Sending a Bivariate Gaussian Source Over a Gaussian MAC With Feedback

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***Abstract*—We study the power-versus-distortion tradeoff for the transmission of a memoryless bivariate Gaussian source over a two-to-one Gaussian multiple-access channel with perfect causal feedback. In this problem, each of two separate transmitters observes a different component of a memoryless bivariate Gaussian source as well as the feedback from the channel output of the previous time-instants. Based on the observed source sequence and the feedback, each transmitter then describes its source component to the common receiver via an average-power constrained Gaussian multiple-access channel. From the resulting channel output, the receiver wishes to reconstruct each source component with the least possible expected squared-error distortion. We study the set of distortion pairs that can be achieved by the receiver on the two source components. We present sufficient conditions and necessary conditions for the achievability of a distortion pair. These conditions are expressed in terms of the source correlation and of the signal-to-noise ratio (SNR) of the channel. In several cases the necessary conditions and sufficient conditions are shown to agree. In particular, we show that if the channel SNR is below a certain threshold, then an uncoded transmission scheme that ignores the feedback is optimal. Thus, below this SNR-threshold, feedback is useless. We also derive the optimal high-SNR asymptotics.**

***Index Terms*—Achievable distortion, combined source-channel coding, correlated sources, feedback, Gaussian multiple-access channel with feedback, Gaussian source, mean squared-error distortion, multiple-access channel with feedback, uncoded transmission.**

I. INTRODUCTION

THIS paper is a sequel to [1], where a bivariate Gaussian source is to be transmitted over a Gaussian multiple-access channel. The new element here is the presence of perfect causal feedback from the channel output to each of the transmitters. As in [1], our interest is in the power-versus-distortion tradeoff.

Our setup consists of a memoryless bivariate Gaussian source and a two-to-one Gaussian multiple-access channel (MAC) with perfect causal feedback. Each of the two transmitters in the multiple-access channel observes a different component of the source as well as feedback from the previous channel outputs. Based on the feedback and the observed source sequence, each transmitter then describes its source component to the common receiver via an average-power constrained Gaussian multiple-access channel. Based on the channel output sequence, the receiver wishes to reconstruct each source component with the least possible expected squared-error distortion. Our interest is in characterizing the pairs of squared-error distortions that can be achieved simultaneously on the two source components.

We present sufficient conditions and necessary conditions for the achievability of a distortion pair. These conditions are expressed in terms of the source correlation and the signal-to-noise ratio (SNR) of the channel. In several cases the necessary conditions and sufficient conditions are shown to agree. In particular, we show that if the channel SNR is below a certain threshold, then an uncoded transmission scheme is optimal, and feedback is useless. We also show that, in general, the source-channel separation approach is suboptimal, but that it is asymptotically optimal as the transmit power tends to infinity.

II. PROBLEM STATEMENT

A. Setup

Our setup is illustrated in Fig. 1. A memoryless bivariate Gaussian source is connected to a two-to-one Gaussian multiple-access channel with perfect causal feedback. Each transmitter of the multiple-access channel observes one of the source components and wishes to describe it to the common receiver. The source symbols produced at time $k \in \mathbb{Z}$ are denoted by $(S_{1,k}, S_{2,k})$. The source output pairs $\{(S_{1,k}, S_{2,k})\}$ are independent identically distributed (i.i.d.) zero-mean Gaussians of covariance matrix

$$\mathsf{K}_{SS} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \quad (1)$$

where $\rho \in [-1,1]$ and $0 < \sigma_i^2 < \infty$, $i \in \{1,2\}$. The sequence $\{S_{1,k}\}$ of the first source component is observed by Transmitter 1 and the sequence $\{S_{2,k}\}$ of the second source component is observed by Transmitter 2. The two source components are to be described over the multiple-access channel to the common receiver by means of the channel input sequences $\{X_{1,k}\}$ and $\{X_{2,k}\}$, where $x_{1,k} \in \mathbb{R}$ and $x_{2,k} \in \mathbb{R}$. The corresponding time-k channel output is given by

$$Y_k = X_{1,k} + X_{2,k} + Z_k \quad (2)$$

where Z_k is the time-k additive noise term, and where $\{Z_k\}$ are i.i.d. zero-mean variance-N Gaussian random variables that are independent of the source sequence.

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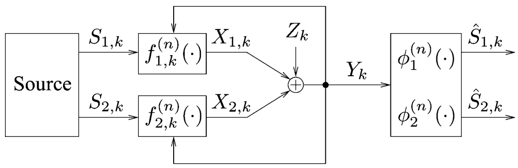

Fig. 1. Bivariate Gaussian source with one-to-two Gaussian multiple-access channel with feedback.

We consider block encoding schemes and denote the blocklength by n and the associated n-sequences in boldface, e.g., $\mathbf{S}_1 = (S_{1,1}, S_{1,2}, \ldots, S_{1,n})$. Transmitter $i \in \{1,2\}$ is described by a sequence of functions $f_{i,k}^{(n)}: \mathbb{R}^n \times \mathbb{R}^{k-1} \rightarrow \mathbb{R}$, $k = 1, \ldots, n$, which, for every time instant $k \in \mathbb{Z}$ produce the channel input $X_{i,k}$ from the source sequence $\mathbf{S}_i$ and the past channel outputs $Y^{k-1} = (Y_1, \ldots, Y_{k-1})$, i.e.,

$$X_{i,k} = f_{i,k}^{(n)}\left(\mathbf{S}_i, Y^{k-1}\right), \quad i \in \{1,2\}. \tag{3}$$

The channel input sequences are subjected to expected average power constraints

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}[X_{i,k}^2] \leq P_i, \quad i \in \{1,2\} \tag{4}$$

for some given $P_i > 0$.

The receiver is described by two functions $\phi_i^{(n)}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $i \in \{1,2\}$, each of which forms an estimate $\hat{\mathbf{S}}_i$ of the respective source sequence $\mathbf{S}_i$ based on the observed channel output sequence $\mathbf{Y}(= Y^n)$. Thus,

$$\hat{\mathbf{S}}_i = \phi_i^{(n)}\left(\mathbf{Y}\right), \quad i \in \{1,2\}. \tag{5}$$

We are interested in the pairs of expected squared-error distortions that can be achieved simultaneously on the source-pair as the blocklength n tends to infinity. In view of this, we next define the notion of achievability.

B. Achievability of Distortion Pairs

Definition II.1: Given $\sigma_1, \sigma_2 > 0$; $\rho \in [-1,1]$; $P_1, P_2 > 0$; and $N > 0$ we say that the tuple $(D_1,D_2,\sigma_1^2,\sigma_2^2,\rho,P_1,P_2,N)$ is *achievable* if there exists a sequence of encoding functions $\left(\{f_{1,k}^{(n)}\}_{k=1}^n, \{f_{2,k}^{(n)}\}_{k=1}^n\right)$ as in (3), satisfying the average power constraints (4), and a sequence of reconstruction pairs $\left(\phi_1^{(n)}, \phi_2^{(n)}\right)$ as in (5), such that the average distortions resulting from these encoding and reconstruction functions satisfy

$$\varlimsup_{n \rightarrow \infty} \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[\left(S_{i,k} - \hat{S}_{i,k}\right)^2\right] \leq D_i, \quad i \in \{1,2\}$$

whenever for all $k \in \{1, 2, \ldots, n\}$

$$Y_k = f_{1,k}^{(n)}(\mathbf{S}_1, Y^{k-1}) + f_{2,k}^{(n)}(\mathbf{S}_2, Y^{k-1}) + Z_k$$

where $\{(S_{1,k},S_{2,k})\}$ are i.i.d. zero-mean bivariate Gaussian vectors of covariance matrix K_{SS} as in (1) and $\{Z_k\}$ are i.i.d. zero-mean variance-N Gaussians that are independent of $\{(S_{1,k},S_{2,k})\}$.

For given σ_1^2, σ_2^2, ρ, P_1, P_2, and N, we wish to find the set of pairs (D_1,D_2) such that $(D_1,D_2,\sigma_1^2,\sigma_2^2,\rho,P_1,P_2,N)$ is achievable. Sometimes, we shall refer to this set as the distortion region associated with $(\sigma_1^2,\sigma_2^2,\rho,P_1,P_2,N)$. In that sense, we shall often say, with respect to some $(\sigma_1,\sigma_2,\rho,P_1,P_2,N)$, that the pair (D_1,D_2) is achievable, instead of saying that the tuple $(D_1,D_2,\sigma_1^2,\sigma_2^2,\rho,P_1,P_2,N)$ is achievable.

C. Normalization

For the described problem we now note that, without loss in generality, the source law given in (1) can be restricted to a simpler form. This restriction simplifies the statement and the derivation of our results.

Reduction II.1: For the problem stated in Sections II-A and II-B, there is no loss in generality in restricting the source law to satisfy

$$\sigma_1^2 = \sigma_2^2 = \sigma^2 \quad \text{and} \quad \rho \in [0,1]. \tag{6}$$

Proof: The proof is almost identical to that of Reduction II.1 in [1] and is thus omitted. □

In view of Reduction II.1, we henceforth assume that the source law additionally satisfies (6).

D. "Symmetric Version" and a Convexity Property

The "symmetric version" of our problem corresponds to the case where the transmitters are subjected to the same power constraint, and where we seek to achieve the same distortion on each source component. That is, $P_1 = P_2 = P$, and we are interested in the minimal distortion

$$D^*(\sigma^2, \rho, P, N) \triangleq \inf\left\{D: (D, D, \sigma^2, \sigma^2, \rho, P, P, N) \text{ is achievable}\right\} \tag{7}$$

that is simultaneously achievable on $\{S_{1,k}\}$ and on $\{S_{2,k}\}$.

We conclude this section with a convexity property of the achievable distortions.

