THEORY AND TEST PROCEDURE FOR SYMMETRIES IN THE FREQUENCY RESPONSE OF COMPLEX TWO-DIMENSIONAL DELTA OPERATOR FORMULATED DISCRETE-TIME SYSTEMS

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ABSTRACT

This paper provides the theory and an efficient tabular algorithm to test for various symmetries in the magnitude response of two-dimensional (2-D) complex-coefficient delta operator formulated discrete-time systems. In general, centro symmetry is not preserved in the complex case. The conditions under which this is preserved is discussed in the paper. It is to be noted that as the sampling period (Δ) goes to zero, the symmetry conditions merge with that of 2-D continuous-time case.

1 .INTRODUCTION

A well-known technique to reduce the complexity of the design and implementation of 2-D filters is to make use of the various symmetries that might be present in the frequency response of these filters. Symmetry conditions for delta-operator formulated real polynomials have already been defined in [1]. This paper will extend the definitions to a broader class of complex-coefficient delta- polynomials and provide the algorithm to test for these symmetries. It is well known that delta-operator formulation provides various advantages over traditional shift-operator based systems such as better finite word length effects. It also provides unification between discrete- and continuous-time systems. [3] Thus, an efficient algorithm to test for symmetries for 2-D delta- polynomials will greatly simplify the design process of 2-D delta-operator formulated filters.

2. PRELIMINARIES

The relation between the transform variables (z_1,z_2) for q-operator based discrete time system and (γ_1,γ_2) corresponding to delta operator based discrete time system (DODT) is given by $\gamma_i = (z_i - 1)/\Delta$, where i=1,2 and Δ is the sampling period. An inverse polynomial in delta domain is obtained by substituting $-\gamma_i/(1+\Delta\gamma_i)$ for γ_i and multiplying the polynomial by factors of $(1+\Delta\gamma_i)^k$ to cancel the denominator. A polynomial is called a self-inverse if its inverse is equal to the original polynomial. A polynomial can be self-inverse w.r.t. to γ_1 or γ_2 or both γ_1 and γ_2.
If $P(\gamma_1, \gamma_2)$ is a 2-D delta(δ)-domain polynomial with complex coefficients, its frequency response is given by $P\left(\frac{e^{j\omega_1\Delta}-1}{\Delta},\frac{e^{j\omega_2\Delta}-1}{\Delta}\right)$ where ω_1 and ω_2 are the radian frequencies. Its magnitude squared function of the frequency response is given by

$$F(\omega_1,\omega_2) = P\left(\frac{e^{j\omega_1\Delta}-1}{\Delta},\frac{e^{j\omega_2\Delta}-1}{\Delta}\right)\cdot P^*\left(\frac{e^{-j\omega_1\Delta}-1}{\Delta},\frac{e^{-j\omega_2\Delta}-1}{\Delta}\right)$$
$$= P(\gamma_1,\gamma_2)\cdot P^*\left(\frac{-\gamma_1}{1+\Delta\gamma_1},\frac{-\gamma_2}{1+\Delta\gamma_2}\right)\Bigg|_{\gamma_i=\frac{e^{j\omega_i\Delta}-1}{\Delta},\ i=1,2} \quad (1)$$

where P^* denotes the complex conjugate of P.
Now, we shall discuss the various types of symmetries in the magnitude squared function of $P(\gamma_1, \gamma_2)$.

3. SYMMETRY CONDITIONS ON $P(\gamma_1,\gamma_2)$

Centro Symmetry:

$P(\gamma_1, \gamma_2)$ possesses centro symmetry $F(\omega_1,\omega_2)=F(-\omega_1,-\omega_2)$ in its magnitude response if $P(\gamma_1, \gamma_2)$ can be expressed as

$$P(\gamma_1, \gamma_2)=P_1(\gamma_1, \gamma_2)\cdot P_2(\gamma_1, \gamma_2)$$

where

$$P_1(\gamma_1,\gamma_2) = P_1\left(\frac{-\gamma_1}{1+\Delta\gamma_1},\frac{-\gamma_2}{1+\Delta\gamma_2}\right)\cdot(1+\Delta\gamma_1)^{J_{11}}\cdot(1+\Delta\gamma_2)^{J_{12}} \quad (2)$$

and $P_2(\gamma_1,\gamma_2) = P_2^*(\gamma_1,\gamma_2)$, i.e. P_2 is a real polynomial (3)
J_{11} and J_{12} are the degrees of γ_1 and γ_2 in P_1 respectively. P_2^* denotes the complex conjugate of P_2.

