Sending a Bivariate Gaussian Over a Gaussian MAC

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***Abstract*—We study the power-versus-distortion tradeoff for the distributed transmission of a memoryless bivariate Gaussian source over a two-to-one average-power limited Gaussian multiple-access channel. In this problem, each of two separate transmitters observes a different component of a memoryless bivariate Gaussian source. The two transmitters then describe their source component to a common receiver via an average-power constrained Gaussian multiple-access channel. From the output of the multiple-access channel, the receiver wishes to reconstruct each source component with the least possible expected squared-error distortion. Our interest is in characterizing the distortion pairs that are simultaneously achievable on the two source components. We focus on the "equal bandwidth" case, where the source rate in source-symbols per second is equal to the channel rate in channel-uses per second. We present sufficient conditions and necessary conditions for the achievability of a distortion pair. These conditions are expressed as a function of the channel signal-to-noise ratio (SNR) and of the source correlation. In several cases, the necessary conditions and sufficient conditions are shown to agree. In particular, we show that if the channel SNR is below a certain threshold, then an uncoded transmission scheme is optimal. Moreover, we introduce a "source-channel vector-quantizer" scheme which is asymptotically optimal as the SNR tends to infinity.**

***Index Terms*—Achievable distortion, combined source-channel coding, correlated sources, Gaussian multiple-access channel, Gaussian source, mean squared-error distortion, multiple-access channel, uncoded transmission.**

I. Introduction

We study the power-versus-distortion tradeoff for the distributed transmission of a memoryless bivariate Gaussian source over a two-to-one average-power limited Gaussian multiple-access channel. In this problem, each of two separate transmitters observes a different component of a memoryless bivariate Gaussian source. The two transmitters then separately describe their source component to a common receiver via an average-power constrained Gaussian multiple-access channel. From the output of the multiple-access channel, the receiver wishes to reconstruct each source component with the least possible expected squared-error distortion. Our interest is in characterizing the distortion pairs that are simultaneously achievable on the two source components. The focus is on the "equal bandwidth" case where the source rate in source-symbols per second is equal to the channel rate in channel-uses per second.

We present sufficient conditions and necessary conditions for the achievability of a distortion pair. These conditions are expressed as a function of the channel signal-to-noise ratio (SNR) and of the source correlation. In several cases the necessary conditions and sufficient conditions are shown to agree, thus yielding a full characterization of the achievable distortions. In particular, we show that if the channel SNR is below a certain threshold (that we compute), then an uncoded transmission scheme is optimal. This uncoded result is reminiscent of Goblick's result [1] that for the transmission of a Gaussian source over an AWGN channel the minimal squared-error distortion is achieved by uncoded transmission, but in our setting, uncoded transmission is only optimal for some SNRs. For communication at higher SNRs, we introduce a "source-channel vector-quantizer" scheme, which we show is asymptotically optimal as the SNR tends to infinity.

Our problem can be viewed as a lossy Gaussian version of the problem addressed by Cover, El Gamal, and Salehi [2] (see also [3], [4]) in which a bivariate finite-alphabet source is to be transmitted losslessly over a two-to-one multiple-access channel. Our problem is also related to the quadratic Gaussian two-terminal source-coding problem [5], [6] and to the quadratic Gaussian CEO problem [7], [8]. In both of these problems, correlated Gaussians are described distributedly to a central receiver. However, in the quadratic Gaussian CEO problem the interest is in reconstructing a single Gaussian random variable that underlies the observations of the different transmitters, rather than reconstructing each transmitter's observation itself, but, more importantly, the above two problems are source-coding problems whereas ours is one of combined source-channel coding. We emphasize that, as our results show, source-channel separation is suboptimal for our setting.

The problem of transmitting correlated sources over multiple-access channels has so far only been studied sparsely. One of the first results is due to Cover, El Gamal, and Salehi [2] who presented sufficient conditions for the lossless transmission of a finite-alphabet bivariate source over a multiple-access channel. Later, several variations of this problem were considered. Salehi [9] studied a lossy version of the problem with a finite-alphabet source and arbitrary distortion measures on each source component. For this problem he derived sufficient conditions for the achievability of a distortion pair. More recently, another variation where the two source components are binary with Hamming distortion and where the multiple-access channel is Gaussian was considered by Murugan, Gopala, and El Gamal [10] who derived sufficient conditions for the

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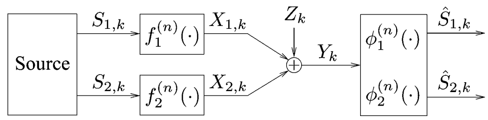

Fig. 1. Bivariate Gaussian source with one-to-two Gaussian multiple-access channel.

achievability of a distortion pair. In [11], Gündüz *et al.* studied the transmission of correlated sources over several multiuser channel models with correlated receiver side-information. Necessary and sufficient conditions for the optimality of source-channel separation were obtained and shown to agree for certain source and side-information structures. In [12]–[14], Rajesh *et al.* studied the transmission of correlated sources over a multiple-access channel with side-information. Gastpar [15] considered a combined source-channel coding analog of the quadratic Gaussian CEO problem. In this problem, distributed transmitters observe independently corrupted versions of the same univariate Gaussian source. These transmitters are connected to a central receiver by means of a many-to-one Gaussian multiple-access channel. The central receiver wishes to reconstruct the original univariate source as accurately as possible. For this problem, Gastpar showed that the minimal expected squared-error distortion is achieved by an uncoded transmission scheme. The extension of our problem to the case where perfect causal feedback from the receiver to each transmitter is available is studied in [16] (see also [17]).

II. Problem Statement

A. Setup

Our setup is illustrated in Fig. 1. A memoryless bivariate Gaussian source is connected to a two-to-one Gaussian multiple-access channel. Each transmitter observes one of the source components and wishes to describe it to the common receiver. The source symbols produced at time $k \in \mathbb{Z}$ are denoted by $(S_{1,k}, S_{2,k})$. The source symbols $\{(S_{1,k}, S_{2,k})\}$ are independent identically distributed (IID) zero-mean Gaussians of covariance matrix

$$\mathsf{K}_{SS} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \quad (1)$$

where $\rho \in [-1, 1]$ and where $0 < \sigma_1^2,\ \sigma_2^2 < \infty$. The sequence $\{S_{1,k}\}$ of the first source component is observed by Transmitter 1 and the sequence $\{S_{2,k}\}$ of the second source component is observed by Transmitter 2. The two source components are to be described over the multiple-access channel to the common receiver by means of the channel input sequences $\{X_{1,k}\}$ and $\{X_{2,k}\}$, where $x_{1,k} \in \mathbb{R}$ and $x_{2,k} \in \mathbb{R}$. The corresponding time-k channel output is given by

$$Y_k = X_{1,k} + X_{2,k} + Z_k \quad (2)$$

where Z_k is the time-k additive noise term, and where $\{Z_k\}$ are IID zero-mean variance-N Gaussian random variables that are independent of the source sequence.

For the transmission of the source $\{S_{1,k}, S_{2,k}\}$, we consider block encoding schemes and denote the block-length by n and the corresponding n-sequences in boldface, e.g., $\mathbf{S}_1 = (S_{1,1}, S_{1,2}, \ldots, S_{1,n})$. Transmitter i is modeled as a function $f_i^{(n)}\colon \mathbb{R}^n \to \mathbb{R}^n$ which produces the channel input sequence $\mathbf{X}_i$ based on the observed source sequence $\mathbf{S}_i = (S_{i,1}, S_{i,2}, \ldots, S_{i,n})$, i.e.,

$$\mathbf{X}_i = f_i^{(n)}(\mathbf{S}_i), \quad i \in \{1, 2\}. \quad (3)$$

The channel input sequences are subjected to expected average-power constraints

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\Big[X_{i,k}^2\Big] \le P_i, \quad i \in \{1, 2\} \quad (4)$$

for some given $P_i > 0$.

The decoder consists of two functions $\phi_i^{(n)}\colon \mathbb{R}^n \to \mathbb{R}^n$, $i \in \{1, 2\}$, which form estimates $\hat{\mathbf{S}}_i$ of the respective source sequences $\mathbf{S}_i$, based on the observed channel output sequence $\mathbf{Y}$, i.e.,

$$\hat{\mathbf{S}}_i = \phi_i^{(n)}(\mathbf{Y}), \quad i \in \{1, 2\}. \quad (5)$$

Our interest is in the pairs of expected squared-error distortions that can be achieved simultaneously on the source-pair as the blocklength n tends to infinity. In view of this, we next define the notion of achievability.

B. Achievable Distortion Pairs

Definition II.1: Given $\sigma_1,\ \sigma_2 > 0$, $\rho \in [-1, 1]$, $P_1,\ P_2 > 0$, and $N > 0$ we say that the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is *achievable* if there exists a sequence of encoding functions $\{f_1^{(n)}, f_2^{(n)}\}$ as in (3), satisfying the average-power constraints(4), and a sequence of reconstruction pairs $\{\phi_1^{(n)}, \phi_2^{(n)}\}$ as in (5), such that the average distortions resulting from these encoding and reconstruction functions satisfy

$$\varlimsup_{n \to \infty} \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[\left(S_{i,k} - \hat{S}_{i,k}\right)^2\right] \le D_i, \quad i \in \{1, 2\}$$

whenever

$$\mathbf{Y} = f_1^{(n)}(\mathbf{S}_1) + f_2^{(n)}(\mathbf{S}_2) + \mathbf{Z}$$

and where $\{(S_{1,k}, S_{2,k})\}$ are IID zero-mean bivariate Gaussian vectors of the covariance matrix K_{SS} in (1) and $\{Z_k\}$ are IID zero-mean variance-N Gaussians that are independent of $\{(S_{1,k}, S_{2,k})\}$.

The problem we address here is, for given $\sigma_1^2,\ \sigma_2^2,\ \rho$, $P_1,\ P_2$, and N, to find the set of pairs (D_1, D_2) such that $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable. Sometimes, we will refer to the set of all (D_1, D_2) such that $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable as the distortion region associated with $(\sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$. In that sense, we will often say, with respect to some $(\sigma_1, \sigma_2, \rho, P_1, P_2, N)$, that the pair (D_1, D_2) is achievable, instead of saying that the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2,\ \rho, P_1, P_2, N)$ is achievable.

C. Normalization

We now show that, without loss in generality, the source law in (1) can be restricted to a simpler form. This restriction will simplify the statement of our results and their derivations.

Reduction II.1: For the problem stated in Sections II-A and II-B, there is no loss in generality in restricting the source law to satisfy

$$\sigma_1^2 = \sigma_2^2 = \sigma^2 \quad \text{and} \quad \rho \in [0,1]. \tag{6}$$

Proof: The proof follows by noting that the described problem has certain symmetry properties with respect to the source law. We prove the reductions on the source variance and on the correlation coefficient separately.

i) The restriction to non-negative correlation coefficients $\rho \in [0,1]$ incurs no loss in generality because the optimal distortion region depends on the correlation coefficient only via its absolute value $|\rho|$. That is, the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable if, and only if, the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, -\rho, P_1, P_2, N)$ is achievable. To see this, note that if $\{f_1^{(n)}, f_2^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\}$ achieves the distortion (D_1, D_2) for the source of correlation coefficient ρ, then $\{\tilde{f}_1^{(n)}, f_2^{(n)}, \tilde{\phi}_1^{(n)}, \phi_2^{(n)}\}$, where

$$\tilde{f}_1^{(n)}(\mathbf{S}_1) = f_1^{(n)}(-\mathbf{S}_1) \quad \text{and} \quad \tilde{\phi}_1^{(n)}(\mathbf{Y}) = -\phi_1^{(n)}(\mathbf{Y})$$

achieves (D_1, D_2) on the source with correlation coefficient $-\rho$.

ii) The restriction to equal variances $\sigma_1^2 = \sigma_2^2 = \sigma^2$ incurs no loss of generality because the distortion region scales linearly with the source variances. That is, the tuple $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable if, and only if, for every $\alpha_1, \alpha_2 \in \mathbb{R}^+$, the tuple $(\alpha_1 D_1, \alpha_2 D_2, \alpha_1\sigma_1^2, \alpha_2\sigma_2^2, \rho, P_1, P_2, N)$ is achievable. This can be seen as follows. If $\{f_1^{(n)}, f_2^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}\}$ achieves $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$, then the combination of the encoders

$$\tilde{f}_i^{(n)}(\mathbf{S}_i) = f_i^{(n)}\left(\frac{\mathbf{S}_i}{\sqrt{\alpha_i}}\right), \quad i \in \{1,2\}$$

with the reconstructors

$$\tilde{\phi}_i^{(n)}(\mathbf{Y}) = \sqrt{\alpha_i} \cdot \phi_i^{(n)}(\mathbf{Y}), \quad i \in \{1,2\}$$

achieves the tuple $(\alpha_1 D_1, \alpha_2 D_2, \alpha_1\sigma_1^2, \alpha_2\sigma_2^2, \rho, P_1, P_2, N)$, and by an analogous argument it follows that if $(\alpha_1 D_1, \alpha_2 D_2, \alpha_1\sigma_1^2, \alpha_2\sigma_2^2, \rho, P_1, P_2, N)$ is achievable, then also $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ is achievable. □

In view of Reduction II.1 we assume for the remainder that the source law additionally satisfies (6).

D. "Symmetric Version" and a Convexity Property

The "symmetric version" of our problem corresponds to the case where the transmitters are subjected to the same power constraint, and where we seek to achieve the same distortion on each source component. That is, $P_1 = P_2 = P$, and we are interested in the minimal distortion that is simultaneously achievable on $\{S_{1,k}\}$ and on $\{S_{2,k}\}$

$$D^*(\sigma^2, \rho, P, N) \triangleq \inf\{D \colon (D, D, \sigma^2, \sigma^2, \rho, P, P, N) \text{ is achievable}\}.$$

In this case, we define the SNR as P/N and seek the distortion $D^*(\sigma^2, \rho, P, N)$.

We conclude this section with a convexity property of the achievable distortions.

Remark II.1: If both $(D_1, D_2, \sigma_1^2, \sigma_2^2, \rho, P_1, P_2, N)$ and $(\tilde{D}_1, \tilde{D}_2, \sigma_1^2, \sigma_2^2, \rho, \tilde{P}_1, \tilde{P}_2, N)$ are achievable, then

$$\Big(\lambda D_1 + \bar{\lambda}\tilde{D}_1, \lambda D_2 + \bar{\lambda}\tilde{D}_2, \sigma_1^2, \sigma_2^2, \rho, \lambda P_1 + \bar{\lambda}\tilde{P}_1, \lambda P_2 + \bar{\lambda}\tilde{P}_2, N\Big)$$

is also achievable for every $\lambda \in [0,1]$, where $\bar{\lambda} = (1-\lambda)$.

Proof: Follows by a time-sharing argument. □

III. Preliminaries: Sending a Bivariate Gaussian Over an AWGN Channel

In this section we lay the ground for our main results. We study a point-to-point analog of the multiple-access problem described in Section II-A. More concretely, we consider the transmission of a memoryless bivariate Gaussian source, subject to expected squared-error distortion on each source component, over the additive white Gaussian noise (AWGN) channel. For this problem, we characterize the power-versus-distortion tradeoff and show that below a certain SNR threshold, an uncoded transmission scheme is optimal. This problem is simpler than our multiple-access problem because here source-channel separation is optimal.

A. Problem Statement

The setup considered in this section is illustrated in Fig. 2. It differs from the multiple-access problem of Section II-A in that the two source sequences $\mathbf{S}_1$ and $\mathbf{S}_2$ are observed and transmitted jointly by one single transmitter and not by two distributed transmitters. Thus, the channel input sequence $\mathbf{X}$ is a function $f^{(n)} \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ of the source sequences $(\mathbf{S}_1, \mathbf{S}_2)$, i.e.,

$$\mathbf{X} = f^{(n)}(\mathbf{S}_1, \mathbf{S}_2). \tag{7}$$

This channel input sequence is subjected to an average-power constraint

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[X_k^2\right] \le P \tag{8}$$

for some given $P > 0$.

The remainder of the problem statement is as in the multiple-access problem. The source law is assumed to be given by (1) and to satisfy (6). The reconstruction functions are as defined in (5), and the achievability of distortion pairs is defined analogously to Section II-A. Our interest is in the set of achievable distortion pairs (D_1, D_2).

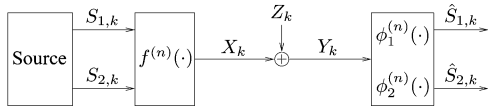

Fig. 2. Bivariate Gaussian source with additive white Gaussian noise channel.

B. Rate-Distortion Function of a Bivariate Gaussian

Denoting the rate-distortion function of the source $\{(S_{1,k}, S_{2,k})\}$ by $R_{S_1,S_2}(D_1, D_2)$, the set of achievable distortion pairs is given by all pairs (D_1, D_2) satisfying

$$R_{S_1,S_2}(D_1, D_2) \leq \frac{1}{2}\log_2\left(1 + \frac{P}{N}\right). \tag{9}$$

We next compute the rate-distortion function $R_{S_1,S_2}(D_1, D_2)$.