Remark II.1: If both $(D_1,D_2,\sigma_1^2,\sigma_2^2,\rho,P_1,P_2,N)$ and $(\tilde{D}_1,\tilde{D}_2,\sigma_1^2,\sigma_2^2,\rho,\tilde{P}_1,\tilde{P}_2,N)$ are achievable, then

$$\Big(\lambda D_1 + \bar{\lambda}\tilde{D}_1, \lambda D_2 + \bar{\lambda}\tilde{D}_2, \sigma_1^2, \sigma_2^2, \rho, \lambda P_1 + \bar{\lambda}\tilde{P}_1, \lambda P_2 + \bar{\lambda}\tilde{P}_2, N\Big)$$

is also achievable for every $\lambda \in [0,1]$, where $\bar{\lambda} = (1-\lambda)$.

Proof: Follows by a time-sharing argument. □

III. Main Results

A. Necessary Condition for Achievability of (D_1,D_2)

To state our necessary condition we first introduce three rate-distortion functions. They are: the rate-distortion function $R_{S_1,S_2}(D_1,D_2)$ on $\{(S_{1,k},S_{2,k})\}$; the rate-distortion function $R_{S_1|S_2}(D_1)$ on $\{S_{1,k}\}$ when the component $\{S_{2,k}\}$ is given as side-information to both Encoder 1 and Decoder 1; and the rate-distortion function $R_{S_2|S_1}(D_2)$ on $\{S_{2,k}\}$ when the component $\{S_{1,k}\}$ is given as side-information to both Encoder 2 and Decoder 2. For $\{(S_{1,k}, S_{2,k})\}$ jointly Gaussian as in (1) with $\sigma_1^2 = \sigma_2^2 = \sigma^2$, the two latter functions are given by

$$R_{S_1|S_2}(D_1) = \frac{1}{2}\log_2^+\left(\frac{\sigma^2(1-\rho^2)}{D_1}\right) \tag{8}$$

$$R_{S_2|S_1}(D_2) = \frac{1}{2}\log_2^+\left(\frac{\sigma^2(1-\rho^2)}{D_2}\right) \tag{9}$$

where we define $\log_2^+(x) = \max\{0, \log_2(x)\}$. The function $R_{S_1,S_2}(D_1, D_2)$ is given in the following theorem.

Theorem III.1 (Xiao, Luo [5]; Lapidoth, Tinguely [2]; Tinguely [4]): The rate-distortion function $R_{S_1,S_2}(D_1, D_2)$ is given by

$$R_{S_1,S_2}(D_1,D_2) = \begin{cases} \frac{1}{2}\log_2^+\left(\frac{\sigma^2}{D_{\min}}\right), & \text{if } (D_1,D_2) \in \mathfrak{D}_1 \\ \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1 D_2}\right), & \text{if } (D_1,D_2) \in \mathfrak{D}_2 \\ \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1 D_2 - (\rho\sigma^2 - \varrho(D_1,D_2))^2}\right), & \text{if } (D_1,D_2) \in \mathfrak{D}_3. \end{cases} \tag{10}$$

where

$$\varrho(D_1,D_2) = \sqrt{(\sigma^2 - D_1)(\sigma^2 - D_2)} \tag{11}$$

where $D_{\min} = \min\{D_1, D_2\}$, and where the regions $\mathfrak{D}_1$, $\mathfrak{D}_2$, and $\mathfrak{D}_3$ are given by the equation shown at the bottom of the page.

Our necessary condition is now as follows.

Theorem III.2: A necessary condition for the achievability of $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2, N)$ is the existance of some $\hat{\rho} \in [0,1]$ such that

$$R_{S_1,S_2}(D_1,D_2) \le \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\hat{\rho}\sqrt{P_1 P_2}}{N}\right) \tag{12}$$

$$R_{S_1|S_2}(D_1) \le \frac{1}{2}\log_2\left(1 + \frac{P_1(1-\hat{\rho}^2)}{N}\right) \tag{13}$$

$$R_{S_2|S_1}(D_2) \le \frac{1}{2}\log_2\left(1 + \frac{P_2(1-\hat{\rho}^2)}{N}\right). \tag{14}$$

Proof: See Appendix A. □

We now specialize Theorem III.2 to the symmetric case. To this end, we first substitute the rate-distortion functions $R_{S_1,S_2}(D_1,D_2)$, $R_{S_1|S_2}(D_1)$, $R_{S_2|S_1}(D_2)$ on the LHS of (12)–(14) by their explicit forms given in (10), (8), and (9), respectively. Substituting (D, D) for (D_1, D_2) in (10) and (12) yields that if (D, D) is achievable, then

$$D \ge \begin{cases} \frac{1}{2}\sigma^2\left(\frac{N(1+\rho)}{N+2P(1+\hat{\rho})} + (1-\rho)\right), & \text{if } \frac{P}{N} \le \frac{\rho}{1-\rho^2} \\ \sigma^2\sqrt{\frac{N(1-\rho^2)}{N+2P(1+\hat{\rho})}}, & \text{if } \frac{P}{N} > \frac{\rho}{1-\rho^2}. \end{cases} \tag{15}$$

Similarly, from (8) and (13) [or (9) and (14)] we obtain that if (D, D) is achievable, then

$$D \ge \sigma^2 \frac{N(1-\rho^2)}{N + P(1-\hat{\rho}^2)}. \tag{16}$$

Denoting the RHS of (15) by $\xi(\sigma^2, \rho, P, N, \hat{\rho})$ and the RHS of (16) by $\psi(\sigma^2, \rho, P, N, \hat{\rho})$, yields the following lower bound on $D^*(\sigma^2, \rho, P, N)$.

Corollary III.1: In the symmetric case

$$D^*(\sigma^2,\rho,P,N) \ge \min_{0\le\hat{\rho}\le 1} \max\left\{\xi(\sigma^2,\rho,P,N,\hat{\rho}), \psi(\sigma^2,\rho,P,N,\hat{\rho})\right\}.$$

The minimization over $\hat{\rho}$ is discussed in the following remark.

Remark III.1: For $P/N \le \rho^2/(2(1-\rho)(1+2\rho))$ the minimum in Corollary III.1 is achieved by $\hat{\rho}^* = 1$, and for all larger P/N the minimum is achieved by the $\hat{\rho}^*$ satisfying

$$\xi(\sigma^2,\rho,P,N,\hat{\rho}^*) = \psi(\sigma^2,\rho,P,N,\hat{\rho}^*).$$

$$\mathfrak{D}_1 = \Big\{(D_1,D_2) : \left(0 \le D_1 \le \sigma^2(1-\rho^2),\ D_2 \ge \sigma^2(1-\rho^2) + \rho^2 D_1\right)$$
$$\text{or } \left(\sigma^2(1-\rho^2) < D_1 \le \sigma^2, D_2 \ge \sigma^2(1-\rho^2) + \rho^2 D_1, D_2 \le \frac{D_1 - \sigma^2(1-\rho^2)}{\rho^2}\right)\Big\}$$

$$\mathfrak{D}_2 = \left\{(D_1,D_2) : 0 \le D_1 \le \sigma^2(1-\rho^2), 0 \le D_2 < (\sigma^2(1-\rho^2) - D_1)\frac{\sigma^2}{\sigma^2 - D_1}\right\}$$

$$\mathfrak{D}_3 = \Big\{(D_1,D_2) : \left(0 \le D_1 \le \sigma^2(1-\rho^2), (\sigma^2(1-\rho^2) - D_1)\frac{\sigma^2}{\sigma^2 - D_1} \le D_2 < \sigma^2(1-\rho^2) + \rho^2 D_1\right)$$
$$\text{or } \left(\sigma^2(1-\rho^2) < D_1 \le \sigma^2, \frac{D_1 - \sigma^2(1-\rho^2)}{\rho^2} < D_2 < \sigma^2(1-\rho^2) + \rho^2 D_1\right)\Big\}.$$

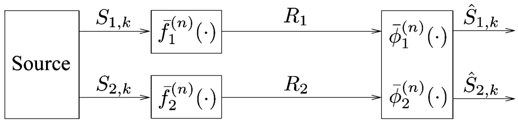

Fig. 2. Distributed source coding problem for a bivariate Gaussian source.

As $P/N \longrightarrow \infty$ it can be shown that $\hat{\rho}^*$ tends to one and hence Corollary III.1 yields

$$\varliminf_{P/N \longrightarrow \infty} \sqrt{\frac{P}{N}} D^*(\sigma^2, \rho, P, N) \geq \sigma^2 \sqrt{\frac{1-\rho^2}{4}}. \tag{17}$$

In the next section, we show that the $\lim\ \inf$ in (17) is a limit, and that it is achieved by source-channel separation.

B. Source-Channel Separation

We now consider the set of distortion pairs that are achieved by combining the optimal scheme for the corresponding source-coding problem with the optimal scheme for the corresponding channel-coding problem. The source-coding problem is illustrated in Fig. 2. The two source components are observed by two separate encoders. These two encoders wish to describe their source sequence to the common receiver by means of individual rate-limited and error-free bit pipes. The receiver estimates each of the sequences subject to expected squared-error distortion. A detailed description of this problem can be found in [6] and [7]. The associated rate-distortion region is given in the next theorem.

Theorem III.3 (Oohama [6]; Wagner, Tavildar, and Viswanath [7]): For the Gaussian two-terminal source coding problem (with source components of unit variances) a distortion-pair (D_1, D_2) is achievable if, and only if,

$$(R_1, R_2) \in \mathcal{R}_1(D_1) \cap \mathcal{R}_2(D_2) \cap \mathcal{R}_{\text{sum}}(D_1, D_2),$$

where the expressions for the regions $\mathcal{R}_1(D_1)$, $\mathcal{R}_2(D_2)$, and $\mathcal{R}_{\text{sum}}(D_1, D_2)$ are shown in the first equation at the bottom of the page, with

$$\beta(D_1, D_2) = 1 + \sqrt{1 + \frac{4\rho^2 D_1 D_2}{(1-\rho^2)^2}}.$$

The capacity region $\mathcal{C}_{\text{FB}}(P_1, P_2, N)$ of the Gaussian multiple-access channel with feedback was derived in [8] and is restated in the following theorem.