Another way of stating this, centro symmetry is present if $P(\gamma_1, \gamma_2)$ can be expressed as any one or a combination of the following forms:

(a) $e^{j\vartheta}P_R$ where P_R is a real-coefficient polynomial. (4)

(b) $$\left[P_A\left(\frac{\gamma_1^2}{1+\Delta\gamma_1},\frac{\gamma_2^2}{1+\Delta\gamma_2}\right)+\gamma_1\gamma_2 P_B\left(\frac{\gamma_1^2}{1+\Delta\gamma_1},\frac{\gamma_2^2}{1+\Delta\gamma_2}\right)\right]\cdot (1+\Delta\gamma_1)^{J1}\cdot(1+\Delta\gamma_2)^{J2} \quad (5)$$

(c) $$\left[\gamma_1 P_C\left(\frac{\gamma_1^2}{1+\Delta\gamma_1},\frac{\gamma_2^2}{1+\Delta\gamma_2}\right)+\gamma_2 P_D\left(\frac{\gamma_1^2}{1+\Delta\gamma_1},\frac{\gamma_2^2}{1+\Delta\gamma_2}\right)\right]\cdot (1+\Delta\gamma_1)^{J3}\cdot(1+\Delta\gamma_2)^{J4} \quad (6)$$

where J_1, J_2 and J_3, J_4 are the powers of γ_1, γ_2 for the polynomials in (b) and (c) respectively. Note that P_A, P_B, P_C and P_D are self-inverse w.r.t. to both γ_1 and γ_2.

Quadrantal Symmetry:

$P(\gamma_1, \gamma_2)$ possesses quadrantal symmetry $F(\omega_1,\omega_2)= F(-\omega_1,\omega_2)$ in its magnitude response if $P(\gamma_1, \gamma_2)$ can be expressed as

$$P(\gamma_1, \gamma_2)=P_1(\gamma_1, \gamma_2)\cdot P_2(\gamma_1, \gamma_2)$$

where $$P_1(\gamma_1,\gamma_2) = P_1\left(\frac{-\gamma_1}{1+\Delta\gamma_1},\gamma_2\right)\cdot(1+\Delta\gamma_1)^{J_{11}} \quad (7)$$

$$P_2(\gamma_1,\gamma_2) = P_2^*\left(\gamma_1,\frac{-\gamma_2}{1+\Delta\gamma_2}\right)\cdot(1+\Delta\gamma_2)^{J_{22}} \quad (8)$$

J_{11} is the degree of γ_1 in P_1 and J_{22} is the degree of γ_2 in P_2. Once again, P_2^* denotes the complex conjugate of P_2. Figure 1.1 depicts the quadrantal symmetry in ω_1, ω_2 plane for the polynomial $P(\gamma_1,\gamma_2)$ satisfying Conditions 7 and/or 8. If $P(\gamma_1,\gamma_2)$ also satisfies any one of the conditions in equations (4) - (6), then Figure 1.2 depicts the quadrantal symmetry.

