $\{\mathbf{x}_i\}_{i=1}^k$. Again, without loss of generality, we assume $\sigma^2=1$.

As in the single-user scenario, Gaussian inputs achieve the capacity region of the multiple-access channel [10], [11]. The rate region achieved by independent Gaussian inputs $\mathcal{N}_{\mathbb{C}}(\mathbf{0},\mathsf{K}_i)$ with fixed input covariance matrices $\{\mathsf{K}_i\}_{i=1}^k$ over the MIMO Rician MAC with LOS matrices $\{\mathsf{D}_i\}_{i=1}^k$ is given (see, e.g., [12]) by (20) at the top of this page.

The capacity region of the MIMO Rician MAC, denoted as an explicit function of the powers of the different users and of their corresponding LOS matrices, can be written as

$$\mathcal{C}(\mathcal{E}_1,\ldots,\mathcal{E}_k;\mathsf{D}_1,\ldots,\mathsf{D}_k) = \bigcup_{\{\mathsf{K}_i\}_{i=1}^k}\mathcal{R}(\mathsf{K}_1,\ldots,\mathsf{K}_k;\mathsf{D}_1,\ldots,\mathsf{D}_k) \qquad (21)$$

where the union is over all positive semidefinite matrices $\{\mathsf{K}_i\}_{i=1}^k$ that satisfy the trace constraints $\operatorname{tr}(\mathsf{K}_i)\le\mathcal{E}_i$, $i=1,\ldots,k$. The achievable region corresponding to power-$\mathcal{E}_i$ isotropic Gaussian inputs is given by

$$\mathcal{R}\left(\frac{\mathcal{E}_1}{n_1}\mathsf{I}_{n_1},\ldots,\frac{\mathcal{E}_k}{n_k}\mathsf{I}_{n_k};\mathsf{D}_1,\ldots,\mathsf{D}_k\right).$$

Let now $n_{\text{tot}}=\sum_{i=1}^k n_i$. Then the monotonicity result for the multiple-access scenario can be stated as follows.

Corollary 5: Consider the coherent MIMO Rician MAC (19) and let the k transmitters use $\mathcal{N}_{\mathbb{C}}(\mathbf{0},\mathsf{K}_i)$ inputs with fixed covariance matrices $\mathsf{K}_i\in\mathcal{H}^+(n_i)$, $i=1,\ldots,k$. Let $\mathsf{D}=[\mathsf{D}_1,\ldots,\mathsf{D}_k]$ and $\tilde{\mathsf{D}}=[\tilde{\mathsf{D}}_1,\ldots,\tilde{\mathsf{D}}_k]$ be two $m\times n_{\text{tot}}$ partitioned LOS matrices with $\mathsf{D}_i,\tilde{\mathsf{D}}_i\in\mathbb{C}^{m\times n_i}$ $(i=1,\ldots,k)$ such that

$$\mathsf{D}^\dagger\mathsf{D}\succeq\tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}. \qquad (22)$$

Then the rate region achieved over the MIMO Rician MAC with partitioned LOS matrix $\tilde{\mathsf{D}}$ is contained in the rate region achieved over the MIMO Rician MAC with partitioned LOS matrix D, i.e.,

$$\mathcal{R}(\mathsf{K}_1,\ldots,\mathsf{K}_k;\tilde{\mathsf{D}}_1,\ldots,\tilde{\mathsf{D}}_k)\subseteq\mathcal{R}(\mathsf{K}_1,\ldots,\mathsf{K}_k;\mathsf{D}_1,\ldots,\mathsf{D}_k). \qquad (23)$$

In particular, if $\mathsf{K}_i=(\mathcal{E}_i/n_i)\mathsf{I}_{n_i}$ we obtain the analogous result for isotropic inputs

$$\mathcal{R}\left(\frac{\mathcal{E}_1}{n_1}\mathsf{I}_{n_1},\ldots,\frac{\mathcal{E}_k}{n_k}\mathsf{I}_{n_k};\tilde{\mathsf{D}}_1,\ldots,\tilde{\mathsf{D}}_k\right) \subseteq\mathcal{R}\left(\frac{\mathcal{E}_1}{n_1}\mathsf{I}_{n_1},\ldots,\frac{\mathcal{E}_k}{n_k}\mathsf{I}_{n_k};\mathsf{D}_1,\ldots,\mathsf{D}_k\right). \qquad (24)$$