Theorem III.4 (Ozarow [8]): The capacity region $\mathcal{C}_{\text{FB}}(P_1, P_2, N)$ of the Gaussian multiple-access channel with perfect feedback is shown in the second equation at the bottom of the page.

The distortions achievable by source-channel separation are now given in the following corollary.

Corollary III.2: A distortion pair (D_1, D_2) is achievable by source-channel separation whenever

$$\mathcal{R}(D_1, D_2) \cap \mathcal{C}_{\text{FB}}(P_1, P_2, N) \neq \emptyset. \tag{18}$$

Proof: The result essentially follows by considering a transmission scheme that first describes the source sequences with bits using the source-coding scheme of Theorem III.3 and then transmits the bits over the Gaussian MAC with feedback using the channel-coding scheme of Theorem III.4. Key is the observation that the coding scheme of Theorem III.4 achieves not only an arbitrarily small *average* probability of error, but also an arbitrarily small *maximal* probability of error. This observation is critical because the messages sent over the MAC are not, in general, equally likely: their law is determined by source law and by the source-encoder.

Consider some (D_1, D_2) and P_1, P_2, N satisfying (18). Let (R_1', R_2') be a rate pair in $\mathcal{R}(D_1, D_2) \cap \mathcal{C}_{\text{FB}}(P_1, P_2, N)$ such that the distortion pair resulting on $(\mathbf{S}_1, \mathbf{S}_2)$ from the

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$$\begin{aligned}
\mathcal{R}_1(D_1) &= \left\{ (R_1, R_2) : R_1 \geq \frac{1}{2} \log_2^+ \left[\frac{1}{D_1} (1 - \rho^2 (1 - 2^{-2R_2})) \right] \right\} \\
\mathcal{R}_2(D_2) &= \left\{ (R_1, R_2) : R_2 \geq \frac{1}{2} \log_2^+ \left[\frac{1}{D_2} (1 - \rho^2 (1 - 2^{-2R_1})) \right] \right\} \\
\mathcal{R}_{\text{sum}}(D_1, D_2) &= \left\{ (R_1, R_2) : R_1 + R_2 \geq \frac{1}{2} \log_2^+ \left[\frac{(1-\rho^2)\beta(D_1, D_2)}{2 D_1 D_2} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_{\text{FB}}(P_1, P_2, N) = \bigcup_{0 \leq \bar{\rho} \leq 1} \Bigg\{ (R_1, R_2) : R_1 &\leq \frac{1}{2} \log_2 \left(1 + \frac{P_1 (1 - \bar{\rho}^2)}{N} \right) \\
R_2 &\leq \frac{1}{2} \log_2 \left(1 + \frac{P_2 (1 - \bar{\rho}^2)}{N} \right) \\
R_1 + R_2 &\leq \frac{1}{2} \log_2 \left(1 + \frac{P_1 + P_2 + 2\bar{\rho}\sqrt{P_1 P_2}}{N} \right) \Bigg\}.
\end{aligned}$$

source-coding scheme of [6] at rate (R_1', R_2') is (D_1,D_2). We now show that the separate source-channel coding scheme that combines the rate-(R_1', R_2') source-code of [6] with the rate-(R_1', R_2') channel-code of [8] results in a distortion pair (D_1', D_2') that approaches (D_1,D_2) as the blocklength n tends to infinity. To this end, denote by $(\mathbf{U}_1^*, \mathbf{U}_2^*)$ the quantized version of $(\mathbf{S}_1, \mathbf{S}_2)$ produced by the source encoder, and denote by $(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)$ the guess of the pair $(\mathbf{U}_1^*, \mathbf{U}_2^*)$ that is made by the source decoder based on the indices received from the channel decoder. By [6, Equation (55), p. 1920], the reconstruction pair $(\hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2)$ is

$$\hat{\mathbf{S}}_i = \xi_{i1}\hat{\mathbf{U}}_1 + \xi_{i2}\hat{\mathbf{U}}_2, \quad i \in \{1,2\}$$

where for all $i,j \in \{1,2\}$ we can assume without loss of optimality that $\xi_{ij} \in [0,1]$ whenever $i = j$ and that $\xi_{ij} \in [0,\rho]$ whenever $i \neq j$.

To show that the distortion pair (D_1', D_2') resulting from this separation-based scheme approaches (D_1,D_2), we now use a genie-aided argument. Let $(\hat{\mathbf{S}}_1^{\mathrm{G}}, \hat{\mathbf{S}}_2^{\mathrm{G}})$ be the reconstruction pair resulting from a genie-aided scheme where the decoder is provided with the correct source reconstructions $(\mathbf{U}_1^*, \mathbf{U}_2^*)$ so

$$\hat{\mathbf{S}}_i^{\mathrm{G}} = \xi_{i1}\mathbf{U}_1^* + \xi_{i2}\mathbf{U}_2^*, \quad i \in \{1,2\}$$

for ξ_{ij} as above. Because this genie-aided scheme is not affected by the transmission errors that might occur in the channel coding part, it achieves the distortion pair (D_1,D_2). Hence, to prove Corollary III.2 it now remains to verify that whenever $(R_1', R_2') \in \mathcal{C}_{\mathrm{FB}}(P_1,P_2,N)$, the differences $\mathsf{E}[\|\mathbf{S}_i - \hat{\mathbf{S}}\|^2] - \mathsf{E}[\|\mathbf{S}_i - \hat{\mathbf{S}}^{\mathrm{G}}\|^2]$, $i \in \{1,2\}$ of the distortions resulting from the two schemes vanishes as the blocklength n tends to infinity. This can be established in much the same way that Proposition D.1 is proved using Lemmas D.10—D.13 in Appendix D of [1]. The details are omitted here. □

From the sufficient condition of Corollary III.2 and the necessary condition of Theorem III.2, we can now derive the optimal high-SNR asymptotics. To state these asymptotics, we denote by (D_1^*,D_2^*) an arbitrary distortion pair resulting from an optimal scheme.

Theorem III.5 (High-SNR Distortion): The high-SNR asymptotic behavior of (D_1^*,D_2^*) is given by

$$\lim_{N \to 0} \frac{P_1 + P_2 + 2\sqrt{P_1P_2}}{N} D_1^* D_2^* = \sigma^4(1-\rho^2)$$

provided that $D_1^*, D_2^* \leq \sigma^2$, and that

$$\lim_{N \to 0} \frac{N}{P_1 D_1^*} = 0 \quad \text{and} \quad \lim_{N \to 0} \frac{N}{P_2 D_2^*} = 0. \tag{19}$$

Proof: See Appendix B. □

Remark III.2: The asymptotics of Theorem III.5 are almost the same as those in [1, Theorem IV.5] for the setup without feedback. The only difference is that in the case with feedback the power term $P_1 + P_2 + 2\rho\sqrt{P_1P_2}$ is replaced by $P_1 + P_2 + 2\sqrt{P_1P_2}$. This stems from the fact that with feedback, as $P/N \to \infty$, the cooperation between the transmitters can be almost full.

Remark III.3: Note that under source-channel separation, which achieves the high-SNR asymptotics, the cooperation between the transmitters takes place only at the channel-coding level. The source-coding is performed in a distributed manner.

To conclude this section we restate Theorem III.5 more specifically for the symmetric case. Since there $D_1^* = D_2^* = D^*(\sigma^2,\rho,P,N)$, condition (19) is implicitly satisfied. Thus, we have the following corollary.

Corollary III.3: In the symmetric case

$$\lim_{\frac{P}{N} \to \infty} \sqrt{\frac{P}{N}} D^*(\sigma^2,\rho,P,N) = \sigma^2\sqrt{\frac{1-\rho^2}{4}}.$$

C. Uncoded Scheme

We now revisit the uncoded scheme of [1, Section IV-C], which was shown to be optimal for the setup without feedback whenever the SNR is below a certain threshold. For our setup with feedback, we show that this scheme is still optimal whenever the SNR is below the threshold of [1, Theorem IV.3]. Below this SNR-threshold, feedback is thus useless.[1] Note, however, that feedback is beneficial for the source-channel separation approach because, even if noisy, it increases the capacity region of the Gaussian multiple-access channel [9].

The uncoded scheme operates as follows. Encoder $i \in \{1,2\}$ produces a time-k channel input $X_{i,k}$ which is a scaled version of the time-k source output $S_{i,k}$. The scaling is such that the average power constraint of the channel (4) is satisfied. That is

$$X_{i,k}^{\mathrm{u}} = \sqrt{\frac{P_i}{\sigma^2}} S_{i,k} \quad \text{for all} \;\; k \in \{1,2,\ldots,n\}.$$

The decoder reconstructs the source output $S_{i,k}$ by performing the MMSE estimate of $S_{i,k}, i \in \{1,2\}, k \in \{1,2,\ldots,n\}$, based on the time-$k$ channel output Y_k. That is,

$$\hat{S}_{i,k}^{\mathrm{u}} = \mathsf{E}[S_{i,k}|Y_k].$$

The expected distortions $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ resulting from this uncoded scheme as well as its optimality below a certain SNR-threshold are stated in the following theorem.