Theorem III.1: The rate-distortion function $R_{S_1,S_2}(D_1, D_2)$ is given by

$$R_{S_1,S_2}(D_1, D_2) = \begin{cases} \frac{1}{2}\log_2^+\left(\frac{\sigma^2}{D_{\min}}\right), & \text{if } (D_1, D_2) \in \mathcal{D}_1 \\ \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1 D_2}\right), & \text{if } (D_1, D_2) \in \mathcal{D}_2 \\ \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1 D_2 - (\rho\sigma^2 - \varrho(D_1, D_2))^2}\right), & \text{if } (D_1, D_2) \in \mathcal{D}_3 \end{cases} \tag{10}$$

where

$$\varrho(D_1, D_2) = \sqrt{(\sigma^2 - D_1)(\sigma^2 - D_2)} \tag{11}$$

$\log_2^+(x) = \max\{0, \log_2(x)\}$, where $D_{\min} = \min\{D_1, D_2\}$, and where, using the shorthand notation $\upsilon = \sigma^2(1-\rho^2)$, the regions $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ are given by

$$\mathcal{D}_1 = \Bigg\{(D_1, D_2) : \left(0 \leq D_1 \leq \upsilon, D_2 \geq \upsilon + \rho^2 D_1\right) \text{ or } \left(\upsilon < D_1 \leq \sigma^2, D_2 \geq \upsilon + \rho^2 D_1, D_2 \leq \frac{D_1 - \upsilon}{\rho^2}\right)\Bigg\}$$

$$\mathcal{D}_2 = \Bigg\{(D_1, D_2) : 0 \leq D_1 \leq \upsilon, \quad 0 \leq D_2 < (\upsilon - D_1)\frac{\sigma^2}{\sigma^2 - D_1}\Bigg\}$$

$$\mathcal{D}_3 = \Bigg\{(D_1, D_2) : \left(0 \leq D_1 \leq \upsilon, \quad (\upsilon - D_1)\frac{\sigma^2}{\sigma^2 - D_1} \leq D_2 < \upsilon + \rho^2 D_1\right) \text{ or } \left(\upsilon < D_1 \leq \sigma^2, \frac{D_1 - \upsilon}{\rho^2} < D_2 < \upsilon + \rho^2 D_1\right)\Bigg\}.$$

Proof: By [26, Theorem 2, p. 856], the rate-distortion function $R_{S_1,S_2}(D_1, D_2)$ is given by

$$R_{S_1,S_2}(D_1, D_2) = \min_{\substack{P_{\hat{S}_1,\hat{S}_2|S_1,S_2}: \\ \mathsf{E}\left[(S_1 - \hat{S}_1)^2\right] \leq D_1 \\ \mathsf{E}\left[(S_2 - \hat{S}_2)^2\right] \leq D_2}} I(S_1, S_2; \hat{S}_1, \hat{S}_2). \tag{12}$$

To prove Theorem III.1 it remains to solve (12) for all distortion pairs $(D_1, D_2) \in (0, \sigma^2] \times (0, \sigma^2]$. One solution was presented in [20]. An alternative approach can be found in [18, Appendix A] and [19, Appendix A.2]. □

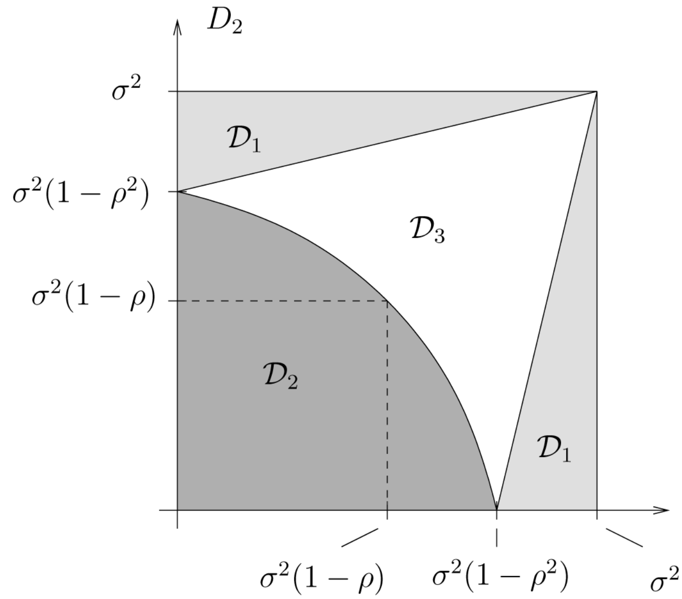

Fig. 3. Regions $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$.

The regions $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ are illustrated in Fig. 3 and can be interpreted as follows. In the region $\mathcal{D}_1$ it is optimal to only describe the component that needs to be reconstructed with the smaller distortion and to then scale the result in order to reconstruct the other component. In the region $\mathcal{D}_2 \cup \mathcal{D}_3$, the distortion pairs (D_1, D_2) can be achieved with the least possible rate $R_{S_1,S_2}(D_1, D_2)$ by first computing two independent linear combinations V_1 and V_2 of the source pair (S_1, S_2), and then quantizing (V_1, V_2) according to the reverse waterfilling principle. For the distortion pairs (D_1, D_2) in $\mathcal{D}_3$ only one of the linear combinations V_1, V_2 is quantized (the one with the larger variance), and for the distortion pairs (D_1, D_2) in $\mathcal{D}_2$ both V_1 and V_2 are quantized. For more details on this source coding procedure see [18, Appendix A] and [19, Appendix A.2].

Remark III.1: Let $R_{S_1}(D_1)$ denote the rate-distortion function for the source component $\{S_{1,k}\}$, i.e.,

$$R_{S_1}(D_1) = \frac{1}{2}\log_2^+\left(\frac{\sigma^2}{D_1}\right)$$

and let $R_{S_2|S_1}(D_2)$ denote the rate-distortion function for $\{S_{2,k}\}$ when $\{S_{1,k}\}$ is given as side-information to both the encoder and the decoder, i.e.,

$$R_{S_2|S_1}(D_2) = \frac{1}{2}\log_2^+\left(\frac{\sigma^2(1-\rho^2)}{D_2}\right).$$

Then, for every $(D_1, D_2) \in \mathcal{D}_2$ the rate-distortion function $R_{S_1,S_2}(D_1, D_2)$ satisfies

$$\begin{aligned} R_{S_1,S_2}(D_1, D_2) &= \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1 D_2}\right) \\ &\overset{a)}{=} \frac{1}{2}\log_2^+\left(\frac{\sigma^2}{D_1}\right) + \frac{1}{2}\log_2^+\left(\frac{\sigma^2(1-\rho^2)}{D_2}\right) \\ &= R_{S_1}(D_1) + R_{S_2|S_1}(D_2) \end{aligned}$$

where $a)$ holds because for $(D_1, D_2) \in \mathcal{D}_2$ we have $D_1 \le \sigma^2$ and $D_2 \le \sigma^2(1-\rho^2)$.

C. Optimal Uncoded Scheme

As an alternative to the separation-based approach, we now present an uncoded scheme that, for all SNR below a certain threshold, is optimal. The optimality of this uncoded scheme will be useful for understanding a similar result in the multiple-access problem.

The uncoded scheme can be described as follows. At every time instant $k \in \{1, 2, \dots, n\}$, the transmitter produces a channel input X_k^{u} of the form

$$X_k^{\mathrm{u}}(\alpha,\beta) = \sqrt{\frac{P}{\sigma^2(\alpha^2+2\rho\alpha\beta+\beta^2)}}\,(\alpha S_{1,k} + \beta S_{2,k})$$

for some $\alpha, \beta \in \mathbb{R}$. From the resulting channel output Y_k, the receiver forms a minimum mean squared-error (MMSE) estimate $\hat{S}_{i,k}^{\mathrm{u}}$, $i \in \{1,2\}$, of the source sample $S_{i,k}$, i.e.,

$$\hat{S}_{i,k}^{\mathrm{u}} = \mathsf{E}[S_{i,k}|Y_k], \quad i \in \{1,2\}.$$

The corresponding expected distortions on $\{S_{1,k}\}$ and on $\{S_{2,k}\}$ are

$$\tilde{D}_i^{\mathrm{u}}(\alpha,\beta) = \sigma^2 \frac{\xi_i(\alpha,\beta,\rho,P,N)}{(P+N)^2(\alpha^2+2\rho\alpha\beta+\beta^2)}, \quad i \in \{1,2\}$$

where

$$\begin{aligned}\xi_1(\alpha,\beta,\rho,P,N) &= P^2\beta^2(1-\rho^2) \\ &\quad + PN\big(\alpha^2+2\rho\alpha\beta+\beta^2(2-\rho^2)\big) \\ &\quad + N^2\big(\alpha^2+2\rho\alpha\beta+\beta^2\big) \\ \xi_2(\alpha,\beta,\rho,P,N) &= P^2\alpha^2(1-\rho^2) \\ &\quad + PN\big(\beta^2+2\rho\alpha\beta+\alpha^2(2-\rho^2)\big) \\ &\quad + N^2\big(\alpha^2+2\rho\alpha\beta+\beta^2\big).\end{aligned}$$

The optimality of this uncoded scheme below a certain SNR-threshold is stated next. To this end, define

$$\Gamma\left(D_1,\sigma^2,\rho\right) = \begin{cases} \frac{\sigma^4(1-\rho^2)-2D_1\sigma^2(1-\rho^2)+D_1^2}{D_1(\sigma^2(1-\rho^2)-D_1)}, & \text{if } 0 < D_1 < \sigma^2(1-\rho^2) \\ +\infty, & \text{else.}\end{cases} \tag{13}$$

Proposition III.1: Let (D_1, D_2) be an achievable distortion pair for the point-to-point setting. If

$$\frac{P}{N} \le \Gamma\left(D_1,\sigma^2,\rho\right) \tag{14}$$

then there exist $\alpha^*, \beta^* \ge 0$ such that

$$\tilde{D}_1^{\mathrm{u}}(\alpha^*,\beta^*) \le D_1 \quad \text{and} \quad \tilde{D}_2^{\mathrm{u}}(\alpha^*,\beta^*) \le D_2.$$

Proof: See Appendix A. □

In the symmetric case, Proposition III.1 simplifies as follows.

Corollary III.1: Let $D > 0$ be such that (D, D) is an achievable distortion pair for the point-to-point problem. If

$$\frac{P}{N} \le \frac{2\rho}{1-\rho} \tag{15}$$

then the pair (D, D) is achieved by the uncoded scheme with time-k channel input

$$X_k^{\mathrm{u}}(\alpha,\alpha) = \sqrt{\frac{P}{2\sigma^2(1+\rho)}}\,(S_{1,k}+S_{2,k}), \quad k \in \{1,2,\dots,n\}.$$

Corollary III.1 can also be verified without relying on Proposition III.1. This is discussed in the following remark.

Remark III.2: The distortions resulting from the uncoded scheme with any choice of (α, β) such that $\alpha = \beta$ are

$$\tilde{D}_i^{\mathrm{u}}(\alpha,\alpha) = \sigma^2\frac{P(1-\rho)+2N}{2(P+N)}, \quad i \in \{1,2\}.$$

By evaluating the necessary and sufficient condition of (9) for the case where $D_1 = D_2 = D$, it follows that this is indeed the minimal achievable distortion for all P/N satisfying (15).

We now return to our multiple-access problem.

IV. Main Results

A. Necessary Condition for the Achievability of (D_1, D_2)

Theorem IV.1: A necessary condition for the achievability of a distortion pair (D_1, D_2) is

$$R_{S_1,S_2}(D_1,D_2) \le \frac{1}{2}\log_2\left(1+\frac{P_1+P_2+2\rho\sqrt{P_1P_2}}{N}\right). \tag{16}$$

Proof: See Appendix B. □

Remark IV.1: Theorem IV.1 can be extended to a wider class of sources and distortion measures. Indeed, if the source is any memoryless bivariate source (not necessarily zero-mean Gaussian) and if the fidelity measures $d_1(s_1, \hat{s}_1) \ge 0$ and $d_2(s_2, \hat{s}_2) \ge 0$ that are used to measure the distortion in reconstructing each of the source components are arbitrary, then the pair (D_1, D_2) is achievable with powers P_1, P_2 only if

$$\inf_{\substack{P_{\hat{S}_1,\hat{S}_2|S_1,S_2}: \\ \mathsf{E}[d_1(S_1,\hat{S}_1)]\le D_1 \\ \mathsf{E}[d_2(S_2,\hat{S}_2)]\le D_2}} I(S_1,S_2;\hat{S}_1,\hat{S}_2) \le \frac{1}{2}\log\left(1+\frac{P_1+P_2+2\rho_{\max}\sqrt{P_1P_2}}{N}\right) \tag{17}$$

where $\rho_{\max}$ is the Hirschfeld–Gebelein–Rényi maximal correlation between S_1 and S_2, i.e.,

$$\rho_{\max} = \sup \mathsf{E}\left[g(S_1)h(S_2)\right] \tag{18}$$

where the supremum is over all functions $g(\cdot)$, $h(\cdot)$ satisfying

$$\mathsf{E}[g(S_1)] = \mathsf{E}[h(S_2)] = 0 \tag{19}$$

and

$$\mathsf{E}\left[g^2(S_1)\right] = \mathsf{E}\left[h^2(S_2)\right] = 1. \tag{20}$$

For the bivariate Gaussian memoryless source, condition (17) reduces to (16) because in this case $\rho_{\max}$ is equal to ρ [21, Lemma 10.2, p. 182].

Remark IV.2: The necessary condition of Theorem IV.1 corresponds to the necessary and sufficient condition for the achievability of a distortion pair (D_1, D_2) when the source $\{(S_{1,k}, S_{2,k})\}$ is transmitted over a point-to-point AWGN channel of input power constraint $P_1 + P_2 + \rho\sqrt{P_1 P_2}$ (see (9)). This is no coincidence. The proof of Theorem IV.1 (see Appendix B) indeed consists of reducing the multiple-access problem to the problem of transmitting the source $\{(S_{1,k}, S_{2,k})\}$ over an AWGN channel of input power constraint $P_1 + P_2 + \rho\sqrt{P_1 P_2}$.

Theorem IV.1 also generalizes to the multivariate case with more than two source components.

Proposition IV.1: Consider the extention of our problem (as described in Section II) to the multivariate case with ν jointly Gaussian source components, each of zero-mean and variance σ^2, and ν corresponding transmitters. Denote the source output ν-tuple at time k by $(S_{1,k}, S_{2,k}, \ldots, S_{\nu,k})$, the correlation coefficient between the source components $S_{i,k}$ and $S_{j,k}$ by ρ_{ij}, the channel input power constraint associated to source component/transmitter $i \in \{1, 2, \ldots, \nu\}$ by P_i, and the distortion on source component $i \in \{1, 2, \ldots, \nu\}$ by D_i. Finally, denote the rate-distortion function on the source ν-tuple by $R_{S_1,\ldots S_\nu}(D_1, \ldots, D_\nu)$. Then, a necessary condition for the achievability of a distortion tuple $(D_1, D_2, \ldots D_\nu)$ is that

$$R_{S_1,\ldots,S_\nu}(D_1, \ldots, D_\nu) \le \frac{1}{2}\log_2\left(1 + \frac{\sum_{i=1}^{\nu} P_i + 2\sum_{i=1}^{\nu-1}\sum_{j=i+1}^{\nu} |\rho_{ij}|\sqrt{P_i P_j}}{N}\right). \tag{21}$$

Proof: See Appendix B-B. □

We now specialize Theorem IV.1 to the symmetric case. We combine the explicit form of the rate-distortion function in (10) with (16) and substitute (D, D) for (D_1, D_2) to obtain:

Corollary IV.1: In the symmetric case

$$D^*(\sigma^2, \rho, P, N) \ge \begin{cases} \sigma^2 \frac{P(1-\rho^2)+N}{2P(1+\rho)+N} & \text{for } \frac{P}{N} \in \left(0, \frac{\rho}{1-\rho^2}\right] \\ \sigma^2 \sqrt{\frac{(1-\rho^2)N}{2P(1+\rho)+N}} & \text{for } \frac{P}{N} > \frac{\rho}{1-\rho^2}. \end{cases}$$

Corollary IV.1 concludes the section on the necessary condition for the achievability of a distortion pair (D_1, D_2). We now compare this necessary condition to several sufficient conditions. The first sufficient condition that we consider is based on conventional source-channel separation.

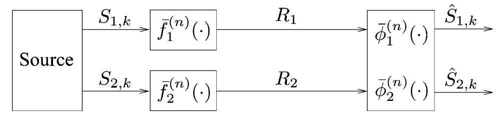

Fig. 4. Distributed source coding problem for a bivariate Gaussian source.

B. Source-Channel Separation

As a benchmark we now consider the set of distortion pairs that are achieved by combining the optimal scheme for the corresponding source-coding problem with the optimal scheme for the corresponding channel-coding problem.

The corresponding source-coding problem is illustrated in Fig. 4. The two source components are observed by two separate encoders. These two encoders wish to describe their source sequence to the common receiver by means of individual rate-limited and error-free bit pipes. The receiver estimates each of the sequences subject to expected squared-error distortion. A detailed description of this problem can be found in [5], [6]. The associated rate-distortion region is given in the next theorem.

Theorem IV.2 (Oohama [5]; Wagner, Tavildar, Viswanath [6]): For the Gaussian two-terminal source coding problem (with source components of unit variances) a distortion-pair (D_1, D_2) is achievable if, and only if

$$(R_1, R_2) \in \mathcal{R}_1(D_1) \cap \mathcal{R}_2(D_2) \cap \mathcal{R}_{\text{sum}}(D_1, D_2)$$

where

$$\mathcal{R}_1(D_1) = \left\{(R_1, R_2) : R_1 \ge \frac{1}{2}\log_2^+\left[\frac{1}{D_1}\left(1 - \rho^2\left(1 - 2^{-2R_2}\right)\right)\right]\right\}$$

$$\mathcal{R}_2(D_2) = \left\{(R_1, R_2) : R_2 \ge \frac{1}{2}\log_2^+\left[\frac{1}{D_2}\left(1 - \rho^2\left(1 - 2^{-2R_1}\right)\right)\right]\right\}$$

$$\mathcal{R}_{\text{sum}}(D_1, D_2) = \left\{(R_1, R_2) : R_1 + R_2 \ge \frac{1}{2}\log_2^+\left[\frac{(1-\rho^2)\beta(D_1, D_2)}{2D_1 D_2}\right]\right\}$$

with

$$\beta(D_1, D_2) = 1 + \sqrt{1 + \frac{4\rho^2 D_1 D_2}{(1-\rho^2)^2}}.$$

The distortions achievable by source-channel separation now follow from combining Theorem IV.2 with the capacity of the Gaussian multiple-access channel (see, e.g., [22] and [23]). We state here the explicit expression for the resulting distortion pairs only for the symmetric case.

Corollary IV.2: In the symmetric case, a distortion D is achievable by source-channel separation if, and only if

$$D \geq \sigma^2 \frac{\sqrt{N(N+2P(1-\rho^2))}}{2P+N}.$$

We next consider several combined source-channel coding schemes. The first scheme is an uncoded scheme.

C. Uncoded Scheme

In this section, we consider an uncoded transmission scheme, which, as we show, is optimal below a certain SNR-threshold.

The uncoded scheme operates as follows. At every time instant k, Encoder $i \in \{1,2\}$ produces as channel input $X_{i,k}$ a scaled version of the time-k source output $S_{i,k}$. The corresponding scaling is such that the average-power constraint of the channel is satisfied. That is

$$X_{i,k}^{\mathrm{u}} = \sqrt{\frac{P_i}{\sigma^2}} S_{i,k}, \quad k \in \{1,2,\ldots,n\}.$$

Based on the resulting time-k channel output Y_k, the decoder then performs an MMSE estimate $\hat{S}_{i,k}^{\mathrm{u}}$ of the source output $S_{i,k}$, $i \in \{1,2\}$, $k \in \{1,2,\ldots,n\}$. That is

$$\hat{S}_{i,k}^{\mathrm{u}} = \mathsf{E}\left[S_{i,k}|Y_k\right], \quad k \in \{1,2,\ldots,n\}.$$

The expected distortions $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ resulting from this uncoded scheme as well as the optimality of the scheme below a certain SNR-threshold are given in the following theorem.