And, by taking the union over all choices of K_i satisfying $\operatorname{tr}(\mathsf{K}_i)\le\mathcal{E}_i$ in (23), we obtain

$$\mathcal{C}(\mathcal{E}_1,\ldots,\mathcal{E}_k;\tilde{\mathsf{D}}_1,\ldots,\tilde{\mathsf{D}}_k)\subseteq\mathcal{C}(\mathcal{E}_1,\ldots,\mathcal{E}_k;\mathsf{D}_1,\ldots,\mathsf{D}_k). \qquad (25)$$

Note that for the MAC scenario the monotonicity of the isotropic Gaussian input information rates (24) and the monotonicity of the capacity region (25) cannot be stated simply in terms of the singular values of the partitioned LOS matrix nor in terms of the singular values of the different users' LOS matrices. However, using the rotational invariance (5) we can weaken the condition (22) for (24) and (25) to hold to the requirement that there exist k unitary matrices $\mathsf{U}_1\in\mathcal{U}(n_1)$, $\ldots$, $\mathsf{U}_k\in\mathcal{U}(n_k)$ such that

$$\mathsf{D}^\dagger\mathsf{D}\succeq[\tilde{\mathsf{D}}_1\mathsf{U}_1,\ldots,\tilde{\mathsf{D}}_k\mathsf{U}_k]^\dagger[\tilde{\mathsf{D}}_1\mathsf{U}_1,\ldots,\tilde{\mathsf{D}}_k\mathsf{U}_k].$$

To derive Corollary 5 from Corollary 2 we need some more notation. For any subset $\mathcal{S}\subseteq\{1,\ldots,k\}$ of cardinality s containing the ordered elements $1\le i_1<i_2<\ldots<i_s\le k$ we define the partitioned matrices

$$\mathsf{D}_{\mathcal{S}}\triangleq[\mathsf{D}_{i_1},\ldots,\mathsf{D}_{i_s}]$$

and

$$\mathbb{H}_{\mathcal{S}}\triangleq[\mathbb{H}_{i_1},\ldots,\mathbb{H}_{i_s}]$$

and the block diagonal matrix

$$\mathsf{K}_{\mathcal{S}}\triangleq\operatorname{diag}(\mathsf{K}_{i_1},\ldots,\mathsf{K}_{i_s}).$$

With this notation we can write the achievable rate region (20) corresponding to the different users employing zero-mean circularly symmetric complex Gaussian inputs of respective covariance matrices $\mathsf{K}_1, \ldots, \mathsf{K}_k$ as

$$\mathcal{R}(\mathsf{K}_1, \ldots, \mathsf{K}_k; \mathsf{D}_1, \ldots, \mathsf{D}_k) = \Bigg\{ (R_1, \ldots, R_k) \in \mathbb{R}_+^k : \sum_{i \in \mathcal{S}} R_i \leq \mathsf{E}\left[\log\det\left(\mathsf{I}_m + (\mathbb{H}_\mathcal{S} + \mathsf{D}_\mathcal{S})\mathsf{K}_\mathcal{S}(\mathbb{H}_\mathcal{S} + \mathsf{D}_\mathcal{S})^\dagger\right)\right], \quad \forall \mathcal{S} \subseteq \{1, \ldots, k\} \Bigg\}$$

or, using (2),

$$\mathcal{R}(\mathsf{K}_1, \ldots, \mathsf{K}_k; \mathsf{D}_1, \ldots, \mathsf{D}_k) = \Bigg\{ (R_1, \ldots, R_k) \in \mathbb{R}_+^k : \sum_{i \in \mathcal{S}} R_i \leq \mathcal{I}(\mathsf{K}_\mathcal{S}, \mathsf{D}_\mathcal{S}), \quad \forall \mathcal{S} \subseteq \{1, \ldots, k\} \Bigg\}. \quad (26)$$