Theorem III.6: The distortion pairs $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ resulting from the described uncoded scheme are given by

$$D_1^{\mathrm{u}} = \sigma^2 \frac{(1-\rho^2)P_2 + N}{P_1 + P_2 + 2\rho\sqrt{P_1P_2} + N}$$
$$D_2^{\mathrm{u}} = \sigma^2 \frac{(1-\rho^2)P_1 + N}{P_1 + P_2 + 2\rho\sqrt{P_1P_2} + N}.$$

These distortion pairs $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ are optimal, i.e., lie on the boundary of the distortion region, whenever

$$P_2(1-\rho^2)^2\Big(P_1 + 2\rho\sqrt{P_1P_2}\Big) \leq N\rho^2\Big(2P_2(1-\rho^2) + N\Big). \tag{20}$$

Proof: The expressions for D_1^{u} and D_2^{u} are derived in [1, Appendix C]. The optimality of the uncoded scheme is

[1]By the simple structure of the uncoded scheme, it follows that feedback is useless not only in terms of performance, but also in terms of delay and complexity.

proven in Appendix C. For the particular case where P_1, P_2, N satisfy (20) with equality, the optimality can also be verified directly from Theorem III.2. To this end, it suffices to notice that for $(D_1, D_2) = (D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$, the necessary condition of Theorem III.2 is satisfied with equality for $\hat{\rho}^* = \rho$. It thus follows that for any (D_1', D_2') satisfying $D_1' \leq D_1^{\mathrm{u}}$ and $D_2' < D_2^{\mathrm{u}}$ or $D_1' < D_1^{\mathrm{u}}$ and $D_2' \leq D_2^{\mathrm{u}}$ the necessary condition of Theorem III.2 is violated for every $\hat{\rho} \in [-1, 1]$. Hence, the uncoded scheme is optimal. □

Corollary III.4: Source-channel separation is in general suboptimal.

Proof: This can be verified by comparing the achievable distortions given in Corollary III.2 with the achievable distortions given in Theorem III.6. For example, in the symmetric case it can be verified that for all $\rho > 0$ and $P/N \leq \rho/(1-\rho^2)$, the smallest distortions achievable by source-channel separation (Corollary III.2) are strictly larger than the distortions resulting from the optimal uncoded scheme (Theorem III.6). □

Remark III.4: From Theorem III.6 it follows that if P_1, P_2, N satisfy (20) with a strict inequality, then the necessary condition of Theorem III.2 is not sufficient. This is due to the constraints (13) and (14) which are loose at low SNR and is best seen in the symmetric case. In the symmetric case, Theorem III.2 (cf. Equations (13) and (14)) yields that for (D, D) to be achievable, it is necessary that D satisfy

$$D \geq \sigma^2(1-\rho^2)\frac{N}{N + P(1-\hat{\rho}^2)} \tag{21}$$

i.e., that (16) hold. Since $\hat{\rho} \in [0, 1]$, the RHS of (21) is upper bounded by $\sigma^2(1-\rho^2)$. Thus, for sufficiently low SNRs the constraint of (21) is inactive, and the only active constraint is the one of (15). But, if only (15) is active, then $\hat{\rho}^* = 1$, which corresponds to fully cooperating transmitters, and thus, yields a loose lower bound on $D^*(\sigma^2, \rho, P, N)$ at low SNRs.

We conclude the section on our main results by restating Theorem III.6 more specifically for the symmetric case.

Corollary III.5: In the symmetric case

$$D^*(\sigma^2, \rho, P, N) = \sigma^2\frac{P(1-\rho^2)+N}{2P(1+\rho)+N}, \quad \frac{P}{N} \leq \frac{\rho}{1-\rho^2}. \tag{22}$$

IV. Summary

We studied the power-versus-distortion tradeoff for the transmission of a memoryless bivariate Gaussian source over a two-to-one average-power limited Gaussian multiple-access channel with perfect causal feedback. In this problem, each of two separate transmitters observes a different component of a memoryless bivariate Gaussian source as well as the feedback from the channel output of the previous time-instants. Based on the observed source sequence and the feedback, each transmitter then describes its source component to the common receiver via an average-power constrained Gaussian multiple-access channel. From the resulting channel output, the receiver wishes to reconstruct both source components with the least possible expected squared-error distortion. Our interest was in the set of distortion pairs that can be achieved by the receiver on the two source components. Our main results were :

- a necessary condition (Theorem III.2) for the achievability of a distortion pair (D_1,D_2);
- the high-SNR asymptotic behavior (Theorem III.5) of optimal transmission schemes, which in the symmetric case (Corollary III.3) is given by

$$\lim_{P/N \to \infty} \sqrt{\frac{P}{N}} D^*(\sigma^2, \rho, P, N) = \sigma^2\sqrt{\frac{1-\rho^2}{4}}$$

  and which is shown to be achievable by source-channel separation;
- the optimality, for all SNRs below a certain threshold, of an uncoded transmission scheme, which ignores the feedback (Theorem III.6). In the symmetric case, this optimality result (Corollary III.5) is given by

$$D^*(\sigma^2, \rho, P, N) = \sigma^2\frac{P(1-\rho^2)+N}{2P(1+\rho)+N}, \quad \frac{P}{N} \leq \frac{\rho}{1-\rho^2}.$$

Appendix I
Proof of Theorem III.2

To prove the necessary condition of Theorem III.2 for the achievability of a distortion pair (D_1,D_2), we use the following two lemmas.

Lemma A.1: For our multiple-access setup with feedback, let $\{X_{1,k}\}$, $\{X_{2,k}\}$, and $\{Y_k\}$ be the channel inputs and channel outputs of a coding scheme achieving some distortion pair (D_1,D_2). Then, for every $\delta > 0$ there exists an integer $n_0(\delta) > 0$ such that for all $n > n_0(\delta)$

$$nR_{S_1,S_2}(D_1+\delta, D_2+\delta) \leq \sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k) \tag{23}$$

$$nR_{S_1|S_2}(D_1+\delta) \leq \sum_{k=1}^{n} I(X_{1,k}; Y_k|X_{2,k}) \tag{24}$$

$$nR_{S_2|S_1}(D_2+\delta) \leq \sum_{k=1}^{n} I(X_{2,k}; Y_k|X_{1,k}). \tag{25}$$

Proof: The proofs of (23)–(25) follow along the lines of the proof for the univariate analog (see e.g., [10, p. 15]). The main ingredients in those derivations are the convexity of the rate-distortion functions and the data-processing inequality. We start with the proof of (23). By the definition of an achievable distortion pair (D_1,D_2) (Definition II.1) and by the monotonicity of $R_{S_1,S_2}(\Delta_1, \Delta_2)$ in (Δ_1,Δ_2), we have that for every $\delta > 0$ there exists an $n_0(\delta) > 0$ such that for every $n > n_0(\delta)$

$$\begin{aligned}
&R_{S_1,S_2}(D_1+\delta, D_2+\delta) \\
&\leq R_{S_1,S_2}\left(\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[(S_{1,k}-\hat{S}_{1,k})^2], \frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[(S_{2,k}-\hat{S}_{2,k})^2]\right) \\
&\overset{(a)}{\leq} \sum_{k=1}^{n}\frac{1}{n}R_{S_1,S_2}\Big(\underbrace{\mathsf{E}[(S_{1,k}-\hat{S}_{1,k})^2]}_{d_{1,k}}, \underbrace{\mathsf{E}[(S_{2,k}-\hat{S}_{2,k})^2]}_{d_{2,k}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{k=1}^{n} \min_{\substack{P_{T_1,T_2|S_1,S_2}:\\ \mathsf{E}[(S_1-T_1)^2]\le d_{1,k}\\ \mathsf{E}[(S_2-T_2)^2]\le d_{2,k}}} I(S_1,S_2;T_1,T_2)\\
&\le \frac{1}{n}\sum_{k=1}^{n} I(S_{1,k},S_{2,k};\hat{S}_{1,k},\hat{S}_{2,k})\\
&= \frac{1}{n}\sum_{k=1}^{n} h(S_{1,k},S_{2,k}) - \sum_{k=1}^{n} h(S_{1,k},S_{2,k}|\hat{S}_{1,k},\hat{S}_{2,k})\\
&\le \frac{1}{n}\sum_{k=1}^{n} h(S_{1,k},S_{2,k}) - \sum_{k=1}^{n} h(S_{1,k},S_{2,k}|\hat{\mathbf{S}}_1,\hat{\mathbf{S}}_2,S_{1,1}^{k-1},S_{2,1}^{k-1})\\
&= \frac{1}{n}h(\mathbf{S}_1,\mathbf{S}_2) - h(\mathbf{S}_1,\mathbf{S}_2|\hat{\mathbf{S}}_1,\hat{\mathbf{S}}_2)\\
&= \frac{1}{n}I(\mathbf{S}_1,\mathbf{S}_2;\hat{\mathbf{S}}_1,\hat{\mathbf{S}}_2)\\
&\overset{(b)}{\le} \frac{1}{n}I(\mathbf{S}_1,\mathbf{S}_2;\mathbf{Y}) \qquad (26)
\end{aligned}
$$

where in step (a) we have used of the convexity of $R_{S_1,S_2}(D_1,D_2)$, and in step (b) we have used the data-processing inequality. The RHS of (26) can be further bounded as follows

$$
\begin{aligned}
I(\mathbf{S}_1,\mathbf{S}_2;\mathbf{Y}) &= h(\mathbf{Y}) - h(\mathbf{Y}|\mathbf{S}_1,\mathbf{S}_2)\\
&= h(\mathbf{Y}) - \sum_{k=1}^{n} h(Y_k|\mathbf{S}_1,\mathbf{S}_2,Y^{k-1})\\
&\le h(\mathbf{Y}) - \sum_{k=1}^{n} h(Y_k|\mathbf{S}_1,\mathbf{S}_2,Y^{k-1},X_{1,k},X_{2,k})\\
&\overset{(a)}{=} h(\mathbf{Y}) - \sum_{k=1}^{n} h(Y_k|X_{1,k},X_{2,k})\\
&\le \sum_{k=1}^{n} h(Y_k) - \sum_{k=1}^{n} h(Y_k|X_{1,k},X_{2,k})\\
&= \sum_{k=1}^{n} I(X_{1,k},X_{2,k};Y_k) \qquad (27)
\end{aligned}
$$

where inequality (a) follows because given the channel inputs $X_{1,k}$, $X_{2,k}$, the channel output Y_k is independent of $(\mathbf{S}_1,\mathbf{S}_2,Y^{k-1})$. Inequalities (26) and (27) combine to prove (23).