Theorem IV.3: The distortion pairs $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ resulting from the described uncoded scheme are given by

$$D_1^{\mathrm{u}} = \sigma^2 \frac{(1-\rho^2)P_2 + N}{P_1 + P_2 + 2\rho\sqrt{P_1P_2} + N} \tag{22}$$

$$D_2^{\mathrm{u}} = \sigma^2 \frac{(1-\rho^2)P_1 + N}{P_1 + P_2 + 2\rho\sqrt{P_1P_2} + N}. \tag{23}$$

These distortion pairs are optimal, i.e., lie on the boundary of the distortion region, whenever

$$P_2(1-\rho^2)^2\Big(P_1 + 2\rho\sqrt{P_1P_2}\Big) \leq N\rho^2\Big(2P_2(1-\rho^2)+N\Big). \tag{24}$$

Proof: The evaluation of $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ leading to (22) and (23) is given in Appendix C. Based on the expressions for D_1^{u} and D_2^{u} the optimality of the uncoded scheme now follows from verifying that for all P_1, P_2 and N satisfying (24) the corresponding distortion pair $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ satisfies the necessary condition (16) of Theorem IV.1 with equality. To verify this, one can first verify that for all P_1, P_2 and N satisfying (24) we have $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}) \in \mathcal{D}_3$. □

Remark IV.3: The optimality of the uncoded scheme can also be derived in a more conceptual way. To see this, denote by $\mathcal{D}_{\mathrm{MAC}}(\sigma^2,\rho,P_1,P_2,N)$ the distortion region for our multiple-access problem, and by $\mathcal{D}_{\mathrm{PTP}}(\sigma^2,\rho,P,N)$ the distortion region for the point-to-point problem of Section III. The optimality of the uncoded scheme for the multiple-access problem now follows from combining the following three statements.

A)

$$\mathcal{D}_{\mathrm{MAC}}\big(\sigma^2,\rho,P_1,P_2,N\big) \subseteq \mathcal{D}_{\mathrm{PTP}}\Big(\sigma^2,\rho,P_1+P_2+2\rho\sqrt{P_1P_2},N\Big).$$

Statement A) is nothing but a restatement of Theorem IV.1 and Remark IV.2.

B) For the point-to-point problem of Section III with power constraint $P = P_1 + P_2 + 2\rho\sqrt{P_1P_2}$, let $(\tilde{D}_1^{\mathrm{u}}(\alpha,\beta), \tilde{D}_2^{\mathrm{u}}(\alpha,\beta))$ be a distortion pair resulting from the uncoded scheme of Section III-C. For every α, $\beta > 0$, resulting in a channel input sequence $\{X_k\}$ that satisfies the power constraint (8) with equality, we have that if

$$\frac{P_1+P_2+2\rho\sqrt{P_1P_2}}{N} \leq \Gamma(\tilde{D}_1^{\mathrm{u}}(\alpha,\beta),\sigma^2,\rho)$$

where Γ is the threshold function defined in (13), then $(\tilde{D}_1^{\mathrm{u}}(\alpha,\beta), \tilde{D}_2^{\mathrm{u}}(\alpha,\beta))$ lies on the boundary of $\mathcal{D}_{\mathrm{PTP}}(\sigma^2,\rho,P_1+P_2+2\rho\sqrt{P_1P_2},N)$.

Statement B) follows by Proposition III.1 and because the set of distortion pairs $(\tilde{D}_1^{\mathrm{u}}(\alpha,\beta), \tilde{D}_2^{\mathrm{u}}(\alpha,\beta))$ resulting from all $\alpha,\beta > 0$ for which (8) is satisfied with equality, is a convex line segment in the (D_1,D_2)-plane, and, thus, every such pair $(\tilde{D}_1^{\mathrm{u}}(\alpha,\beta), \tilde{D}_2^{\mathrm{u}}(\alpha,\beta))$ lies on the boundary of the distortion region.

C) Let $(\tilde{D}_1^{\mathrm{u}}(\alpha,\beta), \tilde{D}_2^{\mathrm{u}}(\alpha,\beta))$ be the distortion pair resulting from the uncoded scheme for the point-to-point problem, and let $(D_1^{\mathrm{u}}, D_2^{\mathrm{u}})$ be the distortion pair resulting from the uncoded scheme for the multiple-access problem. Then, if

$$\alpha = \sqrt{\frac{P_1}{\sigma^2}} \quad \text{and} \quad \beta = \sqrt{\frac{P_2}{\sigma^2}}$$

then

$$\big(\tilde{D}_1^{\mathrm{u}}(\alpha,\beta), \tilde{D}_2^{\mathrm{u}}(\alpha,\beta)\big) = \big(D_1^{\mathrm{u}}, D_2^{\mathrm{u}}\big).$$

Statement C) follows since in the multiple-access problem with channel inputs $\alpha S_{1,k}$ and $\beta S_{2,k}$, the channel output

$$Y_k = \alpha S_{1,k} + \beta S_{2,k} + Z_k$$

mimics the channel output of the uncoded scheme for the point-to-point problem with channel input $\alpha S_{1,k} + \beta S_{2,k}$. Thus, while in the multiple-access problem the encoders cannot cooperate, the channel performs the addition for them, and since the reconstructors are the same in the multiple-access problem and the point-to-point problem, the resulting distortions are the same in both problems.

Combining Statements A), B), and C) gives that if

$$\frac{P_1+P_2+2\rho\sqrt{P_1P_2}}{N} \leq \Gamma(D_1^{\mathrm{u}},\sigma^2,\rho) \tag{25}$$

then $(D_1^{\rm u}, D_2^{\rm u})$ lies on the boundary of $\mathcal{D}_{\rm MAC}(\sigma^2, \rho, P_1, P_2, N)$, i.e., the uncoded scheme for the multiple-access problem is optimal. The threshold condition (24) now follows by (25) and from substituting therein the value of $D_1^{\rm u}$ by its explicit expression given in (22).

Remark IV.4: In analogy to Remark IV.3, it can also be shown for the multivariate setup of Proposition IV.1 that if the correlation coefficients ρ_{ij} between the source components are all strictly positive, then uncoded transmission is optimal below some strictly positive SNR-threshold, i.e., that the extension of the scheme described at the beginning of this section to the ν-variate case results in a distortion tuple $\left(D_1^{\rm u}, D_2^{\rm u}, \dots, D_\nu^{\rm u}\right)$ that lies on the boundary of the distortion region of the corresponding problem. The respective statements correponding to A), B), and C) of Remark IV.3 are as follows.

A') The distortion region for the multiple-access problem with power constraints P_i, $i \in \{1, 2, \dots, \nu\}$, is a subset of the distortion region of the associated point-to-point problem with power constraint

$$P = \sum_{i=1}^{\nu} P_i + 2\sum_{i=1}^{\nu-1}\sum_{j=i+1}^{\nu} \rho_{ij}\sqrt{P_i P_j}. \tag{26}$$

Statement A') follows from Proposition IV.1 and the adaptation of Remark IV.2 to the ν-variate case.

B') To every ν-tuple of positive constants $\alpha_1, \dots, \alpha_\nu$ there corresponds some threshold power $P' > 0$ such that the following holds: If $P \le P'$ and the positive constant α is such that the second moment of $\alpha\left(\alpha_1 S_{1,k} + \cdots + \alpha_\nu S_{\nu,k}\right)$ is P, then an uncoded transmission scheme that sends $X_k = \alpha\left(\alpha_1 S_{1,k} + \cdots + \alpha_\nu S_{\nu,k}\right)$ achieves a distortion tuple that is on the boundary of the optimal distortion region of the point-to-point problem with allowed power P.

To prove Statement B') it suffices to show that, for the point-to-point source-coding problem, a scheme where the transmitter only describes to the receiver a linear combination $\alpha_1 S_{1,k} + \cdots + \alpha_\nu S_{\nu,k}$ whenever the available source-coding rate is below some threshold, results in a distortion tuple that lies on the boundary of the optimal distortion region. It then follows by Goblick's result [1] that for the point-to-point problem an uncoded transmission scheme that sends $X_k = \alpha\left(\alpha_1 S_{1,k} + \cdots + \alpha_\nu S_{\nu,k}\right)$ achieves a distortion tuple that is on the boundary of the distortion region whenever the allowed power P is below some threshold P'.

An optimal scheme for the source-coding problem is described in [18, Appendix A.2, pp. 24 and Remark A.2, p. 26]. It consists of scaling the source components with some coefficients $c_1, c_2, \dots, c_\nu > 0$; unitarily decorrelating the tuple $(c_1 S_{1,k}, c_2 S_{2,k}, \dots, c_\nu S_{\nu,k})$ to obtain ν independent random variables $V_1, V_2, \dots, V_\nu$; and then applying the reverse waterfilling principle on $V_1, V_2, \dots, V_\nu$. Combining the generalizations, to the multivariate case, of [18, Remark A.4, p. 27] and [18, Remark A.3, Part ii), p. 26] yields that for every $c_1, c_2, \dots, c_\nu > 0$ the distortion tuple resulting from this scheme lies on the boundary of the optimal distortion region. It, thus, remains to show that for every $\alpha_1, \dots, \alpha_\nu > 0$ there exist $c_1, c_2, \dots, c_\nu > 0$ and some positive rate-threshold below which this scheme reduces to describing to the receiver only the linear combination $\alpha_1 S_{1,k} + \cdots + \alpha_\nu S_{\nu,k}$. To show this, we use the following lemma.

Lemma IV.1: Let $S_{1,k}, \dots, S_{\nu,k}$ be as in Proposition IV.1 with the additional assumption that the pairwise correlations are all positive. To every $\alpha_1, \alpha_2, \dots, \alpha_\nu > 0$ there correspond $c_1, c_2, \dots, c_\nu > 0$ such that: the covariance matrix Λ of $(c_1 S_{1,k}, c_2 S_{2,k}, \dots, c_\nu S_{\nu,k})$ has a largest eigenvalue $\lambda^* = 1$ of algebraic multiplicity 1; corresponding to λ^* there is an eigenvector $\mathbf{u}^*$ of positive components $u_1^*, \dots, u_\nu^*$; and

$$\begin{pmatrix} u_1^* c_1 \\ u_2^* c_2 \\ \vdots \\ u_\nu^* c_\nu \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_\nu \end{pmatrix}. \tag{27}$$

Proof: Let

$$c_i = \sqrt{\frac{\alpha_i}{\sum_{j=1}^{\nu} \alpha_j \rho_{ij}}} \quad i \in \{1, 2, \dots, \nu\} \tag{28}$$

where ρ_{ij} is the correlation coefficient between $S_{i,k}$ and $S_{j,k}$, and define

$$u_i^* = \frac{\alpha_i}{c_i} \quad i \in \{1, 2, \dots, \nu\}. \tag{29}$$

By (29) it follows immediately that (27) holds. Also, for $\mathbf{u}^*$ as defined in (29), and c_i as in (28), it is easily verified that $\mathbf{u}^*$ is an eigenvector of Λ with corresponding eigenvalue 1: one merely computes Λ, substitutes $\mathbf{u}^*$ as given in (29), and verifies that $\Lambda\mathbf{u}^* = \mathbf{u}^*$.

It remains to prove that 1 is the largest eigenvalue of the covariance matrix Λ, and that its algebraic multiplicity is 1. To this end, we first note that the matrix Λ is (componentwise) positive. This follows because the pairwise correlations of $S_{1,k}, \dots, S_{\nu,k}$ are positive and because the c_i's as defined in (28) are positive. Because Λ is positive and because $\mathbf{u}^*$ is a positive eigenvector of Λ, it now follows from [28, Theorem 1.2.2, pp. 5] that 1 is the largest eigenvalue of Λ and that its algebraic multiplicity is 1. □

For given $c_1, c_2, \dots, c_\nu > 0$, consider the result $V_1, V_2, \dots V_\nu$ of decorrelating $(c_1 S_{1,k}, c_2 S_{2,k}, \dots, c_\nu S_{\nu,k})$ using a unitary matrix. Among $V_1, V_2, \dots V_\nu$, the random variable with the largest variance is given by

$$u_1^* c_1 S_{1,k} + u_2^* c_2 S_{2,k} + \cdots + u_\nu^* c_\nu S_{\nu,k} \tag{30}$$

where $\mathbf{u}^*$ is the eigenvector corresponding to the largest eigenvalue of the covariance matrix Λ of $(c_1 S_{1,k}, \dots c_\nu S_{\nu,k})$. Consequently, by Lemma IV.1, c.f. (27), to every $\alpha_1, \alpha_2, \dots, \alpha_\nu > 0$ there correspond some $c_1, c_2, \dots, c_\nu > 0$ such that among the $V_1, V_2, \dots V_\nu$ above, the one with the largest variance is

$$\alpha_1 S_{1,k} + \alpha_2 S_{2,k} + \cdots + \alpha_\nu S_{\nu,k}. \tag{31}$$

Thus, by Lemma IV.1 and by the reverse waterfilling principle, which is used in the optimal source-coding scheme, it follows that for every $\alpha_1, \dots, \alpha_\nu > 0$ there exists some positive rate-threshold below which the optimal source-coding scheme reduces to describing to the receiver only the linear combination $\alpha_1 S_{1,k} + \cdots + \alpha_\nu S_{\nu,k}$.

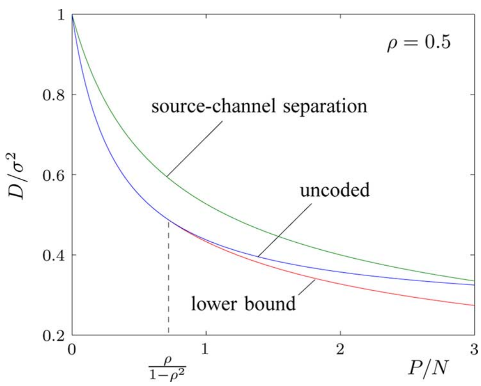

Fig. 5. Upper and lower bounds on $D^*(\sigma^2, \rho, P, N)$ for a source of correlation coefficient $\rho = 0.5$.

C') For the point-to-point problem, the uncoded transmission scheme with channel input $X_k = \alpha_1 S_{1,k} + \cdots + \alpha_{\nu,k} S_{\nu,k}$ and

$$\alpha_i = \sqrt{\frac{P_i}{\sigma^2}}, \quad i \in \{1, 2, \ldots, \nu\}$$

results in the same distortion tuple as the uncoded transmission scheme for the multiple-access problem.

Statement C') holds because first, in the multiple-access case the channel performs the addition of the channel inputs $X_{i,k} = \sqrt{P_i/\sigma^2} S_{i,k}$, $i \in \{1, 2, \ldots, \nu\}$ and, thus, mimics the channel output of the uncoded scheme of the point-to-point problem, and second, in the point-to-point problem and the multiple-access problem the reconstructors of the corresponding uncoded schemes are the same.

That also in the multivariate version of our multiple-access problem uncoded transmission achieves a point on the boundary of the corresponding distortion region now follows from combining Statements A'), B'), and C').

We now specialize Theorem IV.3 to the symmetric case:

Corollary IV.3: In the symmetric case

$$D^* = \sigma^2 \frac{P(1-\rho^2) + N}{2P(1+\rho) + N}, \quad \text{for all } \frac{P}{N} \leq \frac{\rho}{1-\rho^2} \tag{32}$$

where we have used the shorthand notation D^* for $D^*(\sigma^2, \rho, P, N)$. Moreover, for all SNRs below the given threshold, the minimal distortion $D^*(\sigma^2, \rho, P, N)$ is achieved by the uncoded scheme.

The upper and lower bounds on $D^*(\sigma^2, \rho, P, N)$ of Corollaries IV.1–IV.3 are illustrated in Fig. 5 for a source of correlation coefficient $\rho = 0.5$. For SNRs below the threshold of (32) (marked in Fig. 5 by the dashed line) the uncoded approach performs significantly better than the separation-based approach. However, for SNRs above the threshold of (32) the performance of the uncoded scheme deteriorates. By the expressions in (22) and (23), we obtain that in the symmetric case

$$\lim_{P/N \to \infty} D_i^{\mathrm{u}} = \sigma^2 \frac{1-\rho}{2}, \quad i \in \{1, 2\}. \tag{33}$$

That is, as $P/N \to \infty$ the distortion D_i^{u} does not tend to zero. The reason is that as the noise tends to zero, the channel output corresponding to the uncoded scheme tends to $\alpha \mathbf{S}_1 + \beta \mathbf{S}_2$, from which $\mathbf{S}_1$ and $\mathbf{S}_2$ cannot be perfectly recovered.

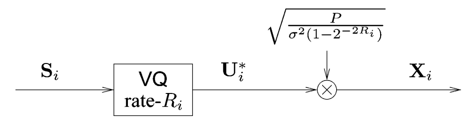

Fig. 6. Encoder of vector-quantizer scheme.

D. Source-Channel Vector-Quantizer Scheme

In this section, we propose a coding scheme that improves on the uncoded scheme at high SNR. It also outperforms the source-channel separation approach. In this scheme the signal transmitted by each encoder is a vector-quantized version of its source sequence. In contrast to the separation-based scheme, the vector-quantized sequences are not mapped to bits before they are transmitted. Instead, the vector-quantized sequences are fed directly to the channel themselves. This transfers some of the correlation from the source to the channel inputs with the channel inputs still being from discrete sets, thereby enabling the decoder to make distinct estimates of $\mathbf{S}_1$ and of $\mathbf{S}_2$. For this scheme, we derive the achievable distortions and, based on those and on the necessary condition of Theorem IV.1, deduce the high SNR asymptotics of an optimal scheme.

The structure of an encoder of our scheme is illustrated in Fig. 6. First, the source sequence $\mathbf{S}_i$ is quantized by a rate-R_i vector-quantizer. The resulting quantized sequence is denoted by $\mathbf{U}_i^*$. For its transmission over the channel, this sequence is scaled so as to satisfy the average-power constraint of (4). That is, the channel input sequence $\mathbf{X}_i$ is given by

$$\mathbf{X}_i = \sqrt{\frac{P}{\sigma^2(1-2^{-2R_i})}} \mathbf{U}_i^*, \quad i \in \{1, 2\}.$$

Based on the channel output $\mathbf{Y}$ resulting from $\mathbf{X}_1$ and $\mathbf{X}_2$, the decoder then estimates the two source sequences $\mathbf{S}_1$ and $\mathbf{S}_2$. It does this in two steps. First, it tries to recover the sequences $\mathbf{U}_1^*$ and $\mathbf{U}_2^*$ from the channel output sequence $\mathbf{Y}$ by performing joint decoding that takes into consideration the correlation between $\mathbf{U}_1^*$ and $\mathbf{U}_2^*$. The resulting decoded sequences are denoted by $\hat{\mathbf{U}}_1$ and $\hat{\mathbf{U}}_2$ respectively. In the second step, the decoder performs approximate MMSE estimates $\hat{\mathbf{S}}_i$, $i \in \{1, 2\}$, of the source sequences $\mathbf{S}_i$ based on $\hat{\mathbf{U}}_1$ and $\hat{\mathbf{U}}_2$, i.e.,

$$\begin{aligned} \hat{\mathbf{S}}_i &= \gamma_{i1} \hat{\mathbf{U}}_1 + \gamma_{i2} \hat{\mathbf{U}}_2 \\ &\approx \mathsf{E}\left[\mathbf{S}_i \middle| \hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2\right]. \end{aligned}$$

A detailed description of the scheme is given in Appendix D.