Corollary 5 now follows from (26) and from Corollary 2 by noting that $\mathsf{D}^\dagger\mathsf{D} \succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$ implies $\mathsf{D}_\mathcal{S}^\dagger\mathsf{D}_\mathcal{S} \succeq \tilde{\mathsf{D}}_\mathcal{S}^\dagger\tilde{\mathsf{D}}_\mathcal{S}$ for all $\mathcal{S} \subseteq \{1, \ldots, k\}$.

IV. Concluding Remarks

In this paper we have found a natural partial ordering of MIMO Rician channels via their LOS matrices. We have shown that for two LOS matrices $\mathsf{D}, \tilde{\mathsf{D}} \in \mathbb{C}^{m\times n}$ we have $\mathsf{D}^\dagger\mathsf{D} \succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$ if, and only if, $\mathcal{I}(\mathsf{K}, \mathsf{D}) \geq \mathcal{I}(\mathsf{K}, \tilde{\mathsf{D}})$ for all $\mathsf{K} \in \mathcal{H}^+(n)$, where $\mathcal{I}(\mathsf{K}, \mathsf{D}) = I(\mathbf{X}; \mathbf{Y}|\mathbb{H})$ is the mutual information induced by a $\mathcal{N}_\mathbb{C}(\mathbf{0}, \mathsf{K})$ input over a coherent MIMO Ricean channel with LOS matrix D. From this result we obtained monotonicity results for isotropic Gaussian input information rates and for channel capacity, not only for the single-user but also for the multiple-access channel.

Our monotonicity results, although intuitive, do not follow straightforwardly from considering a MIMO Gaussian channel $\mathbf{Y} = \mathsf{H}\mathbf{x} + \mathbf{Z}$ with channel matrix $\mathsf{H} \in \mathbb{C}^{m\times n}$ and additive noise $\mathbf{Z}$ with a $\mathcal{N}_\mathbb{C}(\mathbf{0}, \sigma^2\mathsf{I}_m)$ distribution. In that case, for any H an input $\mathbf{X}$ induces the mutual information $I(\mathbf{X}; \mathsf{H}\mathbf{X} + \mathbf{Z})$, and one can show that $\mathsf{H}^\dagger\mathsf{H} \succeq \tilde{\mathsf{H}}^\dagger\tilde{\mathsf{H}}$ if, and only if, $I(\mathbf{X}; \mathsf{H}\mathbf{X} + \mathbf{Z}) \geq I(\mathbf{X}; \tilde{\mathsf{H}}\mathbf{X} + \mathbf{Z})$ for all input distributions $\mathbf{X}$. In the Rician fading scenario, however, we *cannot* conclude from $\mathsf{D}^\dagger\mathsf{D} \succeq \tilde{\mathsf{D}}^\dagger\tilde{\mathsf{D}}$ that for all fading realizations $\mathbb{H} = \mathsf{H}$ we have $(\mathsf{H} + \mathsf{D})^\dagger(\mathsf{H} + \mathsf{D}) \succeq (\mathsf{H} + \tilde{\mathsf{D}})^\dagger(\mathsf{H} + \tilde{\mathsf{D}})$. In fact, if $\mathbb{H}$ has IID zero-mean unit-variance circularly symmetric Gaussian entries, then the probability of the last inequality not being satisfied is strictly positive.

It should also be emphasized that our monotonicity results are proved when the distribution of the granular component, i.e., $\mathbb{H}$, is held fixed. Consequently, as we vary the LOS matrix the output power is not kept constant. These results fail when we adjust the granular component so as to guarantee a fixed output power [13], [14], [15]. In other words, examples can be found where for a fixed input distribution and a fixed LOS matrix the mutual information either increases or decreases with the Rice factor, depending on the particular form of the LOS matrix [16].

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