Diagonal Symmetry

$P(\gamma_1, \gamma_2)$ possesses diagonal symmetry $F(\omega_1,\omega_2)=F(\omega_2,\omega_1)$ in its magnitude response if it can be expressed as

$$P(\gamma_1, \gamma_2)=P_1(\gamma_1, \gamma_2)\cdot P_2(\gamma_1, \gamma_2)$$

where $$P_1(\gamma_1,\gamma_2) = P_1(\gamma_2,\gamma_1) \quad (9)$$

and

$$P_2(\gamma_1,\gamma_2) = P_2^*\left(\frac{-\gamma_2}{1+\Delta\gamma_2},\frac{-\gamma_1}{1+\Delta\gamma_1}\right)\cdot(1+\Delta\gamma_2)^{J_{21}}\cdot(1+\Delta\gamma_1)^{J_{22}} \quad (10)$$

where J_{21} is the degree of γ_1 in $P_2(\gamma_1, \gamma_2)$ and J_{22} is the degree of γ_2 in $P_2(\gamma_1, \gamma_2)$. If in addition to conditions 9 and/or 10, the polynomial satisfies any one of the 3 conditions in eqns (4) - (6), then the complex $P(\gamma_1,\gamma_2)$ will have a diagonal symmetry similar to that of a real 2-D polynomial.

Rotational Symmetry

$P(\gamma_1,\gamma_2)$ possesses rotational symmetry if $F(\omega_1,\omega_2)= F(-\omega_2,\omega_1)$. The conditions for rotational symmetry can be obtained along the same lines as the above symmetries.

Looking at Conditions (2) - (10), we can see that P_1 in Condition (7) is self-inverse w.r.t. γ_1, and P_2 in Condition (8) is self-inverse w.r.t γ_2. P_1 in Condition (2) is a self-inverse w.r.t. to both γ_1 and γ_2. A self-inverse polynomial in both variables is called a para-conjugate polynomial if it involves conjugating the coefficients besides doing the self-inverse operations on both variables. We shall denote the para-conjugate polynomial by P_*, while P^* means simply conjugating the coefficients, no self-inverse operation is involved. Testing for symmetry conditions is equivalent to testing for the presence of self-inverse polynomials w.r.t. either one or the other variable, or both variables. The algorithm described below was originally used to find the para-conjugate (self-inverse with conjugated coefficients) delta-domain polynomials. We shall first introduce the 1-D algorithm and then extend it to 2-D cases.

4. ALGORITHM TO FIND THE PARA-CONJUGATE OF 1-D DELTA DISCRETE POLYNOMIAL

Given $P(\gamma)=\sum_{i=0}^{n} a_i\gamma^i$ where the coefficient a_i can be real or complex. The para-conjugate $P_*(\gamma)=P^*(\gamma)\big|_{\gamma=-\frac{\gamma}{1+\Delta\gamma}}\cdot(1+\Delta\gamma)^n$ (the coefficients of $P(\gamma)$ also need to be conjugated). The tabular algorithm to find $P_*(\gamma)$ is given in Figure 2.

- Step 1: Write the variable γ in ascending power starting with γ^0 in column 0 and ending with γ^n in column n (n is the order of the polynomial). Then write the corresponding coefficient below it, on the 2nd row – a_0 in column 0 and a_n in column n. Draw a dotted line across.
- Step 2: On the next row, write down the conjugate of the coefficient with alternating signs, i.e. $(-1)^i a_i^*$, where i is the column number, column 0 is the leftmost and column n the rightmost column.
- Step 3: Form the remaining rows by multiplying the value in column i of each row by $(n-i)\Delta/k$ where k is the row number below the dotted line. Write this result in column $i+1$ on the next row. This means that each subsequent row formed will have 1 column less; thus the rows formed will look like an inverted triangle. Continue doing this until a row with only 1 column is left.
- Step 4: Add up the values in each column, below the dotted line. Write the result down on the last row.

The complex numbers on the last row are the coefficients of the para-conjugate polynomial $P_*(\gamma)$ in ascending powers of γ.