The distortion pairs achieved by this vector-quantizer scheme are stated in the following theorem.

Theorem IV.4: The distortions achieved by the vector-quantizer scheme are all pairs (D_1, D_2) satisfying

$$D_1 > \sigma^2 2^{-2R_1} \cdot \frac{1-\rho^2(1-2^{-2R_2})}{1-\tilde{\rho}^2}$$

$$D_2 > \sigma^2 2^{-2R_2} \cdot \frac{1-\rho^2(1-2^{-2R_1})}{1-\tilde{\rho}^2}$$

where the rate-pair (R_1, R_2) satisfies

$$R_1 < \frac{1}{2}\log_2\left(\frac{P_1(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)}\right) \quad (34)$$

$$R_2 < \frac{1}{2}\log_2\left(\frac{P_2(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)}\right) \quad (35)$$

$$R_1 + R_2 < \frac{1}{2}\log_2\left(\frac{P_1+P_2+2\tilde{\rho}\sqrt{P_1P_2}+N}{N(1-\tilde{\rho}^2)}\right) \quad (36)$$

and where

$$\tilde{\rho} = \rho\sqrt{(1-2^{-2R_1})(1-2^{-2R_2})}. \quad (37)$$

Proof: See Appendix D. □

Remark IV.5: The coefficient $\tilde{\rho}$ corresponds to the asymptotic average correlation coefficient between two time-k channel inputs $X_{1,k}$ and $X_{2,k}$.

Based on Theorem IV.4, we now derive two more results: we show that for the symmetric version of our problem, source-channel separation is suboptimal also at high SNR, and we determine the precise high-SNR asymptotics of an optimal scheme. We begin with the sub-optimality of source-channel separation. To this end, we restate Theorem IV.4 more specifically for the symmetric case.

Corollary IV.4: In the symmetric case

$$D^*(\sigma^2, \rho, P, N) \leq \sigma^2 2^{-2R}\frac{1-\rho^2(1-2^{-2R})}{1-\rho^2(1-2^{-2R})^2}$$

where

$$R < \frac{1}{4}\log_2\left(\frac{2P(1+\rho(1-2^{-2R}))+N}{N(1-\rho^2(1-2^{-2R})^2)}\right).$$

By comparing the achievable distortion of the vector-quantizer scheme (Corollary IV.4) with the achievable distortion of the separation-based scheme (Corollary IV.2) we obtain:

Corollary IV.5: In the symmetric case with $\rho > 0$, source-channel separation is suboptimal for all $P > 0$.

We turn to the high-SNR asymptotics of an optimal scheme. To this end, let (D_1^*, D_2^*) denote an arbitrary distortion pair resulting from an optimal scheme. For a subset of those distortion pairs, the high SNR behavior is described in the following theorem.

Theorem IV.5 (High-SNR Distortion): The high-SNR asymptotic behavior of (D_1^*, D_2^*) is given by

$$\lim_{N\to 0}\frac{P_1+P_2+2\rho\sqrt{P_1P_2}}{N}D_1^*D_2^* = \sigma^4(1-\rho^2)$$

provided that $D_1^* \leq \sigma^2$ and $D_2^* \leq \sigma^2$, and that

$$\lim_{N\to 0}\frac{N}{P_1D_1^*} = 0 \quad \text{and} \quad \lim_{N\to 0}\frac{N}{P_2D_2^*} = 0. \quad (38)$$

Proof: See Appendix E. □

We restate Theorem IV.5 more specifically for the symmetric case. Since there $D_1^* = D_2^* = D^*(\sigma^2, \rho, P, N)$, condition (38) is implicitly satisfied. Thus:

Corollary IV.6: In the symmetric case

$$\lim_{P/N\to\infty}\sqrt{\frac{P}{N}}D^*(\sigma^2, \rho, P, N) = \sigma^2\sqrt{\frac{1-\rho}{2}}. \quad (39)$$

Remark IV.6: Corollary IV.6 can also be deduced without Theorem IV.5, by comparing the distortion of the vector-quantizer scheme in Corollary IV.4 to the lower bound on $D^*(\sigma^2, \rho, P, N)$ in Corollary IV.1.

For some intuition on the coefficient on the RHS of (39), let us first rewrite (39) as follows:

$$D^*(\sigma^2, \rho, P, N) \approx \sigma^2\sqrt{\frac{N(1-\rho^2)}{2P(1+\rho)}}, \quad \text{as } \frac{P}{N} \gg 1.$$

Next, let us compare this asymptotic behavior to that of two suboptimal schemes: the best separation-based scheme and the suboptimal separation-based scheme that completely ignores the source correlation, i.e., the best scheme where the transmitters and the receiver treat the two source components as if they where independent. Denoting the distortion of the best separation-based scheme by D_{SB} and the distortion of the scheme that ignores the source correlation by D_{IC}, gives

$$D_{\mathrm{SB}} \approx \sigma^2\sqrt{\frac{N(1-\rho^2)}{2P}}, \quad \text{as } \frac{P}{N} \gg 1$$

$$D_{\mathrm{IC}} \approx \sigma^2\sqrt{\frac{N}{2P}}, \quad \text{as } \frac{P}{N} \gg 1.$$

The asymptotic expression for D_{SB} follows by Corollary IV.2 and the asymptotic expression for D_{IC} follows from combining the rate-distortion function of a Gaussian random variable, see, e.g., [24, Theorem 13.3.2, p. 344], with the capacity region of the Gaussian multiple-access channel, see, e.g., [24, Section 14.3.6, p. 403].

The asymptotic behavior can now be understood as follows. The denominator under the square-root corresponds to the average power that the scheme under discussion produces on the sum of the channel inputs $X_{1,k} + X_{2,k}$. In the two separation-based approaches this average power is $2P$, and in the vector-quantizer scheme this average power is $2P(1+\rho)$ as $P/N \to \infty$. The numerator under the square-root consists of the noise variance N multiplied by a coefficient reflecting the gain due to the logical exploitation of the source correlation. For the scheme ignoring the source correlation this coefficient is, by definition of the scheme, equal to 1, i.e., no gain, whereas for the best separation-based scheme and for the vector-quantizer scheme this coefficient is equal to $1 - \rho^2$. The means by which this gain is obtained in the best separation-based scheme and in the vector-quantizer scheme are fundamentally different. In the separation-based scheme the gain is achieved by a generalized form of Slepian–Wolf coding (see [5]), whereas in the vector-quantizer scheme the gain is achieved by joint-typicality decoding that takes into consideration the correlation between the transmitted sequences $\mathbf{U}_1^*$ and $\mathbf{U}_2^*$ (see Theorem IV.4). The corresponding advantage of the vector-quantizer scheme is that by performing the logical exploitation only at the receiver, it additionally allows for exploiting the source correlation in a physical way, i.e., by producing a power boost in the transmitted signal pair.

E. Superposition Approach

The last scheme of this paper is a combination of the previously considered uncoded scheme and vector-quantizer scheme. One way to combine these schemes would be by time- and power-sharing. As stated in Remark II.1, this would result in a convexification of the union of the achievable distortions of the two individual schemes. In this section, we propose a better approach where the two schemes are superimposed. In the symmetric case, this approach results in better performances than time- and power-sharing, and for all SNRs, the resulting distortion is very close to the lower bound on $D^*(\sigma^2,\rho,P,N)$ of Corollary IV.1. We also point out that for the simpler problem of transmitting a univariate memoryless Gaussian source over a point-to-point AWGN channel subject to expected squared-error distortion, a similar superposition approach was shown in [25] to yield a continuum of optimal schemes.

The superimposed scheme can be described as follows. The channel input sequence $\mathbf{X}_i$ produced by Encoder i, $i\in\{1,2\}$, is a linear combination of the source sequence $\mathbf{S}_i$ and its rate-R_i vector-quantized version $\mathbf{U}_i^*$. That is

$$\mathbf{X}_i=\alpha_i\mathbf{S}_i+\beta_i\mathbf{U}_i^* \tag{40}$$

where the sequence $\mathbf{U}_i^*$ is obtained in exactly the same way as in the vector-quantizer scheme, and where the coefficients α_i and β_i are chosen so that the sequence $\mathbf{X}_i$ satisfies the power constraint (4), and so that the receiver can, with high probability, recover the transmitted codeword pair $(\mathbf{U}_1^*,\mathbf{U}_2^*)$. As we shall see, these two conditions will be satisfied as long as α_i and β_i, $i\in\{1,2\}$ satisfy to within some ϵ's and δ's

$$\alpha_i\in\left[0,\frac{P_i}{\sigma^2}\right],\quad \beta_i=\sqrt{\frac{P_i-\alpha_i^2\sigma^2 2^{-2R_i}}{\sigma^2(1-2^{-2R_i})}}-\alpha_i. \tag{41}$$

(For a precise statement see Appendix F).

From the resulting channel output $\mathbf{Y}=\mathbf{X}_1+\mathbf{X}_2+\mathbf{Z}$, the decoder then makes a guess $(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)$ of the transmitted sequences $(\mathbf{U}_1^*,\mathbf{U}_2^*)$. This guess is obtained by joint typicality decoding that takes into consideration the correlation between $\mathbf{U}_1^*$, $\mathbf{U}_2^*$, $\mathbf{S}_1$ and $\mathbf{S}_2$. From the sequences $\hat{\mathbf{U}}_1$, $\hat{\mathbf{U}}_2$, and $\mathbf{Y}$, the decoder then computes approximate MMSE estimates $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$ of the source sequences $\mathbf{S}_1$ and $\mathbf{S}_2$, i.e.,

$$\hat{\mathbf{S}}_i=\gamma_{i1}\hat{\mathbf{U}}_1+\gamma_{i2}\hat{\mathbf{U}}_2+\gamma_{i3}\mathbf{Y},\quad i\in\{1,2\} \tag{42}$$

where the coefficients γ_{ij} are chosen such that $\hat{\mathbf{S}}_i\approx\mathsf{E}\left[\mathbf{S}_i\middle|\mathbf{Y},\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2\right]$. To state the explicit form of coefficients γ_{ij}, define for any rate pair (R_1,R_2), where $R_i\geq 0$, the 3×3 matrix $\mathsf{K}(R_1,R_2)$ by

$$\mathsf{K}(R_1,R_2)\triangleq\begin{pmatrix}\mathsf{k}_{11} & \mathsf{k}_{12} & \mathsf{k}_{13}\\ \mathsf{k}_{12} & \mathsf{k}_{22} & \mathsf{k}_{23}\\ \mathsf{k}_{13} & \mathsf{k}_{23} & \mathsf{k}_{33}\end{pmatrix} \tag{43}$$

where

$$\begin{aligned}\mathsf{k}_{11}&=\sigma^2(1-2^{-2R_1})\\ \mathsf{k}_{12}&=\sigma^2\rho(1-2^{-2R_1})(1-2^{-2R_2})\\ \mathsf{k}_{13}&=(\alpha_1+\beta_1+\alpha_2\rho)\mathsf{k}_{11}+\beta_2\mathsf{k}_{12}\\ \mathsf{k}_{22}&=\sigma^2(1-2^{-2R_2})\\ \mathsf{k}_{23}&=(\alpha_2+\beta_2+\alpha_1\rho)\mathsf{k}_{22}+\beta_1\mathsf{k}_{12}\\ \mathsf{k}_{33}&=\alpha_1^2\sigma^2+2\alpha_1\beta_1\mathsf{k}_{11}+2\alpha_1\alpha_2\rho\sigma^2+2\alpha_1\beta_2\rho\mathsf{k}_{22}\\ &\quad+\beta_1^2\mathsf{k}_{11}+2\beta_1\alpha_2\rho\mathsf{k}_{11}+2\beta_1\beta_2\mathsf{k}_{12}+2\alpha_2\beta_2\mathsf{k}_{22}\\ &\quad+\alpha_2^2\sigma^2+\beta_2^2\mathsf{k}_{22}+N.\end{aligned}$$

The coefficients γ_{ij} are then given by

$$\begin{pmatrix}\gamma_{i1}\\ \gamma_{i2}\\ \gamma_{i3}\end{pmatrix}\triangleq\mathsf{K}^{-1}(R_1,R_2)\begin{pmatrix}\mathsf{c}_{i1}\\ \mathsf{c}_{i2}\\ \mathsf{c}_{i3}\end{pmatrix},\quad i\in\{1,2\} \tag{44}$$

where

$$\begin{aligned}\mathsf{c}_{11}&=\mathsf{k}_{11}\\ \mathsf{c}_{12}&=\rho\mathsf{k}_{22}\\ \mathsf{c}_{13}&=(\alpha_1+\alpha_2\rho)\sigma^2+\beta_1\mathsf{k}_{11}+\beta_2\rho\mathsf{k}_{22}\\ \mathsf{c}_{21}&=\rho\mathsf{k}_{11}\\ \mathsf{c}_{22}&=\mathsf{k}_{22}\\ \mathsf{c}_{23}&=(\alpha_2+\alpha_1\rho)\sigma^2+\beta_1\rho\mathsf{k}_{11}+\beta_2\mathsf{k}_{22}.\end{aligned}$$

The distortions achieved by the superimposed scheme are now given in the following theorem.

Theorem IV.6: The distortions achieved by the superposition approach are all pairs (D_1,D_2) satisfying

$$D_i>\sigma^2-\gamma_{i1}\mathsf{c}_{i1}-\gamma_{i2}\mathsf{c}_{i2}-\gamma_{i3}\mathsf{c}_{i3},\quad i\in\{1,2\}$$

where the rate-pair (R_1,R_2) satisfies

$$\begin{aligned}R_1&<\frac{1}{2}\log_2\left(\frac{\beta_1'^2\mathsf{k}_{11}(1-\tilde{\rho}^2)+N'}{N'(1-\tilde{\rho}^2)}\right)\\ R_2&<\frac{1}{2}\log_2\left(\frac{\beta_2'^2\mathsf{k}_{22}(1-\tilde{\rho}^2)+N'}{N'(1-\tilde{\rho}^2)}\right)\\ R_1+R_2&<\frac{1}{2}\log_2\left(\frac{\beta_1'^2\mathsf{k}_{11}+\beta_2'^2\mathsf{k}_{22}+2\tilde{\rho}\beta_1'\beta_2'\sqrt{\mathsf{k}_{11}\mathsf{k}_{22}}+N'}{N'(1-\tilde{\rho}^2)}\right)\end{aligned}$$

for some α_1, α_2, β_1, and β_2 satisfying (41) and where

$$N'=\alpha_1^2\nu_1+\alpha_2^2\nu_2+2\alpha_1\alpha_2\nu_3+N \tag{45}$$

where

$$\nu_1 = \sigma^2 - (1-a_1\tilde{\rho})^2 \mathsf{k}_{11} - 2(1-a_1\tilde{\rho})a_1\mathsf{k}_{12} - a_1^2\mathsf{k}_{22}$$
$$\nu_2 = \sigma^2 - (1-a_2\tilde{\rho})^2 \mathsf{k}_{22} - 2(1-a_2\tilde{\rho})a_2\mathsf{k}_{12} - a_2^2\mathsf{k}_{11}$$
$$\nu_3 = \rho\sigma^2 - \big((1-a_1\tilde{\rho})(1-a_2\tilde{\rho}) + a_1a_2\big)\mathsf{k}_{12} - (1-a_1\tilde{\rho})a_2\mathsf{k}_{11} - (1-a_2\tilde{\rho})a_1\mathsf{k}_{22}$$

with

$$\beta_1' = \alpha_1(1-a_1\tilde{\rho}) + \beta_1 + \alpha_2 a_2 \quad (46)$$
$$\beta_2' = \alpha_2(1-a_2\tilde{\rho}) + \beta_2 + \alpha_1 a_1 \quad (47)$$

and with

$$a_1 = \frac{\rho 2^{-2R_1}(1-2^{-2R_2})}{\eta_1}, \quad a_2 = \frac{\rho 2^{-2R_2}(1-2^{-2R_1})}{\eta_2} \quad (48)$$

where

$$\eta_1 = (1-2^{-2R_2}) - 2\tilde{\rho}^2\sqrt{(1-2^{-2R_1})(1-2^{-2R_2})} + \tilde{\rho}^2(1-2^{-2R_1}) \quad (49)$$
$$\eta_2 = (1-2^{-2R_1}) - 2\tilde{\rho}^2\sqrt{(1-2^{-2R_1})(1-2^{-2R_2})} + \tilde{\rho}^2(1-2^{-2R_2}). \quad (50)$$

Proof: See Appendix F. □

In the symmetric case where $P_1 = P_2 = P$, $R_1 = R_2 = R$ and where $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$, the matrix $\mathsf{K}(R,R)$ and the coefficients γ_{ij} reduce to

$$\mathsf{K}(R,R) = \begin{pmatrix} \mathsf{k}_1 & \mathsf{k}_2 & \mathsf{k}_3 \\ \mathsf{k}_2 & \mathsf{k}_1 & \mathsf{k}_3 \\ \mathsf{k}_3 & \mathsf{k}_3 & \mathsf{k}_4 \end{pmatrix}$$

where

$$\mathsf{k}_1 = \sigma^2(1-2^{-2R})$$
$$\mathsf{k}_2 = \sigma^2\rho(1-2^{-2R})^2$$
$$\mathsf{k}_3 = (\alpha+\beta+\alpha\rho)\mathsf{k}_1 + \beta\mathsf{k}_2$$
$$\mathsf{k}_4 = 2\alpha\mathsf{c}_3 + 2\beta\mathsf{k}_3 + N$$

and

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \triangleq \mathsf{K}^{-1}(R,R) \begin{pmatrix} \mathsf{c}_1 \\ \mathsf{c}_2 \\ \mathsf{c}_3 \end{pmatrix}$$

where

$$\mathsf{c}_1 = \mathsf{k}_1$$
$$\mathsf{c}_2 = \rho\mathsf{k}_1$$
$$\mathsf{c}_3 = (\alpha\sigma^2 + \beta\mathsf{k}_1)(1+\rho).$$

Thus, in the symmetric case Theorem IV.6 simplifies as follows.