The derivations of (24) and (25) are similar to that of (23). Since there is a symmetry between the derivation of (24) and the derivation of (25), we only give the derivation of (24). By the definition of an achievable distortion pair (D_1,D_2) and by the monotonicity of $R_{S_1|S_2}(\Delta_1)$ in Δ_1, we have that for every $\delta > 0$ there exists an integer $n_0(\delta) > 0$ such that for every $n > n_0(\delta)$

$$
\begin{aligned}
&nR_{S_1|S_2}(D_1+\delta)\\
&\quad\le nR_{S_1|S_2}\left(\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[(S_{1,k}-\hat{S}_{1,k})^2]\right)\\
&\quad\overset{(a)}{\le} n\sum_{k=1}^{n}\frac{1}{n}R_{S_1|S_2}\Big(\underbrace{\mathsf{E}[(S_{1,k}-\hat{S}_{1,k})^2]}_{d_{1,k}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \min_{\substack{P_{T_k|S_{1,k},S_{2,k}}:\\ \mathsf{E}[(S_{1,k}-T_k)^2]\le d_{1,k}}} I(S_{1,k};T_k|S_{2,k})\\
&\le \sum_{k=1}^{n} I(S_{1,k};\hat{S}_{1,k}|S_{2,k})\\
&= \sum_{k=1}^{n} h(S_{1,k}|S_{2,k}) - \sum_{k=1}^{n} h(S_{1,k}|\hat{S}_{1,k},S_{2,k})\\
&= \sum_{k=1}^{n} h(S_{1,k}|S_{1,1}^{k-1},\mathbf{S}_2) - \sum_{k=1}^{n} h(S_{1,k}|\hat{S}_{1,k},S_{2,k})\\
&\le \sum_{k=1}^{n} h(S_{1,k}|S_{1,1}^{k-1},\mathbf{S}_2) - \sum_{k=1}^{n} h(S_{1,k}|\hat{\mathbf{S}}_1,\mathbf{S}_2,S_{1,1}^{k-1})\\
&= \sum_{k=1}^{n} I(S_{1,k};\hat{\mathbf{S}}_1|\mathbf{S}_2,S_{1,1}^{k-1})\\
&= I(\mathbf{S}_1;\hat{\mathbf{S}}_1|\mathbf{S}_2)\\
&\overset{(b)}{\le} I(\mathbf{S}_1,\mathbf{Y}|\mathbf{S}_2) \qquad (28)
\end{aligned}
$$

where step (a) follows by the convexity of $R_{S_1|S_2}(D_1)$ and step (b) follows by the data-processing in equality, i.e.,

$$
\begin{aligned}
I(\mathbf{S}_1;\mathbf{Y},\hat{\mathbf{S}}_1|\mathbf{S}_2) &= I(\mathbf{S}_1;\hat{\mathbf{S}}_1|\mathbf{S}_2) + \underbrace{I(\mathbf{S}_1;\mathbf{Y}|\hat{\mathbf{S}}_1,\mathbf{S}_2)}_{\ge 0}\\
&= I(\mathbf{S}_1;\mathbf{Y}|\mathbf{S}_2) + \underbrace{I(\mathbf{S}_1;\hat{\mathbf{S}}_1|\mathbf{Y},\mathbf{S}_2)}_{=0}.
\end{aligned}
$$

The RHS of (28) can be further bounded as follows

$$
\begin{aligned}
&I(\mathbf{S}_1;\mathbf{Y}|\mathbf{S}_2)\\
&\quad= h(\mathbf{Y}|\mathbf{S}_2) - h(\mathbf{Y}|\mathbf{S}_1,\mathbf{S}_2)\\
&\quad= \sum_{k=1}^{n} h(Y_k|Y^{k-1},\mathbf{S}_2) - \sum_{k=1}^{n} h(Y_k|\mathbf{S}_1,\mathbf{S}_2,Y^{k-1})\\
&\quad\overset{(a)}{=} \sum_{k=1}^{n} h(Y_k|Y^{k-1},\mathbf{S}_2,X_{2,k})\\
&\qquad - \sum_{k=1}^{n} h(Y_k|\mathbf{S}_1,\mathbf{S}_2,Y^{k-1},X_{1,k},X_{2,k})\\
&\quad\overset{(b)}{\le} \sum_{k=1}^{n} h(Y_k|X_{2,k}) - \sum_{k=1}^{n} h(Y_k|X_{1,k},X_{2,k})\\
&\quad= \sum_{k=1}^{n} I(X_{1,k};Y_k|X_{2,k}) \qquad (29)
\end{aligned}
$$

where (a) follows because $X_{i,k}$ is determined by Y^{k-1} and $\mathbf{S}_i$, $i \in \{1,2\}$, and (b) follows because given the channel inputs $X_{1,k}$, $X_{2,k}$, the channel output Y_k is independent of $(\mathbf{S}_1,\mathbf{S}_2,Y^{k-1})$. Inequalities (28) and (29) combine to prove (24). □

Lemma A.2 (Ozarow [8]): Let $\{X_{1,k}\}$ and $\{X_{2,k}\}$ be zero-mean sequences satisfying $\sum_{k=1}^{n}\mathsf{E}[X_{i,k}^2] \le nP_i$, for $i \in \{1,2\}$. Let $Y_k = X_{1,k} + X_{2,k} + Z_k$, where $\{Z_k\}$ are i.i.d. zero-mean variance-N Gaussian, and where for every k, Z_k is independent of $(X_{1,k}, X_{2,k})$. Let $\hat{\rho}_n \in [0,1]$ be given by

$$\hat{\rho}_n \triangleq \frac{\left|\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}[X_{1,k}X_{2,k}]\right|}{\sqrt{\left(\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[X_{1,k}^2]\right)\left(\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[X_{2,k}^2]\right)}}. \tag{30}$$

Then, the RHS of (23)–(25) are upper bounded as in (31)–(33) displayed at the bottom of the page.

Proof: See [8, pp. 627]. □

Proof of Theorem III.2: The proof follows by bounding the expressions on the RHS of (23)–(25) using Lemma A.2; by letting n tend to infinity; and then letting δ tend to zero (from above). □

Appendix II
Proof of Theorem III.5

For $\rho = 1$ the result follows by noting that the multiple-access problem reduces to a point-to-point problem where $D_1^* = D_2^*$. Hence, we shall now assume

$$\rho < 1. \tag{34}$$

The result can then be obtained from the necessary condition for the achievability of a distortion pair (D_1, D_2) in Theorem III.2 and from the sufficient conditions for the achievability of a distortion pair (D_1, D_2) that follow from source-channel separation in Corollary III.2.

By Corollary III.2 it follows that a distortion pair $(\bar{D}_1, \bar{D}_2)$ is achievable if $\bar{D}_1 \le \sigma^2$, $\bar{D}_2 \le \sigma^2$ and

$$\bar{D}_1 \ge \sigma^2 2^{-2R_1}(1-\rho^2) + \sigma^2\rho^2 2^{-2(R_1+R_2)} \tag{35}$$
$$\bar{D}_2 \ge \sigma^2 2^{-2R_2}(1-\rho^2) + \sigma^2\rho^2 2^{-2(R_1+R_2)} \tag{36}$$
$$\bar{D}_1\bar{D}_2 = \sigma^4 2^{-2(R_1+R_2)}(1-\rho^2) + \sigma^4\rho^2 2^{-4(R_1+R_2)} \tag{37}$$

where the rate-pair (R_1, R_2) satisfies for some $\bar{\rho} \in [0,1]$

$$R_1 \le \frac{1}{2}\log_2\left(\frac{P_1(1-\bar{\rho}^2)}{N}\right) \tag{38}$$
$$R_2 \le \frac{1}{2}\log_2\left(\frac{P_2(1-\bar{\rho}^2)}{N}\right) \tag{39}$$
$$R_1 + R_2 \le \frac{1}{2}\log_2\left(\frac{P_1+P_2+2\bar{\rho}\sqrt{P_1P_2}}{N}\right). \tag{40}$$

If we restrict ourselves to distortion pairs $(\bar{D}_1, \bar{D}_2)$ satisfying

$$\lim_{N\to 0}\frac{N}{P_1\bar{D}_1} = 0 \quad \text{and} \quad \lim_{N\to 0}\frac{N}{P_2\bar{D}_2} = 0 \tag{41}$$

and to ρ satisfying (34), then for sufficiently small $N > 0$ the Constraints (35) and (36) become redundant. Consequently, for N sufficiently small, any distortion pair $(\bar{D}_1, \bar{D}_2)$ satisfying (41) and (37), where (R_1, R_2) satisfies (38)–(40) for some $\bar{\rho} \in [0,1]$, is achievable. And because for any fixed $\bar{\rho} \in [0,1)$ as $N \to 0$ the Constraints (38) and (39) become redundant, it follows that any distortion pair satisfying (41) and

$$\lim_{N\to 0}\frac{P_1+P_2+2\bar{\rho}\sqrt{P_1P_2}}{N}\bar{D}_1\bar{D}_2 = \sigma^4(1-\rho^2) \tag{42}$$

for some $\bar{\rho} \in [0,1)$, is achievable. Since $\bar{\rho}$ can be chosen arbitrarily close to 1, simple calculus shows that

$$\lim_{N\to 0}\frac{P_1+P_2+2\sqrt{P_1P_2}}{N}\bar{D}_1\bar{D}_2 = \sigma^4(1-\rho^2) \tag{43}$$

is achievable.