5. ALGORITHM FOR 2-D POLYNOMIAL

$$\text{Let } P(\gamma_1,\gamma_2)=\sum_{i=0}^{M}\sum_{j=0}^{N} q_{ij}\gamma_1^i\gamma_2^j=\sum_{j=0}^{N} R_j(\gamma_1)\gamma_2^j \quad (11)$$

The para-conjugate polynomial is given by $P_*(\gamma_1,\gamma_2)=P^*(\gamma_1,\gamma_2)\big|_{\gamma_i=\frac{-\gamma_i}{1+\Delta_i\gamma_i}}\cdot(1+\Delta_1\gamma_1)^M\cdot(1+\Delta_2\gamma_2)^N$, where i=1,2 (the coefficients of $P(\gamma_1,\gamma_2)$ also need to be conjugated). To use the tabular algorithm to obtain the para-conjugate for this 2-D polynomial, we have the following steps:

- Step 1: Separate the coefficients into the block format shown at the top of Fig 3. If a certain coefficient say q_{kl}=0 (k=0,...M; l=0,...N), insert a zero. This is to ensure each γ_2 block has M coefficients (columns).
- Step 2: Look at each γ_2 block separately and find its para-conjugate with respect to γ_1 using the method described for a 1-D polynomial. i.e. find the para-conjugate with respect to γ_1 for the γ_2^0 block, the γ_2^1 block, all the way to the γ_2^N block. We are determining

the para-conjugate of $R_j(\gamma_1)$ {eqn (11)}. Refer to Fig 3. At this point we have determined the para-conjugate with respect to the γ_1 variable, Sum_{kl} being the coefficients of this para-conjugate.

Step 3: Treat the values under each γ_2 block as a single unit and find the para-conjugate with respect to γ_2, using the method for 1-D polynomial. (Note that, this time, we need not conjugate the coefficients before using the 1-D method. We need only alternate the signs of the coefficients. We only need to conjugate the coefficients *once* using this algorithm and we had already done that in Step 2 above.). Refer to Fig 3.

Step 4: Add up the columns to give the coefficients of the para-conjugate for the 2-D polynomial. Thus,

$$P_*(\gamma_1,\gamma_2) = \sum_{i=0}^{M}\sum_{j=0}^{N} q_{ij*}\gamma_1^{\,i}\gamma_2^{\,j}$$

The above tabular algorithm can be used to test for the various symmetry conditions. For example, Condition 2 is satisfied if the above algorithm yields the same final polynomial as the original. Note that for this case, no conjugating of coefficients is required at the beginning of the tabular algorithm. Condition 7 can be tested by performing the above algorithm , again without conjugating the coefficient, until step 2 (since we are only determining if the polynomial is self-inverse w.r.t. γ_1). Condition 8 can be tested by performing the above steps, except the part to perform the inverse operation on γ_1 (since we are only checking if the polynomial is self-inverse w.r.t. γ_2). To test for Condition 10, the variables need to be switched before using the algorithm. All steps of the algorithm are required. The coefficients need to be conjugated at the beginning also.

6. CONCLUSION

Various symmetries in the magnitude response of 2-D complex coefficient delta polynomials are defined in this paper. The algorithmic test procedure discussed above helps to identify the present of these symmetries.

REFERENCES

[1] H. C. Reddy, P.K. Rajan, G.S. Moschytz, and A.R. Stubberud, "Study of Various Symmetries in the Frequency Response of Two-Dimensional Delta Operator Formulated Discrete-Time Systems", *Proceedings of the IEEE-ISCAS*, vol 2, pp 344-347, May 1996.

[2] M. N. S. Swamy and P. K. Rajan, "Symmetry in 2-D filters and its application," in *Multidimensional Systems: Techniques and Applications* (S.G. Tzafestas, Ed.). New York: Marcel Dekkar, 1986, ch.9.

[3] R. H. Middleton and G. C. Goodwin, "Improved Finite Wordlength Characteristics in Digital Control using Delta Operators", *IEEE Trans. Automatic Control*, vol 31, pp1015-1021, Nov 1986.

[4] H. C. Reddy, G. S. Moschytz and A. R. Stubberud, "All Pass Function Based Stability Test for Delta-operator Formulated Discrete-time Systems", *Proceedings of the IEEE-ISCAS*, vol 2, pp 1492-1495, May 1995.