Corollary IV.7: With the superposition approach in the symmetric case we can achieve the distortion

$$\inf\{\sigma^2 - \gamma_1\mathsf{c}_1 - \gamma_2\mathsf{c}_2 - \gamma_3\mathsf{c}_3\}$$

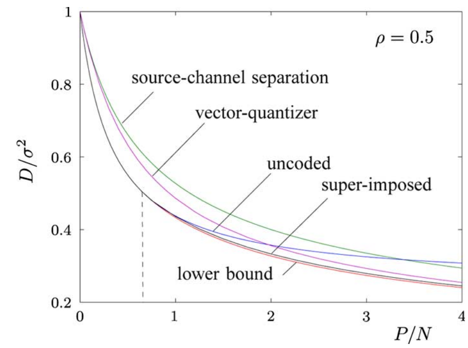

Fig. 7. Upper and lower bounds on $D^*(\sigma^2, \rho, P, N)$ for a source of correlation coefficient $\rho = 0.5$.

where the infimum is over all rates R satisfying

$$R < \frac{1}{4}\log_2\left(\frac{2\beta'^2\mathsf{k}_1(1+\tilde{\rho}) + N'}{N'(1-\tilde{\rho}^2)}\right)$$

for some α and β satisfying

$$\alpha \in \left[0, \frac{P}{\sigma^2}\right] \quad \text{and} \quad \beta = \sqrt{\frac{P - \alpha^2\sigma^2 2^{-2R}}{\sigma^2(1-2^{-2R})}} - \alpha \quad (51)$$

and where

$$\beta' = \alpha\left(1 + \frac{\rho 2^{-2R}}{1-\tilde{\rho}^2}(1-\tilde{\rho})\right) + \beta$$

and

$$N' = 2\alpha^2(\nu_1 + \nu_3) + N$$

with

$$\nu_1 = \sigma^2 2^{-2R}\frac{1-\rho\tilde{\rho}}{1-\tilde{\rho}^2}, \quad \nu_3 = \sigma^2\rho\frac{2^{-4R}}{1-\tilde{\rho}^2}.$$

Fig. 7 illustrates the various bounds on $D^*(\sigma^2, \rho, P, N)$.

V. Summary

We studied the power-versus-distortion tradeoff for the distributed transmission of a memoryless bivariate Gaussian source over a two-to-one average-power limited Gaussian multiple-access channel. In this problem, each of two separate transmitters observes a different component of a memoryless bivariate Gaussian source. The two transmitters then describe their source component to a common receiver via a Gaussian multiple-access channel with average-power constraints on each channel input sequence. From the output of the multiple-access channel, the receiver wishes to reconstruct each source component with the least possible expected squared-error distortion. Our interest was in characterizing the distortion pairs that are simultaneously achievable on the two source components. These pairs are a function of the power constraints and the variance of the additive channel noise, as well as of the source variance and of the correlation coefficient between the two source components.

We first considered a different (nondistributed) problem, which was the point-to-point analog of our multiple-access problem. That is, we studied the power-versus-distortion tradeoff for the transmission of a memoryless bivariate Gaussian source over the AWGN channel, subject to expected squared-error distortion on each source component. For this problem, we determined the set of achievable distortion pairs by deriving the explicit expression for the rate-distortion function of a memoryless bivariate Gaussian source. Moreover, we showed that below a certain SNR-threshold an uncoded transmission scheme is optimal.

For the multiple-access problem, we then derived the following.

- A necessary condition for the achievability of a distortion pair (Theorem IV.1). This condition was obtained by reducing the multiple-access problem to a point-to-point problem. The key step was to upper bound the maximal correlation between the channel inputs by using a result from maximum correlation theory.
- The optimality of an uncoded transmission scheme below a certain SNR-threshold (Theorem IV.3). In the symmetric case, this result becomes (Corollary IV.3)

$$D^*(\sigma^2, \rho, P, N) = \sigma^2 \frac{P(1-\rho^2)+N}{2P(1+\rho)+N}$$

  for all $P/N \leq \rho/(1-\rho^2)$. The strength of the underlying uncoded scheme is that it translates the entire source correlation onto the channel inputs, and thereby boosts the received power of the transmitted signal pair. Its weakness is that it allows the receiver to recover only the sum of the channel inputs.
- A sufficient condition based on a "source-channel vector-quantizer" scheme (Theorem IV.4). The motivation behind this scheme was to overcome the weakness of the uncoded scheme. To this end, rather than transmitting the source components in an uncoded manner, the scheme transmits a scaled version of the optimally vector-quantized source components (without channel coding).
- The precise high-SNR asymptotics of an optimal transmission scheme, which in the symmetric case are given by (Corollary IV.6)

$$\lim_{P/N \longrightarrow \infty} \sqrt{\frac{P}{N}} D^*(\sigma^2, \rho, P, N) = \sigma^2 \sqrt{\frac{1-\rho}{2}}.$$

  The achievability part of this result follows from the "source-channel vector-quantizer" scheme (Theorem IV.4) and the inachievability part from the necessary condition of Theorem IV.1.
- The suboptimality, in the symmetric case, of source-channel separation at all SNRs. This follows by comparing the best separation-based approach (Corollary IV.2) with the achievable distortions from the "source-channel vector-quantizer" scheme (Corollary IV.4).
- A sufficient condition based on a superposition of the uncoded scheme and the vector-quantizer scheme (Theorem IV.6). In the symmetric case this superposition approach was shown to be optimal or close to optimal at all SNRs.

The presented sufficient conditions indicate that for the efficient exploitation of the source correlation it is necessary not only to exploit the source correlation in a logical way, e.g., by Slepian–Wolf-like strategies, but to additionally exploit the source correlation in a physical way. In the considered schemes, this is done by translating the source correlation onto the channel inputs. The logical exploitation of the source correlation is then performed at the receiver-side, e.g., by joint-typicality decoding taking into consideration the correlation between the channel inputs.

Appendix A
Proof of Proposition III.1

Proposition III.1 pertains to the point-to-point problem of Section III, in which the source pair $\{(S_{1,k}, S_{2,k})\}$ is to be transmitted over an AWGN channel. It states that for an achievable distortion pair (D_1, D_2) for which the SNR of the channel satisfies $P/N \leq \Gamma(D_1, \sigma^2, \rho)$, there exist $\alpha^*, \beta^* \geq 0$ such that

$$\tilde{D}_1^{\mathrm{u}}(\alpha^*, \beta^*) \leq D_1 \quad \text{and} \quad \tilde{D}_2^{\mathrm{u}}(\alpha^*, \beta^*) \leq D_2.$$

The essence of Proposition III.1 is that the uncoded scheme proposed in Section III-C achieves every distortion pair (D_1, D_2) in $\mathcal{D}_1 \cup \mathcal{D}_3$ with the least possible transmission power, i.e., with the P for which

$$R_{S_1,S_2}(D_1, D_2) = \frac{1}{2}\log_2\left(1 + \frac{P}{N}\right).$$

In Proposition III.1, the condition $(D_1, D_2) \in \mathcal{D}_1 \cup \mathcal{D}_3$ is merely expressed in form of the threshold $\Gamma(D_1, \sigma^2, \rho)$ on P/N.

We start the proof by showing that the uncoded scheme indeed achieves every $(D_1, D_2) \in \mathcal{D}_1 \cup \mathcal{D}_3$ with the least possible transmission power, respectively at the smallest P/N. To this end, let $\Psi(D_1, D_2)$ be the smallest P/N at which (D_1, D_2) is achievable, i.e.,

$$R_{S_1,S_2}(D_1, D_2) = \frac{1}{2}\log_2(1 + \Psi(D_1, D_2)).$$

We now argue that for every $(D_1, D_2) \in \mathcal{D}_1 \cup \mathcal{D}_3$, there exist α^*, β^* such that the distortions resulting from the uncoded scheme at $P/N = \Psi(D_1, D_2)$ satisfy $(\tilde{D}_1^{\mathrm{u}}(\alpha, \beta), \tilde{D}_2^{\mathrm{u}}(\alpha, \beta)) = (D_1, D_2)$. To this end, we first recall that in [1] it is shown that the minimum expected squared-error transmission of a Gaussian source over a AWGN channel is achieved by uncoded transmission. Next, we recall that for the source coding part of the problem studied in Section III, every (D_1, D_2) in $\mathcal{D}_1 \cup \mathcal{D}_3$ can be achieved with rate $R_{S_1,S_2}(D_1, D_2)$ by optimally vector-quantizing a linear combination of $\mathbf{S}_1$ and $\mathbf{S}_2$ (for details, see [18, Proposition A.1, p. 31]). Thus, since $\{(S_{1,k}, S_{2,k})\}$ are jointly Gaussian, and, therefore, each of their linear combination $\alpha\mathbf{S}_1 + \beta\mathbf{S}_2$ is also Gaussian, it follows in combination with [1] that every distortion pair $(D_1, D_2) \in \mathcal{D}_1 \cup \mathcal{D}_3$ is achieved at $P/N = \Psi(D_1, D_2)$, by sending at every time instant $k \in \{1, 2, \ldots, n\}$

$$X_k^{\mathrm{u}}(\alpha, \beta) = \sqrt{\frac{P}{\sigma^2(\alpha^2 + 2\rho\alpha\beta + \beta^2)}}\,(\alpha S_{1,k} + \beta S_{2,k})$$

with the appropriate $\alpha, \beta \geq 0$.

It remains to derive the threshold function Γ. To this end, first notice that for an arbitrary fixed $D_1 \in [0, \sigma^2]$, the smaller the associated D_2 gets, the larger $\Psi(D_1, D_2)$ becomes, i.e., for a fixed D_1 the function $\Psi(D_1, D_2)$ is decreasing in D_2. Now, for every $D_1 \in [0, \sigma^2]$, let $\bar{D}_2(D_1)$ be the smallest D_2 such that $(D_1, D_2) \in \mathcal{D}_1 \cup \mathcal{D}_3$. Then, for every $D_1 \in [0, \sigma^2]$

$$\Gamma(D_1, \sigma^2, \rho) = \Psi(D_1, \bar{D}_2(D_1)).$$

Hence, it remains to evaluate $\Psi(D_1, \bar{D}_2(D_1))$ for every $D_1 \in [0, \sigma^2]$. Using the shorthand notation $\upsilon = \sigma^2(1-\rho^2)$, we have

$$\bar{D}_2(D_1) = \begin{cases} (\upsilon - D_1)\frac{\sigma^2}{\sigma^2 - D_1}, & \text{if } 0 \leq D_1 \leq \upsilon \\ 0, & \text{if } D_1 > \upsilon\,. \end{cases} \tag{52}$$

For $D_1 > \sigma^2(1-\rho^2)$ it immediately follows that $\Gamma(D_1, \sigma^2, \rho) = \infty$. For $0 \leq D_1 \leq \sigma^2(1-\rho^2)$ the value of $\Psi(D_1, \bar{D}_2(D_1))$, and, hence, the value of $\Gamma(D_1, \sigma^2, \rho)$ follows from solving

$$\frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1\bar{D}_2 - \left(\rho\sigma^2 - \varrho(D_1, \bar{D}_2)\right)^2}\right) = \frac{1}{2}\log_2\left(1+\frac{P}{N}\right) \tag{53}$$

where $\varrho(D_1, D_2)$ is defined in (11), and where we have used the shorthand notation $\bar{D}_2$ for $\bar{D}_2(D_1)$. From (52), we now get

$$\rho\sigma^2 - \sqrt{(\sigma^2 - D_1)(\sigma^2 - \bar{D}_2)} = 0.$$

Thus, (53) reduces to

$$\frac{\sigma^4(1-\rho^2)}{D_1\bar{D}_2} = 1 + \frac{P}{N}. \tag{54}$$

which, by (52), can be rewritten as

$$\frac{P}{N} = \frac{\sigma^4(1-\rho^2) - 2\sigma^2 D_1(1-\rho^2) + D_1^2}{D_1\left(\sigma^2(1-\rho^2) - D_1\right)}. \tag{55}$$

This is the threshold given in Proposition III.1 whenever $0 \leq D_1 \leq \sigma^2(1-\rho^2)$.

To conclude the proof, we justify the restriction to $\alpha \geq 0$ and $\beta \geq 0$. This restriction is made because from the expressions for $\tilde{D}_1^{\mathrm{u}}(\alpha, \beta)$ and $\tilde{D}_2^{\mathrm{u}}(\alpha, \beta)$ it follows that it incurs no loss in performance. This is so, since $\rho \geq 0$, and, thus, the uncoded transmission scheme with the choice of (α, β) such that $\alpha\beta < 0$ yields a distortion that is uniformly worse than the choice of $(|\alpha|, |\beta|)$, and every distortion pair achievable with $\alpha, \beta < 0$, is also achievable with $(|\alpha|, |\beta|)$. Thus, without loss in performance, we can limit ourselves to $\alpha, \beta \geq 0$. □

Appendix B
Proof of Theorem IV.1

We begin with a reduction.

Reduction B.1: There is no loss in optimality in restricting the encoding functions to satisfy

$$\mathsf{E}[X_{i,k}] = 0, \quad \text{for } i \in \{1, 2\}, \text{ and all } k \in \mathbb{Z}. \tag{56}$$

Proof: We show that for every achievable tuple $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2, N)$, there exists a scheme with encoding functions satisfying (56) that achieves this tuple. To this end, let $(D_1, D_2, \sigma^2, \sigma^2, \rho, P_1, P_2, N)$ be an arbitrary achievable tuple. Further, let $\{f_1^{(n)}\}$, $\{f_2^{(n)}\}$, $\{\phi^{(n)}\}$ be sequences of encoding and decoding functions achieving this tuple. If the encoding functions $\{f_1^{(n)}\}$, $\{f_2^{(n)}\}$ do not satisfy (56), then they can be adapted as follows. Before sending the codewords over the channel, the mean of the codewords is subtracted so as to satisfy (56), and at the channel output this subtraction is corrected by adding this term to the received sequence before decoding. □

In view of Reduction B.1, we restrict ourselves, for the remainder of this proof to encoding functions that satisfy (56). The key element in the proof of Theorem IV.1 is the following.

Lemma B.1: Any scheme satisfying condition (56) and the original power constraints (4), also satisfies

$$\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}[(X_{1,k} + X_{2,k})^2] \leq P_1 + P_2 + 2\rho\sqrt{P_1 P_2}. \tag{57}$$

Proof: See Appendix B-A. □

Based on Lemma B.1, the proof of Theorem IV.1 is now obtained by relaxing the original problem as follows. First, the power constraint of (4) is replaced by the power constraint of (57). Then, under the power constraint of (57), the two transmitters are allowed to fully cooperate. These two relaxations reduce the original multiple-access problem to a point-to-point problem where the source sequence $\{(S_{1,k}, S_{2,k})\}$ is to be transmitted over an AWGN channel of power constraint $P_1 + P_2 + 2\rho\sqrt{P_1 P_2}$ and noise variance N. For this point-to-point problem, a necessary condition for the achievability of a distortion pair (D_1, D_2) follows by source-channel separation, and is

$$R_{S_1,S_2}(D_1, D_2) \leq \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2}}{N}\right). \tag{58}$$

It is now easy to conclude that (58) is also a necessary condition for the achievability of a distortion pair (D_1, D_2) in the original multiple-access problem. This simply follows since (58) is a necessary condition for the achievability of a distortion pair (D_1, D_2) in a relaxed version of the multiple-access problem. This concludes the proof of Theorem IV.1. □

A. Proof of Lemma B.1

The key to Lemma B.1 is as follows:

Lemma B.2: For any coding scheme with encoding functions of the form (3) that satisfy the power constraints (4) and condition (56) of Reduction B.1, and where the encoder input sequences are jointly Gaussian as in (1) with non-negative correlation coefficient ρ and equal variances $\sigma_1^2 = \sigma_2^2 = \sigma^2$ (Reduction II.1), any time-k encoder outputs $X_{1,k}$ and $X_{2,k}$ satisfy

$$\mathsf{E}\left[X_{1,k}X_{2,k}\right] \leq \rho\sqrt{\mathsf{E}\left[X_{1,k}^2\right]}\sqrt{\mathsf{E}\left[X_{2,k}^2\right]}. \tag{59}$$

Proof: Lemma B.2 follows from two results from Maximum Correlation Theory. These results are stated now.

Theorem B.1 (Witsenhausen [27]): Consider a sequence of pairs of random variables $\{(W_{1,k}, W_{2,k})\}$, where the pairs are independent (not necessarily identically distributed). Then

$$\sup_{g_1^{(n)}, g_2^{(n)}} \mathsf{E}\left[g_1^{(n)}(\mathbf{W}_1) g_2^{(n)}(\mathbf{W}_2)\right] \leq \sup_{\substack{1\leq k\leq n \\ g_{1,k}, g_{2,k}}} \mathsf{E}\Big[g_{1,k}(W_{1,k}) g_{2,k}(W_{2,k})\Big] \quad (60)$$

where the supremum on the LHS of (60) is taken over all functions $g_i^{(n)} : \mathbb{R}^n \to \mathbb{R}$, satisfying

$$\mathsf{E}\left[g_i^{(n)}(\mathbf{W}_i)\right] = 0, \quad i \in \{1,2\}$$

and

$$\mathsf{E}\left[\left(g_i^{(n)}(\mathbf{W}_i)\right)^2\right] = 1, \quad i \in \{1,2\}$$

and the supremum on the RHS of (60) is taken over all functions $g_{i,k} : \mathbb{R} \to \mathbb{R}$, satisfying

$$\mathsf{E}\left[g_{i,k}^{(n)}(W_{i,k})\right] = 0, \quad i \in \{1,2\}$$

and

$$\mathsf{E}\left[\left(g_{i,k}^{(n)}(W_{i,k})\right)^2\right] = 1, \quad i \in \{1,2\}.$$