Next, let $\left(D_1^*(\sigma^2,\rho,P_1,P_2,N), D_2^*(\sigma^2,\rho,P_1,P_2,N)\right)$ be a distortion pair resulting from an arbitrary optimal scheme for the corresponding SNR, and let (D_1^*, D_2^*) be the corresponding shorthand notation for this distortion pair. By Theorem III.2, we have that

$$R_{S_1,S_2}(D_1,D_2) \le \frac{1}{2}\log_2\left(1+\frac{P_1+P_2+2\sqrt{P_1P_2}}{N}\right). \tag{44}$$

If (D_1^*, D_2^*) satisfies

$$\lim_{N\to 0}\frac{N}{P_1 D_1^*} = 0 \quad \text{and} \quad \lim_{N\to 0}\frac{N}{P_2 D_2^*} = 0 \tag{45}$$

then for N sufficiently small

$$R_{S_1,S_2}(D_1^*, D_2^*) = \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1^*D_2^*}\right) \tag{46}$$

by Theorem III.1 and because $(D_1^*, D_2^*) \in \mathfrak{D}_2$. From (44) and (46) we thus get that if (D_1^*, D_2^*) satisfies (45), then

$$\lim_{N\to 0}\frac{P_1+P_2+2\sqrt{P_1P_2}}{N}D_1^*D_2^* \ge \sigma^4(1-\rho^2). \tag{47}$$

Combining (43) with (47) yields Theorem III.5. □

$$\sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k) \le \frac{n}{2}\log_2\left(1+\frac{P_1+P_2+2\hat{\rho}_n\sqrt{P_1P_2}}{N}\right) \tag{31}$$
$$\sum_{k=1}^{n} I(X_{1,k}; Y_k | X_{2,k}) \le \frac{n}{2}\log_2\left(1+\frac{P_1(1-\hat{\rho}_n^2)}{N}\right) \tag{32}$$
$$\sum_{k=1}^{n} I(X_{2,k}; Y_k | X_{1,k}) \le \frac{n}{2}\log_2\left(1+\frac{P_2(1-\hat{\rho}_n^2)}{N}\right). \tag{33}$$

Appendix III
Proof of Theorem III.6

Theorem III.6 states that if P_1, P_2, N satisfy (20), then the uncoded scheme is optimal, i.e., no pair (D_1,D_2) satisfying $D_1 \leq D_1^{\mathrm{u}}$ and $D_2 < D_2^{\mathrm{u}}$ or satisfying $D_1 < D_1^{\mathrm{u}}$ and $D_2 \leq D_2^{\mathrm{u}}$ is achievable. For P_1, P_2, N satisfying (20) with equality this was proven right after Theorem III.6. Thus, here we restrict ourselves to P_1, P_2, N satisfying (20) with strict inequality.

We show the inachievability of any (D_1,D_2) satisfying $D_1 < D_1^{\mathrm{u}}$ and $D_2 \leq D_2^{\mathrm{u}}$. The inachievability of any (D_1,D_2) satisfying $D_1 \leq D_1^{\mathrm{u}}$ and $D_2 < D_2^{\mathrm{u}}$ follows by similar arguments and is therefore omitted. The main step in our proof follows by contradiction. More precisely, we show that a contradiction arises from the following assumption.

Assumption C.1 (Leading to a contradiction): For P_1, P_2, N satisfying (20) with strict inequality, there exist encoding rules $\{f_{i,k}^{(n)}\}$ satisfying the average power constraints (4), which, when combined with the optimal conditional expectation reconstructors

$$\hat{\mathbf{S}}_i = \mathsf{E}[\mathbf{S}_i|\mathbf{Y}], \quad i \in \{1,2\} \tag{48}$$

result in

$$\varlimsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathsf{E}[(S_{i,k} - \hat{S}_{i,k})^2] \triangleq D_i^* \quad i \in \{1,2\} \tag{49}$$

such that

$$(D_1^*, D_2^*) \in \mathrm{int}(\mathfrak{D}_3), \quad D_1^* < D_1^{\mathrm{u}} \quad \text{and} \quad D_2^* = D_2^{\mathrm{u}} \tag{50}$$

where we have denoted by $\mathrm{int}(\mathfrak{D}_3)$ the interior of $\mathfrak{D}_3$.

Once a contradiction from Assumption C.1 is established, it will follow that Assumption C.1 is false and the proof of Theorem III.6 will follow in Section C-C.

Assume that Assumption C.1 is true. Let $\{f_{i,k}^{(n)}\}$ be a sequence of encoding functions, with resulting channel inputs $\{X_{1,k}, X_{2,k}\}$ and resulting channel outputs $\{Y_k\}$, which, when combined with the optimal conditional expectation reconstructors $\hat{\mathbf{S}}_1 = \mathsf{E}[\mathbf{S}_1|\mathbf{Y}]$ and $\hat{\mathbf{S}}_2 = \mathsf{E}[\mathbf{S}_2|\mathbf{Y}]$ result in distortions (D_1^*,D_2^*) as defined in (49) and satisfying (50). The contradiction based on Assumption C.1 will be obtained by deriving contradictory lower and upper bounds for the expected squared-error that Transmitter 2 can achieve at the end of the transmission on the sequence $\mathbf{W} \triangleq \mathbf{S}_1 - \rho\mathbf{S}_2$. To this end, let $\varphi^{(n)}(\mathbf{S}_2, \mathbf{Y})$ be some estimator of $\mathbf{W}$ from $(\mathbf{S}_2, \mathbf{Y})$ and let $D_W(\varphi^{(n)})$ be the mean squared-error associated with it:

$$D_W(\varphi^{(n)}) \triangleq \frac{1}{n}\mathsf{E}[\|\mathbf{W} - \varphi^{(n)}(\mathbf{S}_2, \mathbf{Y})\|^2].$$

Based on Assumption C.1, we now derive a lower bound on $D_W(\varphi^{(n)})$.

A) "Lower Bound" on $D_W(\varphi^{(n)})$:

In this section, we show

$$\text{Assumption C.1} \Rightarrow \left(\varliminf_{n \to \infty} D_W(\varphi^{(n)}) > \sigma^2(1-\rho^2)\frac{N}{N+P_1(1-\rho^2)}, \forall \varphi^{(n)}\right). \tag{51}$$

The idea for showing (51) is to exploit the fact that the sequence $\mathbf{W}$ is independent of $\mathbf{S}_2$, and that therefore the only information that Transmitter 2 receives about $\mathbf{W}$ is via the feedback signal $\mathbf{Y}$. Roughly speaking, we then show that if $\mathbf{Y}$ allows for "good" estimates of $\mathbf{S}_1$ and $\mathbf{S}_2$, i.e., if $D_1^* < D_1^{\mathrm{u}}$ and $D_2^* = D_2^{\mathrm{u}}$, then $\mathbf{Y}$ can only contain "little" information about $\mathbf{W}$, and hence Transmitter 2 can only make a coarse estimate of $\mathbf{W}$. The main element in showing (51) is given by the following lemma.

Lemma C.1: Let $\hat{\rho}_n$ be as defined in (30). Then

$$I(\mathbf{S}_1; \mathbf{Y}|\mathbf{S}_2) \leq \frac{n}{2}\log_2\left(1 + \frac{P_1(1-\hat{\rho}_n^2)}{N}\right) \tag{52}$$

and

$$\text{Assumption C.1} \quad \Rightarrow \quad \varliminf_{n \to \infty} \hat{\rho}_n > \rho. \tag{53}$$

Proof: Combining (29) with Inequality (32) of Lemma A.2 establishes (52). It remains to prove that Assumption C.1 implies that $\varliminf_{n \to \infty} \hat{\rho}_n > \rho$. To this end, we recall that from [1, Proof of Theorem IV.1] we have that if P_1, P_2, N satisfy (20), then the tuple $(D_1^{\mathrm{u}},D_2^{\mathrm{u}})$ satisfies [1, Condition (16) of Th. IV.1] with equality, i.e.,

$$R_{S_1,S_2}(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}) = \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\rho\sqrt{P_1P_2}}{N}\right). \tag{54}$$

Next, we notice that since Assumption C.1 guarantees that (D_1^*,D_2^*) is achievable, it follows from Lemma A.1 that for every $\delta > 0$ there exists an $n'(\delta) > 0$ such that for all $n > n'(\delta)$ we have

$$\begin{aligned} nR_{S_1,S_2}&(D_1^* + \delta, D_2^* + \delta) \\ &\overset{(a)}{\leq} \sum_{k=1}^{n} I(X_{1,k}, X_{2,k}; Y_k) \\ &\overset{(b)}{\leq} \frac{n}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\hat{\rho}_n\sqrt{P_1P_2}}{N}\right) \end{aligned} \tag{55}$$

where (a) follows from (23) in Lemma A.1, and (b) follows from Lemma A.2. Taking the $\lim\inf$ of (55) yields that for every $\delta > 0$

$$R_{S_1,S_2}(D_1^* + \delta, D_2^* + \delta) \leq \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\hat{\rho}^*\sqrt{P_1P_2}}{N}\right)$$

where $\hat{\rho}^* = \varliminf_{n \to \infty} \hat{\rho}_n$. And since $R_{S_1,S_2}(D_1, D_2)$ is continuous in (D_1,D_2) it follows, upon letting δ tend to zero, that

$$R_{S_1,S_2}(D_1^*, D_2^*) \leq \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\hat{\rho}^*\sqrt{P_1P_2}}{N}\right). \tag{56}$$

By Assumption C.1 and by the strict monotonicity of $R_{S_1,S_2}(D_1,D_2)$ as a function of D_1 in $\mathrm{int}(\mathfrak{D}_3)$, it follows from the hypothesis $D_1^* < D_1^{\mathrm{u}}$ and $D_1^* = D_1^{\mathrm{u}}$ that

$$R_{S_1,S_2}(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}) < R_{S_1,S_2}(D_1^*, D_2^*). \tag{57}$$

Combining (57) with (56) and (54) gives $\varliminf_{n \to \infty} \hat{\rho}_n > \rho$. □

We next prove that

$$D_W(\varphi^{(n)}) \geq \sigma^2(1-\rho^2)2^{-\frac{2}{n}I(\mathbf{S}_1;\mathbf{Y}|\mathbf{S}_2)}. \tag{58}$$

To derive (58), denote by $R_W(D)$ the rate-distortion function for a source of the law of $\mathbf{W}$. We then have 56tit04-stinguely-2040870.xml

$$\begin{aligned}
&nR_W(D_W(\varphi^{(n)}))\\
&\overset{(a)}{\leq} I(\mathbf{W};\varphi^{(n)}(\mathbf{S}_2,\mathbf{Y}))\\
&\overset{(b)}{\leq} I(\mathbf{W};\mathbf{Y},\mathbf{S}_2)\\
&= I(\mathbf{S}_1-\rho\mathbf{S}_2;\mathbf{Y},\mathbf{S}_2)\\
&= h(\mathbf{S}_1-\rho\mathbf{S}_2) - h(\mathbf{S}_1-\rho\mathbf{S}_2|\mathbf{Y},\mathbf{S}_2)\\
&\overset{(c)}{=} h(\mathbf{S}_1-\rho\mathbf{S}_2|\mathbf{S}_2) - h(\mathbf{S}_1-\rho\mathbf{S}_2|\mathbf{Y},\mathbf{S}_2)\\
&= h(\mathbf{S}_1|\mathbf{S}_2) - h(\mathbf{S}_1|\mathbf{Y},\mathbf{S}_2)\\
&= I(\mathbf{S}_1;\mathbf{Y}|\mathbf{S}_2)
\end{aligned} \tag{59}$$

where inequality a) follows by the data-processing inequality and the convexity of $R_W(\cdot)$. Inequality b) follows by the data-processing inequality, and c) follows since $\mathbf{S}_2$ and $\mathbf{S}_1-\rho\mathbf{S}_2$ are independent. Substituting $R_W(D_W(\varphi^{(n)}))$ on the LHS of (59) by its explicit form gives

$$\frac{n}{2}\log_2\left(\frac{\sigma^2(1-\rho^2)}{D_W(\varphi^{(n)})}\right) \leq I(\mathbf{S}_1;\mathbf{Y}|\mathbf{S}_2).$$

Rewriting this inequality establishes (58).