[5] K. Premaratne and A. S. Boujarwah, "An Algorithm for Stability Determination of Two-Dimensional Delta-Operator Formulated Discrete-Time Systems", *Multidimensional Systems and Signal Processing*, 1994.

[6] I. H. Khoo, "Efficient Polynomial Transformation Algorithms for Delta-Operator Formulated Discrete-Time Systems", Masters' Thesis, 1996.

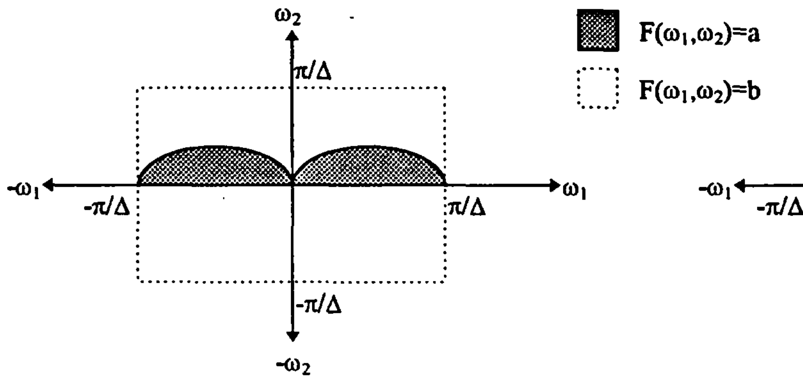

Figure 1.1 – 2-D magnitude response $F(\omega_1,\omega_2)$ for filter satisfying quadrantal symmetry

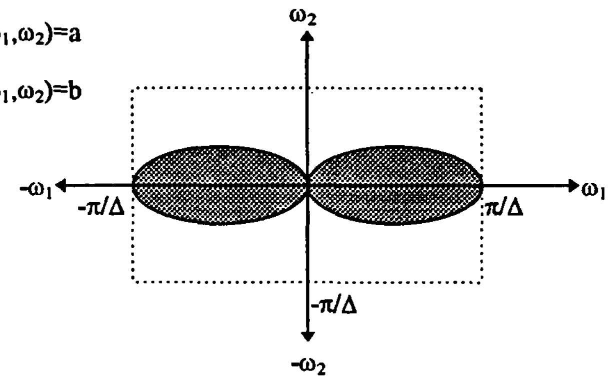

Figure 1.2 – 2-D magnitude response $F(\omega_1,\omega_2)$ for filter satisfying both quadrantal symmetry and centro symmetry

Note: as $\Delta \to 0$, the frequency response domain of 2-D (DODT) system coincides with that of 2-D CT system