Proof: See [27, Theorem 1, p. 105]. □

Lemma B.3: Consider two jointly Gaussian random variables $W_{1,k}$ and $W_{2,k}$ with correlation coefficient ρ_k. Then

$$\sup_{g_{1,k}, g_{2,k}} \mathsf{E}\Big[g_{1,k}(W_{1,k}) g_{2,k}(W_{2,k})\Big] = |\rho_k|$$

where the supremum is taken over all functions $g_{i,k} : \mathbb{R} \to \mathbb{R}$, satisfying

$$\mathsf{E}\left[g_{i,k}(W_{i,k})\right] = 0, \quad i \in \{1,2\}$$

and

$$\mathsf{E}[(g_{i,k}(W_{i,k}))^2] = 1, \quad i \in \{1,2\}.$$

Proof: See [21, Lemma 10.2, p. 182]. □

Lemma B.2 is now merely a consequence of Theorem B.1 and Lemma B.3 applied to our setup. To see this, substitute $\mathbf{W}_1$ and $\mathbf{W}_2$ by the source sequences $\mathbf{S}_1$ and $\mathbf{S}_2$, and let the functions $g_1^{(n)}(\cdot)$ and $g_2^{(n)}(\cdot)$ be the encoding sub-functions that produce the time-k channel inputs $X_{1,k}$ and $X_{2,k}$, i.e., $g_i^{(n)}(\mathbf{S}_i) = X_{i,k}$. Then, for every $k \in \{1, 2, \ldots, n\}$

$$\begin{aligned}\frac{\mathsf{E}[X_{1,k}X_{2,k}]}{\sqrt{\mathsf{E}[X_{1,k}^2]}\sqrt{\mathsf{E}[X_{2,k}^2]}} &\leq \sup_{g_1^{(n)}, g_2^{(n)}} \mathsf{E}\left[g_1^{(n)}(\mathbf{S}_1) g_2^{(n)}(\mathbf{S}_2)\right] \\ &\overset{a)}{\leq} \sup_{\substack{1\leq k\leq n \\ g_{1,k}, g_{2,k}}} \mathsf{E}[g_{1,k}(S_{1,k}) g_{2,k}(S_{2,k})] \\ &\overset{b)}{\leq} \rho \end{aligned} \quad (61)$$

where $a)$ follows from Theorem B.1 and $b)$ follows from Lemma B.3 and from our assumption that $\rho \geq 0$ (Reduction II.1). Thus, for every time k

$$\mathsf{E}[X_{1,k}X_{2,k}] \leq \rho \sqrt{\mathsf{E}[X_{1,k}^2]}\sqrt{\mathsf{E}[X_{2,k}^2]} \quad (62)$$

which is the bound of Lemma B.2. □

Using Lemma B.2, we can now prove the bound of Lemma B.1 as follows:

$$\begin{aligned}&\frac{1}{n}\sum_{k=1}^{n} \mathsf{E}[(X_{1,k} + X_{2,k})^2] \\ &= \frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[X_{1,k}^2] + \frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[X_{1,k}^2] + 2\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[X_{1,k}X_{2,k}] \\ &\leq P_1 + P_2 + 2\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[X_{1,k}X_{2,k}] \\ &\overset{a)}{\leq} P_1 + P_2 + 2\rho\frac{1}{n}\sum_{k=1}^{n}\sqrt{\mathsf{E}[X_{1,k}^2]}\sqrt{\mathsf{E}[X_{2,k}^2]} \\ &\overset{b)}{\leq} P_1 + P_2 + 2\rho\frac{1}{n}\sqrt{\sum_{k=1}^{n}\mathsf{E}[X_{1,k}^2]}\sqrt{\sum_{k=1}^{n}\mathsf{E}[X_{2,k}^2]} \\ &\leq P_1 + P_2 + 2\rho\frac{1}{n}\sqrt{nP_1}\sqrt{nP_2} \\ &= P_1 + P_2 + 2\rho\sqrt{P_1P_2} \end{aligned} \quad (63)$$

where Inequality $a)$ follows by Lemma B.2 and from our assumption that $\rho \geq 0$, and where Inequality $b)$ follows by Cauchy–Schwarz. This concludes the proof of Lemma B.1. □

B. Proof of Proposition IV.1

The proof is a simple generalization of the proof of Theorem IV.1 given above. To see this, we first note that in the mutlivariate case where the correlation coefficients ρ_{ij}, $i, j \in \{1, 2, \ldots, \nu\}$ are not necessarily non-negative, the upper bound of Lemma B.2 on $\mathsf{E}[X_{1,k}X_{2,k}]$ can be written as

$$\mathsf{E}[X_{i,k}X_{j,k}] \leq |\rho_{ij}|\sqrt{\mathsf{E}[X_{i,k}^2]}\sqrt{\mathsf{E}[X_{j,k}^2]} \quad (64)$$

and, as in the derivation of (63), it can be shown that

$$\frac{1}{n}\sum_{k=1}^{n}\mathsf{E}[(X_{1,k} + \cdots + X_{\nu,k})^2] \leq \sum_{i=1}^{\nu} P_i + 2\sum_{i=1}^{\nu-1}\sum_{j=i+1}^{\nu}|\rho_{ij}|\sqrt{P_iP_j}. \quad (65)$$

The proof is then concluded by arguments similar to those in the proof of Theorem IV.1. □

APPENDIX C
DISTORTIONS $(D_1^{\rm u}, D_2^{\rm u})$ OF THE UNCODED SCHEME

The expression for $D_i^{\rm u}$, $i \in \{1,2\}$, is obtained as follows:

$$\begin{aligned} D_i^{\rm u} &= \frac{1}{n}\sum_{k=1}^{n} \mathsf{E}\left[(S_{i,k} - \hat{S}_{i,k}^{\rm u})^2\right] \\ &= \frac{1}{n}\left(\mathsf{E}[S_{i,k}^2] - 2\mathsf{E}[S_{i,k}\hat{S}_{i,k}^{\rm u}] + \mathsf{E}\left[\left(\hat{S}_{i,k}^{\rm u}\right)^2\right]\right) \\ &\overset{a)}{=} \frac{1}{n}\left(\mathsf{E}[S_{i,k}^2] - \mathsf{E}\left[\left(\hat{S}_{i,k}^{\rm u}\right)^2\right]\right) \\ &\overset{b)}{=} \frac{1}{n}\left(\mathsf{E}[S_{i,k}^2] - \frac{(\mathsf{E}[S_{i,k}Y_k])^2}{\mathsf{E}[Y_k^2]}\right) \\ &\overset{c)}{=} \sigma^2 - \sigma^2 \frac{P_1 + 2\rho\sqrt{P_1P_2} + \rho^2 P_2}{P_1 + 2\rho\sqrt{P_1P_2} + P_2 + N} \\ &= \sigma^2 \frac{P_1(1-\rho^2) + N}{P_1 + P_2 + 2\rho\sqrt{P_1P_2} + N} \end{aligned}$$

where in $a)$ we have used that $\hat{S}_{i,k}^{\rm u} = \mathsf{E}[S_{i,k}|Y_k]$ satisfies the Orthogonality Principle; in $b)$ we have used the explicit form of the conditional mean for jointly Gaussians

$$\hat{S}_{i,k}^{\rm u} = \mathsf{E}[S_{i,k}|Y_k] = \frac{\mathsf{E}[S_{1,k}Y_k]}{\mathsf{E}[Y_k^2]}\, Y_k$$

and in $c)$ we have used the calculation

$$\frac{(\mathsf{E}[S_{1,k}Y_k])^2}{\mathsf{E}[Y_k^2]} = \frac{\sigma^2\left(\sqrt{P_1} + \rho\sqrt{P_2}\right)^2}{P_1 + P_2 + 2\rho\sqrt{P_1P_2} + N}.$$

□

APPENDIX D
PROOF OF THEOREM IV.4

In this appendix, we analyze the distortions achievable by the vector-quantizer scheme that was presented in Section IV-D. To start, we give a thorough description of the corresponding coding scheme.

A. Coding Scheme

Fix some $\epsilon > 0$ and rates R_1 and R_2.

Code Construction: Two codebooks $\mathcal{C}_1$ and $\mathcal{C}_2$ are generated independently. Codebook $\mathcal{C}_i$, $i \in \{1,2\}$, consists of 2^{nR_i} codewords $\{\mathbf{U}_i(1), \mathbf{U}_i(2), \ldots, \mathbf{U}_i(2^{nR_i})\}$. The codewords are drawn independently uniformly over the surface of the centered $\mathbb{R}^n$-sphere $\mathcal{S}_i$ of radius $r_i = \sqrt{n\sigma^2(1-2^{-2R_i})}$.

Encoding: Based on the observed source sequence $\mathbf{S}_i$ each encoder produces its channel input $\mathbf{X}_i$ by first vector-quantizing the source sequence $\mathbf{S}_i$ to a codeword $\mathbf{U}_i^* \in \mathcal{C}_i$ and then scaling $\mathbf{U}_i^*$ to satisfy the average-power constraint. To describe the vector-quantizer precisely, denote for every $\mathbf{w}, \mathbf{v} \in \mathbb{R}^n$ where neither $\mathbf{w}$ nor $\mathbf{v}$ are the zero-sequence, the angle between $\mathbf{w}$ and $\mathbf{v}$ by $\sphericalangle(\mathbf{w},\mathbf{v})$, i.e.,

$$\cos\sphericalangle(\mathbf{w},\mathbf{v}) \triangleq \frac{\langle \mathbf{w},\mathbf{v}\rangle}{\|\mathbf{w}\|\|\mathbf{v}\|}. \tag{66}$$

Let $\mathcal{F}(\mathbf{s}_i, \mathcal{C}_i)$ be the set defined in (67), shown at the bottom of the page. The vector-quantizer output $\mathbf{U}_i^*$ is then given as follows: if $\mathcal{F}(\mathbf{s}_i, \mathcal{C}_i) \neq \emptyset$, then $\mathbf{U}_i^*$ is the codeword $\mathbf{U}_i(j) \in \mathcal{F}(\mathbf{s}_i, \mathcal{C}_i)$ that minimizes $|\cos\sphericalangle(\mathbf{u}_i(j), \mathbf{s}_i) - \sqrt{1-2^{-2R_i}}|$, and if $\mathcal{F}(\mathbf{s}_i, \mathcal{C}_i) = \emptyset$, then $\mathbf{U}_i^*$ is the all-zero sequence. This is restated in (68), shown at the bottom of the page. More formally, $\mathbf{U}_i^*$ should be written as $\mathbf{U}_i^*(\mathbf{S}_i, \mathcal{C}_i)$, but we shall usually make these dependencies implicit. The channel input is now given by

$$\mathbf{X}_i = \alpha_i \mathbf{U}_i^* \tag{69}$$

where

$$\alpha_i = \sqrt{\frac{P_i}{\sigma^2(1-2^{-2R_i})}}, \quad i \in \{1,2\}. \tag{70}$$

Since the codebook $\mathcal{C}_i$ is drawn over the centered $\mathbb{R}^n$-sphere of radius $r_i = \sqrt{\sigma^2(1-2^{-2R_i})}$, each channel input $\mathbf{X}_i$ satisfies the average-power constraint individually.

Reconstruction: The receiver's estimate $(\hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2)$ of the source pair $(\mathbf{S}_1, \mathbf{S}_2)$ is derived from the channel output $\mathbf{Y}$ in two steps. First, the receiver makes a guess $(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)$ of the pair $(\mathbf{U}_1^*, \mathbf{U}_2^*)$ by choosing among all "jointly typical pairs" $(\mathbf{U}_1, \mathbf{U}_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ the pair whose linear combination $\alpha_1\mathbf{U}_1 + \alpha_2\mathbf{U}_2$ has the smallest distance to the received sequence $\mathbf{Y}$. More precisely

$$(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2) = \underset{\substack{(\mathbf{U}_1,\mathbf{U}_2)\in\mathcal{C}_1\times\mathcal{C}_2:\\ |\tilde{\rho} - \cos\sphericalangle(\mathbf{u}_1,\mathbf{u}_2)| \leq 7\epsilon}}{\arg\min} \|\mathbf{Y} - (\alpha_1\mathbf{U}_1 + \alpha_2\mathbf{U}_2)\|^2 \tag{71}$$

$$\mathcal{F}(\mathbf{s}_i, \mathcal{C}_i) \triangleq \left\{\mathbf{u}_i \in \mathcal{C}_i : \sqrt{1-2^{-2R_i}}(1-\epsilon) \leq \cos\sphericalangle(\mathbf{s}_i, \mathbf{U}_i) \leq \sqrt{1-2^{-2R_i}}(1+\epsilon)\right\}. \tag{67}$$

$$\mathbf{U}_i^* = \begin{cases} \underset{\substack{\mathbf{U}_i\in\mathcal{C}_i:\\ \mathbf{U}_i\in\mathcal{F}(\mathbf{s}_i,\mathcal{C}_i)}}{\arg\min} \left|\cos\sphericalangle(\mathbf{u}_i(j), \mathbf{S}_i) - \sqrt{1-2^{-2R_i}}\right|, & \text{if } \mathcal{F}(\mathbf{s}_i, \mathcal{C}_i) \neq \emptyset \\ \mathbf{0}, & \text{otherwise} \end{cases} \tag{68}$$

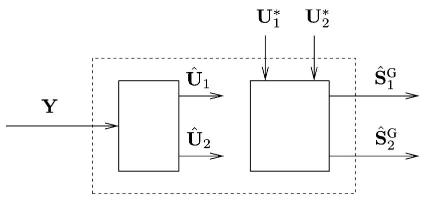

Fig. 8. Genie-aided decoder.

where

$$\tilde{\rho} = \rho\sqrt{(1-2^{-2R_1})(1-2^{-2R_2})}.$$

If the channel output $\mathbf{Y}$ and the codebooks $\mathcal{C}_1$ and $\mathcal{C}_2$ are such that there does not exist a pair $(\mathbf{U}_1, \mathbf{U}_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ that satisfies

$$|\tilde{\rho} - \cos\sphericalangle(\mathbf{u}_1, \mathbf{u}_2)| \leq 7\epsilon \tag{72}$$

then $\hat{\mathbf{U}}_1$ and $\hat{\mathbf{U}}_2$ are chosen to be all-zero.

In the second step, the receiver computes the estimates $(\hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2)$ from the guess $(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)$ by setting

$$\hat{\mathbf{S}}_1 = \gamma_{11}\hat{\mathbf{U}}_1 + \gamma_{12}\hat{\mathbf{U}}_2 \tag{73}$$

$$\hat{\mathbf{S}}_2 = \gamma_{21}\hat{\mathbf{U}}_2 + \gamma_{22}\hat{\mathbf{U}}_1 \tag{74}$$

where

$$\gamma_{11} = \frac{1-\rho^2(1-2^{-2R_2})}{1-\tilde{\rho}^2}, \quad \gamma_{12} = \rho 2^{-2R_1} \tag{75}$$

$$\gamma_{21} = \frac{1-\rho^2(1-2^{-2R_1})}{1-\tilde{\rho}^2}, \quad \gamma_{22} = \rho 2^{-2R_2}. \tag{76}$$

Note that

$$0 < \gamma_{i1} \leq 1 \quad \text{and} \quad 0 < \gamma_{i2} \leq \rho, \quad i \in \{1,2\}. \tag{77}$$

B. Expected Distortion

To analyze the expected distortion we use a genie-aided argument. We first show that, under certain rate constraints, the asymptotic normalized distortion of the proposed scheme remains the same when a certain help from a genie is provided. To derive the achievable distortions it then suffices to analyze the genie-aided version.

1) Genie-Aided Scheme: In the genie-aided scheme, the genie's help is provided to the decoder. An illustration of this genie-aided decoder is given in Fig. 8. The genie provides the decoder with the codeword pair $(\mathbf{U}_1^*, \mathbf{U}_2^*)$. The decoder then estimates the source pair $(\mathbf{S}_1, \mathbf{S}_2)$ based on $(\mathbf{U}_1^*, \mathbf{U}_2^*)$ and ignores the guess $(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)$ produced in the first decoding step. The estimate of this genie-aided decoder is denoted by $(\hat{\mathbf{S}}_1^{\mathrm{G}}, \hat{\mathbf{S}}_2^{\mathrm{G}})$, where

$$\hat{\mathbf{S}}_1^{\mathrm{G}} = \gamma_{11}\mathbf{U}_1^* + \gamma_{12}\mathbf{U}_2^* \tag{78}$$

$$\hat{\mathbf{S}}_2^{\mathrm{G}} = \gamma_{21}\mathbf{U}_2^* + \gamma_{22}\mathbf{U}_1^* \tag{79}$$

with γ_{11}, γ_{12}, γ_{21}, γ_{22} as in (75) and (76). Under certain rate constraints, the normalized asymptotic distortion of this genie-aided scheme is the same as for the originally proposed scheme. This is stated more precisely in the following proposition.

Proposition D.1: For every $\delta > 0$ and $0 < \epsilon < 0.3$ there exists an $n'(\delta, \epsilon) > 0$ such that for all $n > n'(\delta, \epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1\|^2\right] \leq \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] + 2\sigma^2\left(\epsilon + \left(44\sqrt{1+\epsilon} + 61\right)\delta\right)$$

whenever (R_1, R_2) is in the rate region $\mathcal{R}(\epsilon)$ given by

$$\mathcal{R}(\epsilon) = \left\{ R_1 \leq \frac{1}{2}\log_2\left(\frac{P_1(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)} - \kappa_1\epsilon\right) \right.$$
$$R_2 \leq \frac{1}{2}\log_2\left(\frac{P_2(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)} - \kappa_2\epsilon\right)$$
$$\left. R_1 + R_2 \leq \frac{1}{2}\log_2\left(\frac{P_1+P_2+2\tilde{\rho}\sqrt{P_1P_2}+N}{N(1-\tilde{\rho}^2)} - \kappa_3\epsilon\right)\right\}$$

where κ_1, κ_2 and κ_3 depend only on P_1, P_2, N, ζ_1, and ζ_2, where

$$\zeta_1 = \frac{N\tilde{\rho}}{P_1(1-\tilde{\rho}^2)+N}\sqrt{\frac{P_1}{P_2}}$$
$$\zeta_2 = \frac{P_1(1-\tilde{\rho}^2)}{P_1(1-\tilde{\rho}^2)+N}.$$

Proof: See Appendix D-C. □

Corollary D.1: If (R_1, R_2) satisfy

$$R_1 < \frac{1}{2}\log_2\left(\frac{P_1(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)}\right)$$
$$R_2 < \frac{1}{2}\log_2\left(\frac{P_2(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)}\right)$$
$$R_1 + R_2 < \frac{1}{2}\log_2\left(\frac{P_1+P_2+2\tilde{\rho}\sqrt{P_1P_2}+N}{N(1-\tilde{\rho}^2)}\right)$$

then

$$\varlimsup_{n\to\infty}\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1\|^2\right] \leq \varlimsup_{n\to\infty}\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right].$$

Proof: Follows from Proposition D.1 by first letting $n \to \infty$ and then $\epsilon \to 0$ and $\delta \to 0$. □

By Corollary D.1, to analyze the distortion achievable by our scheme it suffices to analyze the genie-aided scheme. This is done in Appendix D-D.

C. Proof of Proposition D.1

The main step in the proof of Proposition D.1 is to show that for every $(R_1, R_2) \in \mathcal{R}(\epsilon)$ and sufficiently large n, the probability of a decoding error, and, thus, the probability of $\hat{\mathbf{S}}_1 \neq \hat{\mathbf{S}}_1^{\mathrm{G}}$, can be made very small. This step is done in the following section. The proof of Proposition D.1 is then completed in Appendix D-C-II.

1) Upper Bound on Probability of a Decoding Error: In this section, we show that for every $(R_1, R_2) \in \mathcal{R}(\epsilon)$ and sufficiently large n, the probability of a decoding error, and, thus, the probability of $\hat{\mathbf{S}}_1 \neq \hat{\mathbf{S}}_1^{\mathrm{G}}$, can be made very small. The hitch is that to upper bound the probability of a decoding error for the proposed scheme, we cannot proceed by the method conventionally used for the multiple-access channel. The reason is that in the conventional analysis of the multiple-access channel it is assumed that the probability of the codewords $\mathbf{u}_i(j)$ does not depend on the realization of the codebook $\mathcal{C}_i$. However, in our combined source-channel coding scheme, the probability of encoder $i \in \{1,2\}$ producing the channel input of index $j \in \{1,2,\ldots,2^{nR_i}\}$ depends not only on the source sequence $\mathbf{s}_i$, but also on the realization of $\mathcal{C}_i$. Another reason the conventional analysis fails is that, conditional on the codebooks $\mathcal{C}_1$ and $\mathcal{C}_2$, the indices produces by the vector-quantizers are dependent.