Lemma C.1 and Inequality (58) combine to prove (51). We next derive an upper bound on $D_W(\varphi^{(n)})$.

B) "Upper Bound" on Minimal $D_W(\varphi^{(n)})$: We now present an estimator $\tilde{\varphi}^{(n)}(\mathbf{S}_2,\mathbf{Y})$ for which we show

$$\text{Assumption C.1} \Rightarrow \left(\varlimsup_{\nu\to\infty} D_W(\tilde{\varphi}^{(n_\nu)}) < \sigma^2(1-\rho^2)\frac{N}{N+P_1(1-\rho^2)}\right) \tag{60}$$

for some monotonically increasing sequence $\{n_\nu\}$ of integers. From Implications (51) and (60) we then conclude that Assumption C.1 is false. The estimator $\tilde{\varphi}^{(n)}(\mathbf{S}_2,\mathbf{Y})$ is given by

$$\begin{aligned}
\tilde{\varphi}^{(n)}(\mathbf{S}_2,\mathbf{Y}) &\triangleq \alpha\hat{\mathbf{S}}_1 - \beta\mathbf{S}_2\\
&= \alpha\mathsf{E}[\mathbf{S}_1|\mathbf{Y}] - \beta\mathbf{S}_2
\end{aligned}$$

where the coefficients α and β are given by

$$\alpha \triangleq \frac{\sigma^2\left(\sqrt{\sigma^2-D_1^*} - \rho\sqrt{\sigma^2-D_2^*}\right)}{D_2^*\sqrt{\sigma^2-D_1^*}} \tag{61}$$

$$\beta \triangleq \frac{\sqrt{(\sigma^2-D_1^*)(\sigma^2-D_2^*)} - \rho(\sigma^2-D_2^*)}{D_2^*} \tag{62}$$

with (D_1^*,D_2^*) as in Assumption C.1. The idea for showing that for this estimator (60) holds, is to exploit the fact that if $\mathbf{Y}$ allows for a "good" estimate $\hat{\mathbf{S}}_1$ of $\mathbf{S}_1$, i.e., if $D_1^* < D_1^{\mathrm{u}}$, then Transmitter 2 can also make a "good" estimate of $\mathbf{W}$, based on $\mathbf{S}_2$ and $\mathbf{Y}$. To show this we first notice that Assumption C.1 implies that there exists a monotonically increasing sequence of integers $\{n_\nu\}$ such that

$$\lim_{\nu\to\infty}\frac{1}{n_\nu}\sum_{k=1}^{n_\nu}\mathsf{E}[(S_{i,k}-\hat{S}_{i,k})^2] = D_i^* \quad i\in\{1,2\}. \tag{63}$$

We now derive (60) using the following two lemmas.

Lemma C.2: For every $\delta > 0$ there exists an $\nu_0(\delta)$ such that for all $\nu > \nu_0(\delta)$ the following inequalities hold:

$$\frac{1}{n_\nu}\sum_{k=1}^{n_\nu}\mathsf{E}[S_{1,k}\hat{S}_{1,k}] \geq \sigma^2 - D_1^* - \delta \tag{64}$$

$$\frac{1}{n_\nu}\sum_{k=1}^{n_\nu}\mathsf{E}[\hat{S}_{1,k}^2] \leq \sigma^2 - D_1^* + \delta \tag{65}$$

$$\frac{1}{n_\nu}\sum_{k=1}^{n_\nu}\mathsf{E}[\hat{S}_{1,k}S_{2,k}] \leq \sqrt{(\sigma^2-D_1^*)(\sigma^2-D_2^*)+\delta(\sigma^2+\delta)}. \tag{66}$$

Proof: See Appendix C–D. □

Lemma C.3: Assumption C.1 and in particular $(D_1^*,D_2^*) \in \mathfrak{D}_3$ implies that the coefficients α and β defined in (61) and (62) satisfy

$$\alpha \geq 0 \quad \text{and} \quad (\rho-\beta) \geq 0. \tag{67}$$

Proof: Follows by noting that for every $(D_1^*,D_2^*) \in \mathfrak{D}_3$

$$D_2^* \geq \begin{cases} \left(\sigma^2(1-\rho^2)-D_1^*\right)\frac{\sigma^2}{\sigma^2-D_1^*}, & \text{if } 0\leq D_1^*\leq\sigma^2(1-\rho^2)\\ \frac{(D_1^*-\sigma^2(1-\rho^2))}{\rho^2}, & \text{if } D_1^* > \sigma^2(1-\rho^2). \end{cases}$$

□

To prove (60), we first rewrite $D_W(\tilde{\varphi}^{(n_\nu)})$ as shown in (68) at the bottom of the next page. Using (68), Lemma C.2 and Lemma C.3, as well as $\mathsf{E}[S_{1,k}^2] = \mathsf{E}[S_{2,k}^2] = \sigma^2$ and $\mathsf{E}[S_{1,k}S_{2,k}] = \rho\sigma^2$, we now get that for P_1, P_2, N satisfying (20) and for every $\delta > 0$ there exists a $\nu_0(\delta) > 0$ such that for all $\nu > \nu_0(\delta)$,

$$\begin{aligned}
&D_W(\tilde{\varphi}^{(n_\nu)})\\
&\leq \sigma^2 - 2\alpha(\sigma^2-D_1^*-\delta) - 2(\rho-\beta)\rho\sigma^2\\
&\quad+\alpha^2(\sigma^2-D_1^*+\delta)\\
&\quad+2\alpha(\rho-\beta)\left(\sqrt{(\sigma^2-D_1^*)(\sigma^2-D_2^*)+\delta(\sigma^2+\delta)}\right)\\
&\quad+(\rho-\beta)^2\sigma^2.
\end{aligned} \tag{69}$$

Letting in (60), the index ν tend to infinity and then $\delta \to 0$, we obtain (70) displayed at the bottom of the page, where in the last step we have replaced the terms α and β by their expressions in (61) and (62). To conclude our upper bound we now make use of one last lemma.

Lemma C.4: For all $(D_1^*, D_2^*) \in \mathrm{int}(\mathfrak{D}_3)$ the expression on the RHS of (70) is strictly increasing in D_1^*.

Proof: Denote by $\tilde{D}_W$ the RHS of (70). The proof follows by showing that for all $(D_1^*, D_2^*) \in \mathrm{int}(\mathfrak{D}_3)$

$$\frac{\partial \tilde{D}_W}{\partial D_1^*} > 0.$$

This follows by direct differentiation and by noting that for $(D_1^*, D_2^*) \in \mathrm{int}(\mathfrak{D}_3)$

$$D_2^* > \begin{cases} \left(\sigma^2(1-\rho^2) - D_1^*\right) \frac{\sigma^2}{\sigma^2 - D_1^*}, & \text{if } 0 \le D_1^* \le \sigma^2(1-\rho^2) \\ \frac{\left(D_1^* - \sigma^2(1-\rho^2)\right)}{\rho^2}, & \text{if } D_1^* > \sigma^2(1-\rho^2). \end{cases}$$

□

Since $D_1^* < D_1^{\mathrm{u}}$ and $D_2^* < D_2^{\mathrm{u}}$ it follows from (70) and Lemma C.4 that $\limsup_{\nu \to \infty} D_W(\tilde{\varphi}^{(n_\nu)})$ is upper bounded as in (71) at the bottom of the page, where the last line follows from replacing D_1^{u} and D_2^{u} by their expressions given in Theorem III.6. Thus, we have proven (60).

C) Concluding the Proof of Theorem III.6: It follows from (51) and (60) that Assumption C.1 is false. We now show that this implies that if P_1, P_2, N satisfy (20) with strict inequality, then no pair (D_1, D_2) satisfying $D_1 < D_1^{\mathrm{u}}$ and $D_2 \le D_2^{\mathrm{u}}$ or satisfying $D_1 \le D_1^{\mathrm{u}}$ and $D_2 < D_2^{\mathrm{u}}$ is achievable. To prove this we assume $\rho > 0$ because for $\rho = 0$ Condition (20) becomes $P_1 P_2 \le 0$ and is therefore never satisfied with strict inequality.

Our arguments are given in the following sequence of statements:

A) If P_1, P_2, N satisfy (20) with strict inequality, then the set of (D_1^*, D_2^*) satisfying (50) is not empty.

Statement A) holds since if P_1, P_2, N satisfy (20) with strict inequality, then $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}) \in \mathrm{int}(\mathfrak{D}_3)$ and $\mathrm{int}(\mathfrak{D}_3) \neq \emptyset$ whenever $\rho \neq 0$.