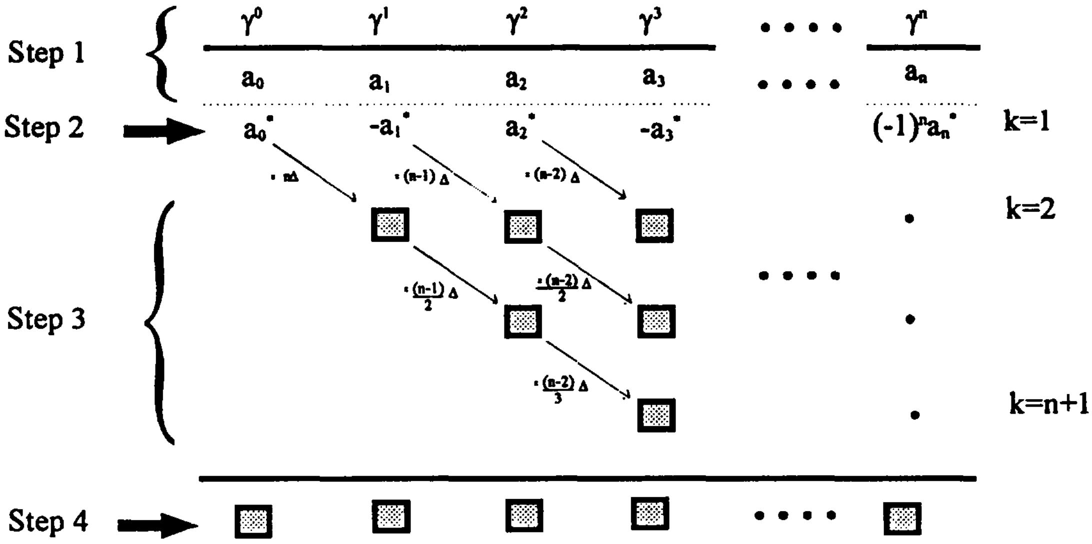

Figure 2. – 1-D algorithm to find the paraconjugate of a delta polynomial

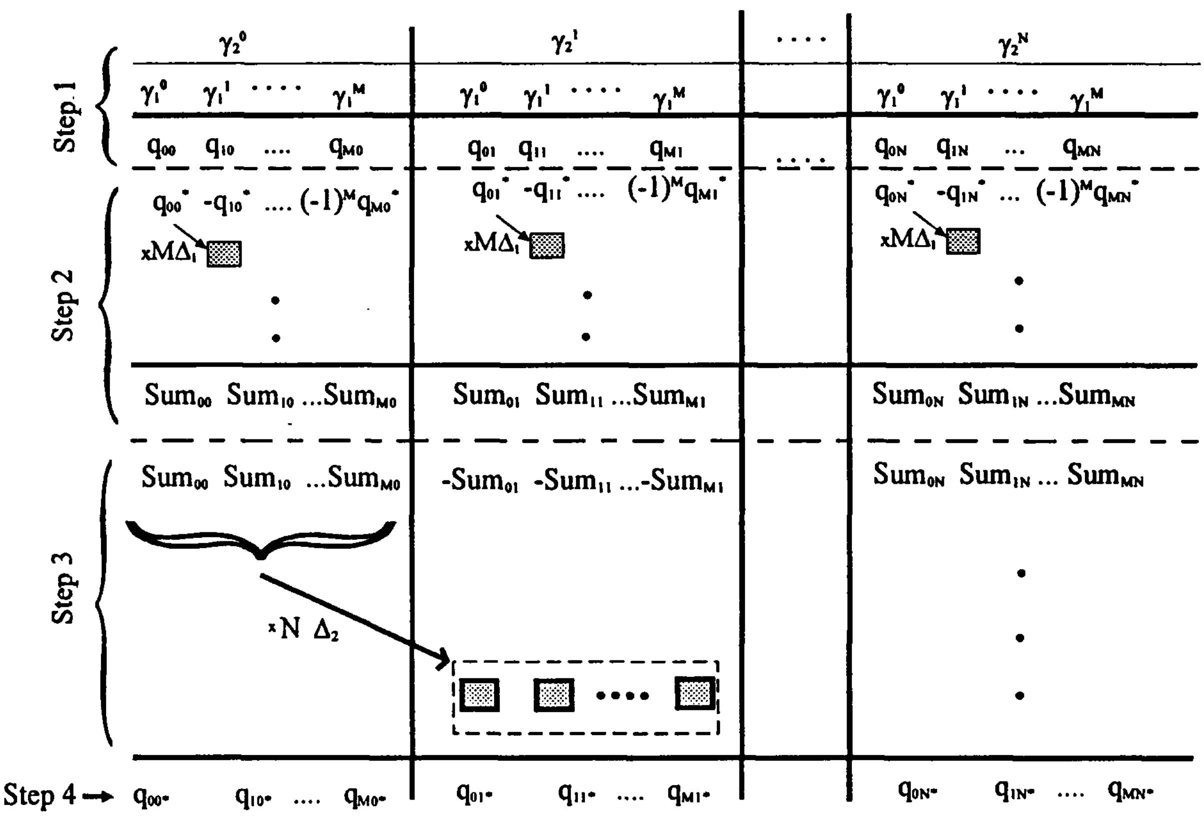

Figure 3. – 2-D algorithm to find the paraconjugate of a delta polynomial