To address these difficulties, we proceed by a geometric approach. To this end, we introduce an error event related to a decoding error at the receiver. This event is denoted by $\mathcal{E}_{\hat{\mathbf{U}}}$ and consists of all tuples $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z})$ for which there exists a pair $(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2) \neq (\mathbf{u}_1^*,\mathbf{u}_2^*)$ in $\mathcal{C}_1 \times \mathcal{C}_2$ that satisfies Condition (72) of the reconstructor, and for which the Euclidean distance between $\alpha_1\tilde{\mathbf{u}}_1 + \alpha_2\tilde{\mathbf{u}}_2$ and $\mathbf{y}$ is smaller or equal to the Euclidean distance between $\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*$ and $\mathbf{y}$. More formally, $\mathcal{E}_{\hat{\mathbf{U}}} = \mathcal{E}_{\hat{\mathbf{U}}_1} \cup \mathcal{E}_{\hat{\mathbf{U}}_2} \cup \mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)}$ where $\mathcal{E}_{\hat{\mathbf{U}}_1}$, $\mathcal{E}_{\hat{\mathbf{U}}_2}$, and $\mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)}$ are given in (80)–(82), shown at the bottom of the page, where $\mathbf{y} \triangleq \alpha_1\mathbf{u}_1^* + \alpha_1\mathbf{u}_1^* + \mathbf{z}$. Note that a decoding error occurs only if $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) \in \mathcal{E}_{\hat{\mathbf{U}}}$. The main result of this section can now be stated as follows.

Lemma D.1: For every $\delta > 0$ and $0.3 > \epsilon > 0$, there exists an $n_4'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n_4'(\delta,\epsilon)$

$$\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] < 11\,\delta, \quad \text{whenever} \quad (R_1,\, R_2) \in \mathcal{R}(\epsilon).$$

To prove Lemma D.1, we introduce three auxiliary error events. The first auxiliary event is denoted by $\mathcal{E}_{\mathbf{S}}$ and corresponds to an atypical source output. It is given in (83), shown at the bottom of the page. The second auxiliary event is denoted by $\mathcal{E}_{\mathbf{Z}}$ and corresponds to an atypical behavior of the additive noise, and is given in (84), shown at the bottom of the page. Finally, the third auxiliary event is denoted by $\mathcal{E}_{\mathbf{X}}$ and corresponds to irregularities at the encoders. That is,

$$\mathcal{E}_{\hat{\mathbf{U}}_1} = \Big\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) : \exists \tilde{\mathbf{u}}_1 \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\} \text{ s.t. } |\tilde{\rho} - \cos\sphericalangle(\tilde{\mathbf{u}}_1,\mathbf{u}_2^*)| \leq 7\epsilon, \text{ and } \|\mathbf{y} - (\alpha_1\tilde{\mathbf{u}}_1 + \alpha_2\mathbf{u}_2^*)\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2\Big\} \tag{80}$$

$$\mathcal{E}_{\hat{\mathbf{U}}_2} = \Big\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) : \exists \tilde{\mathbf{u}}_2 \in \mathcal{C}_2 \setminus \{\mathbf{u}_2^*\} \text{ s.t. } |\tilde{\rho} - \cos\sphericalangle(\mathbf{u}_1^*,\tilde{\mathbf{u}}_2)| \leq 7\epsilon, \text{ and } \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\tilde{\mathbf{u}}_2)\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2\Big\} \tag{81}$$

$$\mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} = \Big\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) : \exists \tilde{\mathbf{u}}_1 \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\} \text{ and } \exists \tilde{\mathbf{u}}_2 \in \mathcal{C}_2 \setminus \{\mathbf{u}_2^*\} \text{ s.t. } |\tilde{\rho} - \cos\sphericalangle(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)| \leq 7\epsilon, \text{ and } \|\mathbf{y} - (\alpha_1\tilde{\mathbf{u}}_1 + \alpha_2\tilde{\mathbf{u}}_2)\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2\Big\} \tag{82}$$

$$\mathcal{E}_{\mathbf{S}} = \left\{(\mathbf{s}_1,\mathbf{s}_2) \in \mathbb{R}^n \times \mathbb{R}^n : \left|\frac{1}{n}\|\mathbf{s}_1\|^2 - \sigma^2\right| > \epsilon\sigma^2 \text{ or } \left|\frac{1}{n}\|\mathbf{s}_2\|^2 - \sigma^2\right| > \epsilon\sigma^2 \text{ or } |\cos\sphericalangle(\mathbf{s}_1,\mathbf{s}_2) - \rho| > \epsilon\rho\right\} \tag{83}$$

$$\mathcal{E}_{\mathbf{Z}} = \left\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) : \left|\frac{1}{n}\|\mathbf{z}\|^2 - N\right| > \epsilon N \text{ or } \frac{1}{n}|\langle\alpha_1\mathbf{u}_1^*(\mathbf{s}_1,\mathcal{C}_1),\mathbf{z}\rangle| > \sqrt{P_1 N}\epsilon \text{ or } \frac{1}{n}|\langle\alpha_2\mathbf{u}_2^*(\mathbf{s}_2,\mathcal{C}_2),\mathbf{z}\rangle| > \sqrt{P_2 N}\epsilon\right\} \tag{84}$$

the event that one of the codebooks contains no codeword satisfying Condition (67) of the vector-quantizer, or that the two quantized sequences $\mathbf{u}_1^*$ and $\mathbf{u}_2^*$ have an "atypical" angle to each other. More formally, $\mathcal{E}_{\mathbf{X}} = \mathcal{E}_{\mathbf{X}_1} \cup \mathcal{E}_{\mathbf{X}_2} \cup \mathcal{E}_{(\mathbf{X}_1,\mathbf{X}_2)}$ where $\mathcal{E}_{\mathbf{X}_1}$, $\mathcal{E}_{\mathbf{X}_2}$, and $\mathcal{E}_{(\mathbf{X}_1,\mathbf{X}_2)}$ are given in (85)–(87), shown at the bottom of the page. To prove Lemma D.1 we now start with the decomposition

$$\begin{aligned}\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] &= \left[\mathcal{E}_{\hat{\mathbf{U}}} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c \cap \mathcal{E}_{\mathbf{Z}}^c\right] \\ &\quad + \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}} | \mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}} \cup \mathcal{E}_{\mathbf{Z}}\right] \Pr\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}} \cup \mathcal{E}_{\mathbf{Z}}\right] \\ &\le \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c \cap \mathcal{E}_{\mathbf{Z}}^c\right] + \Pr\left[\mathcal{E}_{\mathbf{S}}\right] \\ &\quad + \Pr\left[\mathcal{E}_{\mathbf{X}}\right] + \Pr\left[\mathcal{E}_{\mathbf{Z}}\right] \\ &\le \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}_1} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c \cap \mathcal{E}_{\mathbf{Z}}^c\right] \\ &\quad + \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}_2} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c \cap \mathcal{E}_{\mathbf{Z}}^c\right] \\ &\quad + \Pr\left[\mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c \cap \mathcal{E}_{\mathbf{Z}}^c\right] \\ &\quad + \Pr\left[\mathcal{E}_{\mathbf{S}}\right] + \Pr\left[\mathcal{E}_{\mathbf{X}}\right] + \Pr\left[\mathcal{E}_{\mathbf{Z}}\right] \end{aligned} \tag{88}$$

where we have used the shorthand notation $\Pr[\mathcal{E}_\nu]$ for $\Pr[(\mathbf{S}_1,\mathbf{S}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{Z}) \in \mathcal{E}_\nu]$, and where $\mathcal{E}_\nu^c$ denotes the complement of $\mathcal{E}_\nu$. Lemma D.1 now follows from upper bounding the probability terms on the RHS of (88).

Lemma D.2: For every $\delta > 0$ and $\epsilon > 0$ there exists an $n_1'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n_1'(\delta,\epsilon)$

$$\Pr\left[\mathcal{E}_{\mathbf{S}}\right] < \delta.$$

Proof: The proof follows by the weak law of large numbers. □

Lemma D.3: For every $\epsilon > 0$ and $\delta > 0$ there exists an $n_3'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n_3'(\delta,\epsilon)$

$$\Pr\left[\mathcal{E}_{\mathbf{Z}}\right] < \delta.$$

Proof: The proof follows by the weak law of large numbers and since for every $\epsilon > 0$, as $n \to \infty$

$$\sup_{\substack{\mathbf{u}\in\mathbb{R}^n:\\ \|\mathbf{u}\|=\sqrt{n\sigma^2(1-2^{-2R_i})}}} \Pr\left[\frac{1}{n}\left|\langle \alpha_i \mathbf{u}, \mathbf{z}\rangle\right| > \sqrt{P_i N}\epsilon\right] \longrightarrow 0$$

where $i \in \{1,2\}$. □

Lemma D.4: For every $\delta > 0$ and $0.3 > \epsilon > 0$ there exists an $n_2'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n_2'(\delta,\epsilon)$

$$\Pr\left[\mathcal{E}_{\mathbf{X}}\right] < 6\delta.$$

Proof: This result has nothing to do with the channel; it is a result from rate-distortion theory. A proof for our setting is given in Appendix D-E1. □

Lemma D.5: For every $\delta > 0$ and every $\epsilon > 0$ there exists some $n_4''(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n_4''(\delta,\epsilon)$ Conditions (89)–(91), shown at the bottom of the page, hold in which κ_1, κ_2, and κ_3 are positive constants determined by P_1, P_2, and N.

The proof of Lemma D.5 requires some preliminaries. To this end, define

$$\mathbf{w}(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) \triangleq \zeta_1(\mathbf{y} - \alpha_2\mathbf{u}_2^*) + \zeta_2\alpha_2\mathbf{u}_2^* \tag{92}$$

where

$$\zeta_1 = \frac{N\tilde{\rho}}{P_1(1-\tilde{\rho}^2)+N}\sqrt{\frac{P_1}{P_2}} \tag{93}$$

$$\zeta_2 = \frac{P_1(1-\tilde{\rho}^2)}{P_1(1-\tilde{\rho}^2)+N}. \tag{94}$$

In the remainder, we shall use the shorthand notation $\mathbf{w}$ instead of $\mathbf{w}(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z})$. We now start with a lemma that will be used to prove (89).

$$\mathcal{E}_{\mathbf{X}_1} = \left\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2) : \nexists \mathbf{u}_1 \in \mathcal{C}_1 \text{ s.t. } \left|\sqrt{1-2^{-2R_1}} - \cos\sphericalangle(\mathbf{s}_1,\mathbf{u}_1)\right| \le \epsilon\sqrt{1-2^{-2R_1}}\right\} \tag{85}$$

$$\mathcal{E}_{\mathbf{X}_2} = \left\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2) : \nexists \mathbf{u}_2 \in \mathcal{C}_2 \text{ s.t. } \left|\sqrt{1-2^{-2R_2}} - \cos\sphericalangle(\mathbf{s}_2,\mathbf{u}_2)\right| \le \epsilon\sqrt{1-2^{-2R_2}}\right\} \tag{86}$$

$$\mathcal{E}_{(\mathbf{X}_1,\mathbf{X}_2)} = \left\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2) : \left|\tilde{\rho} - \cos\sphericalangle(\mathbf{u}_1^*(\mathbf{s}_1,\mathcal{C}_1),\mathbf{u}_2^*(\mathbf{s}_2,\mathcal{C}_2))\right| > 7\epsilon\right\}. \tag{87}$$

$$\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}_1} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c \cap \mathcal{E}_{\mathbf{Z}}^c\right] \le \delta, \quad \text{if } R_1 < \frac{1}{2}\log_2\left(\frac{P_1(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)} - \kappa_1\epsilon\right) \tag{89}$$

$$\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}_2} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c \cap \mathcal{E}_{\mathbf{Z}}^c\right] \le \delta, \quad \text{if } R_2 < \frac{1}{2}\log_2\left(\frac{P_2(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)} - \kappa_2\epsilon\right) \tag{90}$$

$$\Pr\left[\mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c \cap \mathcal{E}_{\mathbf{Z}}^c\right] \le \delta, \quad \text{if } R_1 + R_2 < \frac{1}{2}\log_2\left(\frac{P_1+P_2+2\tilde{\rho}\sqrt{P_1P_2}+N}{N(1-\tilde{\rho}^2)} - \kappa_3\epsilon\right) \tag{91}$$

Lemma D.6: Let $\varphi_j \in [0,\pi]$ be the angle between $\mathbf{w}$ and $\mathbf{u}_1(j)$, and let the set $\mathcal{E}'_{\hat{\mathbf{U}}_1}$ be defined as

$$\mathcal{E}'_{\hat{\mathbf{U}}_1} \triangleq \Bigg\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) : \exists \mathbf{u}_1(j) \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\} \text{ s.t. } \cos\varphi_j \geq \sqrt{\frac{P_1(1-\tilde{\rho}^2)+N\tilde{\rho}^2}{P_1(1-\tilde{\rho}^2)+N} - \kappa''\epsilon}\Bigg\} \quad (95)$$

where κ'' is a positive constant determined by P_1, P_2, N, ζ_1 and ζ_2. Then

$$\left(\mathcal{E}_{\hat{\mathbf{U}}_1} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}\right) \subseteq \left(\mathcal{E}'_{\hat{\mathbf{U}}_1} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}\right)$$

and, in particular

$$\Pr\Big[\mathcal{E}_{\hat{\mathbf{U}}_1} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}\Big] \leq \Pr\Big[\mathcal{E}'_{\hat{\mathbf{U}}_1} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}\Big].$$

Proof: We first recall that for the event $\mathcal{E}_{\hat{\mathbf{U}}_1}$ to occur, there must exist a codeword $\mathbf{u}_1(j) \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\}$ that satisfies

$$|\tilde{\rho} - \cos\sphericalangle(\mathbf{u}_1(j),\mathbf{u}_2^*)| < 7\epsilon \quad (96)$$

and

$$\|\mathbf{y} - (\alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2^*)\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2. \quad (97)$$

The proof is now based on a sequence of statements related to Condition (96) and Condition (97).

A) For every $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}}$ and every $\mathbf{u} \in \mathcal{S}_1$, where $\mathcal{S}_1$ is the surface area of the codeword sphere of $\mathcal{C}_1$ defined in the code construction

$$|\tilde{\rho} - \cos\sphericalangle(\mathbf{u},\mathbf{u}_2^*)| < 7\epsilon \Rightarrow \left| n\tilde{\rho}\sqrt{P_1P_2} - \langle \alpha_1\mathbf{u}, \alpha_2\mathbf{u}_2^*\rangle \right| \leq 7n\sqrt{P_1P_2}\epsilon. \quad (98)$$

Statement A) follows by rewriting $\cos\sphericalangle(\mathbf{u},\mathbf{u}_2^*)$ as $\langle \mathbf{u},\mathbf{u}_2^*\rangle/(\|\mathbf{u}\|\|\mathbf{u}_2^*\|)$, and then multiplying the inequality on the LHS of (98) by $\|\alpha_1\mathbf{u}\| \cdot \|\alpha_2\mathbf{u}_2^*\|$ and recalling that $\|\alpha_1\mathbf{u}\| = \sqrt{nP_1}$ and that $\|\alpha_2\mathbf{u}_2^*\| = \sqrt{nP_2}$.

B) For every $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$ and every $\mathbf{u} \in \mathcal{S}_1$

$$\|\mathbf{y} - (\alpha_1\mathbf{u} + \alpha_2\mathbf{u}_2^*)\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2 \Rightarrow \langle \mathbf{y} - \alpha_2\mathbf{u}_2^*, \alpha_1\mathbf{u}\rangle \geq nP_1 - n\sqrt{P_1N}\epsilon. \quad (99)$$

Statement B) follows from rewriting the inequality on the LHS of (99) as $\|(\mathbf{y} - \alpha_2\mathbf{u}_2^*) - \alpha_1\mathbf{u}_1(j)\|^2 \leq \|(\mathbf{y} - \alpha_2\mathbf{u}_2^*) - \alpha_1\mathbf{u}_1^*\|^2$ or equivalently as

$$\begin{aligned}\langle \mathbf{y} - \alpha_2\mathbf{u}_2^*, \alpha_1\mathbf{u}_1(j)\rangle &\geq \langle \mathbf{y} - \alpha_2\mathbf{u}_2^*, \alpha_1\mathbf{u}_1^*\rangle \\ &= \langle \alpha_1\mathbf{u}_1^* + \mathbf{z}, \alpha_1\mathbf{u}_1^*\rangle \\ &= \|\alpha_1\mathbf{u}_1^*\|^2 + \langle \mathbf{z}, \alpha_1\mathbf{u}_1^*\rangle. \end{aligned} \quad (100)$$

It now follows from the equivalence of the first inequality in (99) with (100) that for $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) \in \mathcal{E}^c_{\mathbf{Z}}$, the first inequality in (99) can only hold if

$$\langle \mathbf{y} - \alpha_2\mathbf{u}_2^*, \alpha_1\mathbf{u}\rangle \geq nP_1 - n\sqrt{P_1N}\epsilon \quad (101)$$

thus establishing B).

C) For every $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$ and every $\mathbf{u} \in \mathcal{S}_1$, implication (102), shown at the bottom of the page, holds.

Statement C) is obtained as follows:

$$\begin{aligned}&\|\alpha_1\mathbf{u} - \mathbf{w}\|^2 \\ &= \|\alpha_1\mathbf{u}\|^2 - 2\langle \alpha_1\mathbf{u}, \mathbf{w}\rangle + \|\mathbf{w}\|^2 \\ &= \|\alpha_1\mathbf{u}\|^2 - 2\Big(\zeta_1\langle \alpha_1\mathbf{u}, \mathbf{y} - \alpha_2\mathbf{u}_2^*\rangle + \zeta_2\langle \alpha_1\mathbf{u}, \alpha_2\mathbf{u}_2^*\rangle\Big) \\ &\quad + \|\mathbf{w}\|^2 \\ &\overset{a)}{\leq} nP_1 - 2\Big(\zeta_1 n\big(P_1 - \sqrt{P_1N}\epsilon\big) + \zeta_2 n\sqrt{P_1P_2}(\tilde{\rho} - 7\epsilon)\Big) \\ &\quad + \|\mathbf{w}\|^2\end{aligned}$$

where in $a)$ we have used Statement A) and Statement B).

D) For every $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$

$$\|\mathbf{w}\|^2 \leq n\Big(\zeta_1^2 P_1 + 2\zeta_1\zeta_2\sqrt{P_1P_2}\tilde{\rho} + \zeta_2^2(P_1+N) + \kappa\epsilon\Big) \quad (103)$$

where κ depends on P_1, P_2, N, ζ_1, and ζ_2 only.