B) If P_1, P_2, N satisfy (20) with strict inequality, then there do not exist encoding rules, that, when combined with the optimal conditional expectation reconstructors, result in (D_1^*, D_2^*) as defined in (49) satisfying

$$D_1^* < D_1^{\mathrm{u}} \quad \text{and} \quad D_2^* = D_2^{\mathrm{u}}$$

(with (D_1^*, D_2^*) in or outside $\mathrm{int}(\mathfrak{D}_3)$).

Statement B) can be shown by contradiction. If a coding scheme as described in B) were to exist, then by time-sharing it with the uncoded scheme—for which $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}) \in \mathrm{int}(\mathfrak{D}_3)$—and by Statement A), we would obtain

$$\begin{aligned} D_W(\tilde{\varphi}^{(n_\nu)}) &= \frac{1}{n_\nu} \mathsf{E}[\|\mathbf{W} - \tilde{\varphi}(\mathbf{S}_2, \mathbf{Y})\|^2] \\ &= \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}[(S_{1,k} - \rho S_{2,k} - \alpha \hat{S}_{1,k} + \beta S_{2,k})^2] \\ &= \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}[(S_{1,k} - \alpha \hat{S}_{1,k} - (\rho - \beta) S_{2,k})^2] \\ &= \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \Big(\mathsf{E}[S_{1,k}^2] - 2\alpha \mathsf{E}[S_{1,k}\hat{S}_{1,k}] - 2(\rho-\beta)\mathsf{E}[S_{1,k}S_{2,k}] + \alpha^2 \mathsf{E}[\hat{S}_{1,k}^2] \\ &\qquad + 2\alpha(\rho-\beta)\mathsf{E}[\hat{S}_{1,k}S_{2,k}] + (\rho-\beta)^2 \mathsf{E}[S_{2,k}^2] \Big). \end{aligned} \tag{68}$$

$$\begin{aligned} \overline{\lim}_{\nu \to \infty} D_W(\tilde{\varphi}^{(n_\nu)}) &\le \sigma^2 - 2\alpha(\sigma^2 - D_1^*) - 2(\rho-\beta)\rho\sigma^2 + \alpha^2(\sigma^2 - D_1^*) \\ &\quad + 2\alpha(\rho-\beta)\sqrt{(\sigma^2 - D_1^*)(\sigma^2 - D_2^*)} + (\rho-\beta)^2\sigma^2 \\ &= \sigma^2 \frac{2\rho\sqrt{(\sigma^2 - D_1^*)(\sigma^2 - D_2^*)} + D_1^* + D_2^* - \sigma^2(1+\rho^2)}{D_2^*} \end{aligned} \tag{70}$$

$$\begin{aligned} \overline{\lim}_{\nu \to \infty} D_W(\tilde{\varphi}^{(n_\nu)}) &< \sigma^2 \frac{2\rho\sqrt{(\sigma^2 - D_1^{\mathrm{u}})(\sigma^2 - D_2^{\mathrm{u}})} + D_1^{\mathrm{u}} + D_2^{\mathrm{u}} - \sigma^2(1+\rho^2)}{D_2^{\mathrm{u}}} \\ &= \sigma^2 \frac{N(1-\rho^2)}{P_1(1-\rho^2) + N} \end{aligned} \tag{71}$$

a scheme for which (D_1^*,D_2^*) satisfies (50), in contradiction to the fact that Assumption C.1 is false.

C) If P_1, P_2, N satisfy (20) with strict inequality, then there exist no encoding rules, which, when combined with the optimal conditional expectation reconstructors, result in (D_1^*,D_2^*) as defined in (49) such that

$$D_1^* = D_1^{\mathrm{u}} \quad \text{and} \quad D_2^* < D_2^{\mathrm{u}}.$$

Statement C) can be proved using arguments similar to those used to prove Statement B).

D) If P_1, P_2, N satisfy (20) with strict inequality, then there exist no encoding rules, which when combined with the optimal conditional expectation reconstructors, result in (D_1^*,D_2^*) as defined in (49) such that $D_1^* < D_1^{\mathrm{u}}$ and $D_2^* \leq D_2^{\mathrm{u}}$ or such that $D_1^* \leq D_1^{\mathrm{u}}$ and $D_2^* < D_2^{\mathrm{u}}$.

To show Statement D) we proceed by contradiction. To this end, consider two variations of our uncoded scheme. Call these two variations "Scheme U_1" and "Scheme U_2". Let Scheme U_1 be given by the channel inputs

$$X_{1,k}^{\mathrm{u}_1} = \sqrt{\frac{P_1}{\sigma^2}} S_{1,k} \quad \text{and} \quad X_{2,k}^{\mathrm{u}_1} = 0$$

and the optimal conditional expectation reconstructors $\hat{\mathbf{S}}_1 = \mathsf{E}[\mathbf{S}_1 | \mathbf{Y}]$ and $\hat{\mathbf{S}}_2 = \mathsf{E}[\mathbf{S}_2 | \mathbf{Y}]$. The resulting distortion pair $(D_1^{\mathrm{u}_1}, D_2^{\mathrm{u}_1})$ is given by

$$D_1^{\mathrm{u}_1} = \sigma^2 \frac{N}{P_1 + N} \quad D_2^{\mathrm{u}_1} = \sigma^2 \frac{(1-\rho^2)P_1 + N}{P_1 + N}.$$

Similarly, let Scheme U_2 be given by the channel inputs

$$X_{1,k}^{\mathrm{u}_2} = 0 \quad \text{and} \quad X_{2,k}^{\mathrm{u}_2} = \sqrt{\frac{P_2}{\sigma^2}} S_{2,k}$$

and the same optimal conditional expectation reconstructors as for Scheme U_1. The resulting distortion pair $(D_1^{\mathrm{u}_2}, D_2^{\mathrm{u}_2})$ is given by

$$D_1^{\mathrm{u}_2} = \sigma^2 \frac{(1-\rho^2)P_2 + N}{P_2 + N} \quad D_2^{\mathrm{u}_2} = \sigma^2 \frac{N}{P_2 + N}.$$

Now assume that there could exist a coding scheme as described in D). Since $D_2^{\mathrm{u}_1} > D_2^{\mathrm{u}}$ and $D_1^{\mathrm{u}_2} > D_1^{\mathrm{u}}$ it would follow from time-sharing either with Scheme U_1 or Scheme U_2 that Statement B) or Statement C) is false.

E) If P_1, P_2, N satisfy (20) with strict inequality, then there exist no coding scheme resulting in (D_1^*,D_2^*) as defined in (49) such that

$$D_1^* < D_1^{\mathrm{u}} \quad \text{and} \quad D_2^* \leq D_2^{\mathrm{u}}$$

(be the reconstruction rule optimal or not).

Statement E) follows from D) because no reconstructor $\phi_i^{(n)}$ can outperform the optimal conditional expectation reconstructor $\hat{\mathbf{S}}_i = \mathsf{E}[\mathbf{S}_i | \mathbf{Y}]$.

By Statement E) it follows that if P_1, P_2, N satisfy (20) with strict inequality, then no (D_1,D_2) satisfying $D_1 < D_1^{\mathrm{u}}$ and $D_2 \leq D_2^{\mathrm{u}}$ is achievable.

D) Proof of Lemma C.2: By (63) it follows that for every $\delta > 0$ there exists a $\nu_0(\delta) > 0$ such that for all $\nu > \nu_0(\delta)$

$$D_i^* - \delta < \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}[(S_{i,k} - \hat{S}_{i,k})^2] < D_i^* + \delta, \quad i \in \{1,2\}. \tag{72}$$

Using (72), the relation $\mathsf{E}[S_{1,k}^2] = \sigma^2$, and (48) we obtain that

$$\sigma^2 - D_1^* - \delta \leq \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}[S_{1,k}\hat{S}_{1,k}] \leq \sigma^2 - D_1^* + \delta \tag{73}$$

and that

$$\sigma^2 - D_1^* - \delta \leq \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}[\hat{S}_{1,k}^2] \leq \sigma^2 - D_1^* + \delta. \tag{74}$$

This proves Inequalities (64) and (65).

To prove (66) we note that for every $c \in \mathbb{R}$ we can view $c\hat{S}_{1,k}$ as an estimator of $S_{2,k}$ based on $\mathbf{Y}$. As such it cannot outperform the optimal estimator of $S_{2,k}$ given by $\mathbf{Y}$, namely the estimator $\hat{\mathbf{S}}_2 = \mathsf{E}[\mathbf{S}_2 | \mathbf{Y}]$. Consequently, for every $\delta > 0$ it follows by (72) that there exists an $\nu_0(\delta) > 0$ such that for all $\nu > \nu_0(\delta)$ and all $c \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}[(S_{2,k} - c\hat{S}_{1,k})^2] &\geq \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}[(S_{2,k} - \hat{S}_{1,k})^2] \\ &> D_2^* - \delta. \end{aligned} \tag{75}$$

Rewriting (75) gives

$$\sigma^2 - 2c \frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}[S_{2,k}\hat{S}_{1,k}] + c^2(\sigma^2 - D_1^* + \delta) > D_2^* - \delta$$

and choosing

$$c = \sqrt{\frac{\sigma^2 - D_2^* - \delta}{\sigma^2 - D_1^* + \delta}}$$

yields that for all $\nu > \nu_0(\delta)$

$$\begin{aligned} &\frac{1}{n_\nu} \sum_{k=1}^{n_\nu} \mathsf{E}[S_{2,k}\hat{S}_{1,k}] \\ &\quad \leq \sqrt{(\sigma^2 - D_1^* + \delta)(\sigma^2 - D_2^* - \delta)} \\ &\quad = \sqrt{(\sigma^2 - D_1^*)(\sigma^2 - D_2^*) - \delta(D_2^* - D_1^* + \delta)} \\ &\quad \leq \sqrt{(\sigma^2 - D_1^*)(\sigma^2 - D_2^*) + \delta(\sigma^2 + \delta)}. \end{aligned}$$

□

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