Statement D) is obtained as follows:

$$\begin{aligned}\|\mathbf{w}\|^2 &= \zeta_1^2\|\alpha_2\mathbf{u}_2^*\|^2 + 2\zeta_1\zeta_2\langle \alpha_2\mathbf{u}_2^*, \mathbf{y} - \alpha_2\mathbf{u}_2^*\rangle \\ &\quad + \zeta_2^2\|\mathbf{y} - \alpha_2\mathbf{u}_2^*\|^2 \\ &= \zeta_1^2 nP_2 + 2\zeta_1\zeta_2\left(\langle \alpha_2\mathbf{u}_2^*, \alpha_1\mathbf{u}_1^*\rangle - \langle \alpha_2\mathbf{u}_2^*, \mathbf{z}\rangle\right) \\ &\quad + \zeta_2^2\left(\|\alpha_1\mathbf{u}_1^*\|^2 + 2\langle \alpha_1\mathbf{u}_1^*, \mathbf{z}\rangle + \|\mathbf{z}\|^2\right) \\ &\overset{a)}{\leq} \zeta_1^2 nP_1 + 2\zeta_1\zeta_2\left(n\sqrt{P_1P_2}(\tilde{\rho} + 7\epsilon) + n\sqrt{P_2N}\epsilon\right) \\ &\quad + \zeta_2^2\left(nP_1 + 2n\sqrt{P_1N}\epsilon + nN(1+\epsilon)\right) \\ &\leq n\left(\zeta_1^2 P_1 + 2\zeta_1\zeta_2\sqrt{P_1P_2}\tilde{\rho} + \zeta_2^2(P_1+N) + \kappa\epsilon\right)\end{aligned}$$

where in $a)$ we have used that $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) \in \mathcal{E}^c_{\mathbf{Z}}$.

E) For every $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$ and an arbitrary $\mathbf{u} \in \mathcal{S}_1$, implication (104), shown at the bottom of the next page, holds, where we have used the notation

$$\Upsilon(\epsilon) = n\frac{P_1N(1-\tilde{\rho}^2)}{P_1(1-\tilde{\rho}^2)+N} + n\kappa'\epsilon$$

and where κ' only depends on P, N_1, N_2, ζ_1 and ζ_2.

Statement E) follows from combining Statement C) with Statement D) and the explicit values of ζ_1 and ζ_2 given in (93) and (94).

$$\begin{aligned}&\left(|\tilde{\rho} - \cos\sphericalangle(\mathbf{u},\mathbf{u}_2^*)| < 7\epsilon \text{ and } \|\mathbf{y} - (\alpha_1\mathbf{u} + \alpha_2\mathbf{u}_2^*)\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2\right) \\ &\Rightarrow \left(\|\alpha_1\mathbf{u} - \mathbf{w}\|^2 \leq nP_1 - 2\Big(\zeta_1 n\big(P_1 - \sqrt{P_1N}\epsilon\big) + \zeta_2 n\sqrt{P_1P_2}(\tilde{\rho} - 7\epsilon)\Big) + \|\mathbf{w}\|^2\right)\end{aligned} \quad (102)$$

F) For every $\mathbf{u} \in \mathcal{S}_1$, denote by $\varphi \in [0, \pi]$ the angle between $\mathbf{u}$ and $\mathbf{w}$, and let

$$\mathcal{B}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1^*, \mathbf{u}_2^*, \mathbf{z}) \triangleq \left\{ \mathbf{u} \in \mathcal{S}_1^{(n)} : \cos\varphi \geq \sqrt{\frac{P_1(1-\tilde{\rho}^2) + N\tilde{\rho}^2}{P_1(1-\tilde{\rho}^2)+N} - \kappa''\epsilon} \right\}$$

where κ'' only depends on P, N_1, N_2, ζ_1 and ζ_2, and where we assume ϵ sufficiently small such that

$$\frac{P_1(1-\tilde{\rho}^2) + N\tilde{\rho}^2}{P_1(1-\tilde{\rho}^2)+N} - \kappa''\epsilon > 0.$$

Then, for every $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) \in \mathcal{E}_{\mathbf{X}}^c \cap \mathcal{E}_{\mathbf{Z}}^c$, implication (105), shown at the bottom of the page, holds.

Statement F) follows from Statement E) by noting that if $\mathbf{w} \neq \mathbf{0}$ and $1 - \Upsilon(\epsilon)/(nP_1) > 0$, then

$$\left.\begin{array}{c} \|\alpha_1 \mathbf{u}\|^2 = nP_1 \\ \|\alpha_1\mathbf{u} - \mathbf{w}\|^2 \leq \Upsilon(\epsilon) \end{array}\right\} \Rightarrow \cos\sphericalangle(\mathbf{u}, \mathbf{w}) \geq \sqrt{1 - \frac{\Upsilon(\epsilon)}{nP_1}}.$$

To see this, first note that for every $\alpha_1\mathbf{u}$, where $\mathbf{u} \in \mathcal{S}_1$, satisfying the condition on the LHS of (105) lies within a sphere of radius $\sqrt{\Upsilon(\epsilon)}$ centered at $\mathbf{w}$, and for every $\mathbf{u} \in \mathcal{S}_1$ we have that $\alpha_1\mathbf{u}$ also lies on the centered $\mathbb{R}^n$-sphere of radius $\sqrt{nP_1}$. Hence, every $\mathbf{u} \in \mathcal{S}_1^{(n)}$ satisfying the condition on the LHS of (105) lies in the intersection of these two regions, which is a polar cap on the centered sphere of radius $\sqrt{nP_1}$. An illustration of such a polar cap is given in Fig. 9. The area of this polar cap is outer bounded as follows. Let $\mathbf{r}$ be an arbitrary point on the boundary of this polar cap. The half-angle of the polar cap would be maximized if $\mathbf{w}$ and $\mathbf{r} - \mathbf{w}$ would lie perpendicular to each other, as is illustrated in Subplot (b) of Fig. 10. Hence, every $\mathbf{u} \in \mathcal{S}_1^{(n)}$ satisfying the upper conditions of (105) also satisfies

$$\cos\varphi \geq \sqrt{1 - \frac{\Upsilon(\epsilon)}{nP_1}} = \sqrt{\frac{P_1(1-\tilde{\rho}^2) + N\tilde{\rho}^2}{P_1(1-\tilde{\rho}^2)+N} - \kappa''\epsilon}$$

where we assume ϵ sufficiently small such that $1 - \Upsilon(\epsilon)/(nP_1) > 0$ and where $\kappa'' = \kappa'/P_1$.

The proof of Lemma D.6 is now concluded by noticing that the set $\mathcal{E}'_{\hat{\mathbf{U}}_1}$, defined in (95), is the set of tuples $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z})$ for which there exists a $\mathbf{u}_1(j) \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\}$ such that $\mathbf{u}_1(j) \in \mathcal{B}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1^*, \mathbf{u}_2^*, \mathbf{z})$. Thus, by Statement F) and by the definition of $\mathcal{E}_{\hat{\mathbf{U}}_1}$ in (80) it follows that

$$\mathcal{E}_{\hat{\mathbf{U}}_1} \cap \mathcal{E}_{\mathbf{Z}}^c \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c \subseteq \mathcal{E}'_{\hat{\mathbf{U}}_1} \cap \mathcal{E}_{\mathbf{Z}}^c \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c$$

and, therefore

$$\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}_1} \cap \mathcal{E}_{\mathbf{Z}}^c \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c\right] \leq \Pr\left[\mathcal{E}'_{\hat{\mathbf{U}}_1} \cap \mathcal{E}_{\mathbf{Z}}^c \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c\right].$$

□

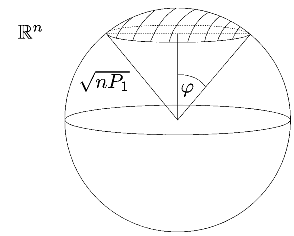

Fig. 9. Polar cap of half angle φ on an $\mathbb{R}^n$-sphere of radius $\sqrt{nP_1}$.

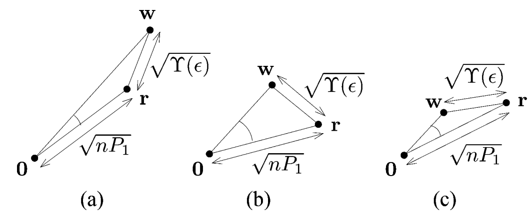

Fig. 10. Half-angle of cap for different constellations of $\mathbf{w}$ and $\mathbf{r}$.

We now state one more lemma that will be used for the proof of (89).

Lemma D.7: For every $\Delta \in (0, 1]$, let the set $\mathcal{G}$ be given by

$$\mathcal{G} = \Big\{(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) : \exists \mathbf{u}_1(j) \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\} \text{ s.t. } \cos\sphericalangle(\mathbf{w}, \mathbf{u}_1(j)) \geq \Delta\Big\}$$

where $\mathbf{w}$ is defined in (92). Then

$$\left(R_1 < -\frac{1}{2}\log_2(1-\Delta^2)\right) \Rightarrow \left(\lim_{n\to\infty} \Pr\left[\mathcal{G}|\mathcal{E}_{\mathbf{X}_1}^c\right] = 0, \quad \epsilon > 0\right). \quad (106)$$

Proof: The proof follows from upper bounding in every point on $\mathcal{S}_1$ the density of every $\mathbf{u}_1(j) \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\}$ and then using a standard argument from sphere-packing. The proof is given in Appendix D-E2. □

$$\left(|\tilde{\rho} - \cos\sphericalangle(\mathbf{u}, \mathbf{u}_2^*)| < 7\epsilon \text{ and } \|\mathbf{y} - (\alpha_1\mathbf{u} + \alpha_2\mathbf{u}_2^*)\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2\right) \Rightarrow \left(\|\alpha_1\mathbf{u} - \mathbf{w}\|^2 \leq \Upsilon(\epsilon)\right) \quad (104)$$

$$\left(|\tilde{\rho} - \cos\sphericalangle(\mathbf{u}, \mathbf{u}_2^*)| < 7\epsilon \text{ and } \|\mathbf{y} - (\alpha_1\mathbf{u} + \alpha_2\mathbf{u}_2^*)\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2\right) \Rightarrow \mathbf{u} \in \mathcal{B}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1^*, \mathbf{u}_2^*, \mathbf{z}) \quad (105)$$

We next state two lemmas for the proof of (91). These lemmas are similar to Lemma D.6 and Lemma D.7.

Lemma D.8: For every sufficiently small $\epsilon > 0$, define the set $\mathcal{E}'_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)}$ as in (107), shown at the bottom of the page, where we have used the notation

$$\Lambda(\epsilon) = \sqrt{\frac{P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1P_2} - \xi'\epsilon}{P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1P_2} + N + \xi_2\epsilon}}$$

and where ξ' and ξ_2 depend only on P_1, P_2, and N. Then, for every sufficiently small $\epsilon > 0$

$$\left(\mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right) \subseteq \left(\mathcal{E}'_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right)$$

and, in particular

$$\Pr\left[\mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right] \leq \Pr\left[\mathcal{E}'_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right].$$

Proof: We first recall that for the event $\mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)}$ to occur, there must exist codewords $\mathbf{u}_1(j) \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\}$ and $\mathbf{u}_2(\ell) \in \mathcal{C}_2 \setminus \{\mathbf{u}_2^*\}$ such that

$$|\tilde{\rho} - \cos\sphericalangle(\mathbf{u}_1(j), \mathbf{u}_2(\ell))| < 7\epsilon \tag{108}$$

and

$$\|\mathbf{y} - (\alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell))\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2. \tag{109}$$

The proof is now based on a sequence of statements related to Condition (108) and Condition (109).

A) For every $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$, implication (110), shown at the bottom of the page, holds, where ξ_1 only depends on P_1, P_2, and N.

Statement A) follows by rewriting the LHS of (110) as

$$\begin{aligned}
&2\langle \mathbf{y}, \alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell)\rangle \\
&\geq 2\langle \mathbf{y}, \alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*\rangle + \|\alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell)\|^2 \\
&\quad - \|\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*\|^2 \\
&= \|\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*\|^2 + 2\langle \mathbf{z}, \alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*\rangle \\
&\quad + \|\alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell)\|^2 \\
&\overset{a)}{\geq} 2n\Big(P_1 + 2\tilde{\rho}\sqrt{P_1P_2}(1 - 7\epsilon) \\
&\qquad + P_2 + \sqrt{P_1N}\epsilon + \sqrt{P_2N}\epsilon\Big) \\
&= 2n\left(P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1P_2} - \xi_1\epsilon\right)
\end{aligned} \tag{111}$$

where in $a)$ we have used that $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$ and that $\|\alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell)\|^2 \geq 0$.

B) For every $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$

$$\|\mathbf{y}\|^2 \leq n\left(P_1 + 2\tilde{\rho}\sqrt{P_1P_2} + P_2 + N + \xi_2\epsilon\right)$$

where ξ_2 only depends on P_1, P_2, and N.

Statement B) is obtained as follows:

$$\begin{aligned}
\|\mathbf{y}\|^2 &= \|\alpha_1\mathbf{u}_1^*\|^2 + 2\langle \alpha_1\mathbf{u}_1^*, \alpha_2\mathbf{u}_2^*\rangle + \|\alpha_2\mathbf{u}_2^*\|^2 \\
&\quad + 2\left(\langle \alpha_1\mathbf{u}_1^*, \mathbf{z}\rangle + \langle \alpha_2\mathbf{u}_2^*, \mathbf{z}\rangle\right) + \|\mathbf{z}\|^2 \\
&\overset{a)}{\leq} nP_1 + 2n\tilde{\rho}\sqrt{P_1P_2}(1 + 7\epsilon) + nP_2 \\
&\quad + 2n\sqrt{P_1N}\epsilon + 2n\sqrt{P_2N}\epsilon + nN(1 + \epsilon) \\
&\leq n\left(P_1 + 2\tilde{\rho}\sqrt{P_1P_2} + P_2 + N + \xi_2\epsilon\right)
\end{aligned}$$

where in $a)$ we have used that $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$.

C) For every $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z})$, implication (112), shown at the bottom of the next page, holds.

Statement C) follows by

$$\begin{aligned}
&\|\alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell)\|^2 \\
&= \|\alpha_1\mathbf{u}_1(j)\|^2 + 2\langle \alpha_1\mathbf{u}_1(j), \alpha_2\mathbf{u}_2(\ell)\rangle + \|\alpha_2\mathbf{u}_2(\ell)\|^2 \\
&\overset{a)}{\leq} nP_1 + 2n\tilde{\rho}\sqrt{P_1P_2}(1 + 7\epsilon) + nP_2 \\
&= n\left(P_1 + 2\tilde{\rho}\sqrt{P_1P_2} + P_2 + \xi_3\epsilon\right)
\end{aligned}$$

where in $a)$ we have used that multiplying the inequality on the LHS of (112) by $\|\alpha_1\mathbf{u}_1(j)\| \cdot \|\alpha_2\mathbf{u}_2(\ell)\|$ and recalling that $\|\alpha_1\mathbf{u}_1(j)\| \leq \sqrt{nP_1}$ and that $\|\alpha_2\mathbf{u}_2(\ell)\| \leq \sqrt{nP_2}$ gives

$$\left|n\sqrt{P_1P_2}\tilde{\rho} - \langle \alpha_1\mathbf{u}_1(j), \alpha_2\mathbf{u}_2(\ell)\rangle\right| < 7n\sqrt{P_1P_2}\epsilon$$

$$\begin{aligned}
\mathcal{E}'_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \triangleq \Big\{&(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) : \exists \mathbf{u}_1(j) \in \mathcal{C}_1 \setminus \{\mathbf{u}_1^*\} \text{ and } \exists \mathbf{u}_2(\ell) \in \mathcal{C}_2 \setminus \{\mathbf{u}_2^*\} \text{ s.t.} \\
&\cos\sphericalangle(\mathbf{u}_1(j), \mathbf{u}_2(\ell)) \geq \tilde{\rho} - 7\epsilon \text{ and } \cos\sphericalangle(\mathbf{y}, \alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell)) \geq \Lambda(\epsilon)\Big\}
\end{aligned} \tag{107}$$

$$\begin{aligned}
&\left(\|\mathbf{y} - (\alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell))\|^2 \leq \|\mathbf{y} - (\alpha_1\mathbf{u}_1^* + \alpha_2\mathbf{u}_2^*)\|^2\right) \\
&\qquad \Rightarrow \Big(\langle \mathbf{y}, \alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell)\rangle \geq n\big(P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1P_2} - \xi_1\epsilon\big)\Big)
\end{aligned} \tag{110}$$

and thus

$$\langle \alpha_1 \mathbf{u}_1(j), \alpha_2 \mathbf{u}_2(\ell) \rangle < n\sqrt{P_1 P_2}\tilde{\rho}(1+7\epsilon)$$

thus establishing C).

D) For every $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$, Implication (113), shown at the bottom of the page, holds.

Statement D) follows by rewriting the expression $\cos \sphericalangle(\mathbf{y}, \alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell))$ as

$$\cos \sphericalangle(\mathbf{y}, \alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell)) = \frac{\langle \mathbf{y}, \alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell) \rangle}{\|\mathbf{y}\| \cdot \|\alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell)\|}$$

and then lower bounding $\langle \mathbf{y}, \alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell) \rangle$ using A) and upper bounding $\|\mathbf{y}\|$ and $\|\alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell)\|$ using B) and C), respectively. Using the shorthand notation

$$\eta_1 = P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1 P_2} + N + \xi_2 \epsilon$$
$$\eta_2 = P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1 P_2} + \xi_3 \epsilon$$

this, yields that for every $(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) \in \mathcal{E}^c_{\mathbf{X}} \cap \mathcal{E}^c_{\mathbf{Z}}$

$$\begin{aligned} &\cos \sphericalangle \left(\mathbf{y}, \alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell)\right) \\ &\quad \geq \frac{P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1 P_2} - \xi_1 \epsilon}{\sqrt{\eta_1}\sqrt{\eta_2}} \\ &\quad \geq \sqrt{\frac{P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1 P_2} - \xi' \epsilon}{P_1 + P_2 + 2\tilde{\rho}\sqrt{P_1 P_2} + N + \xi_2 \epsilon}} \\ &\quad = \Lambda(\epsilon). \end{aligned}$$

Lemma D.8 now follows by D) which gives

$$\left(\mathcal{E}_{(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right) \subseteq \left(\mathcal{E}'_{(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right)$$

and, therefore

$$\Pr\left[\mathcal{E}_{(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)} \middle| \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right] \leq \Pr\left[\mathcal{E}'_{(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)} \middle| \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right].$$

□

We now state the second lemma needed for the proof of (91).

Lemma D.9: For every $\Theta \in (0,1]$ and $\Delta \in (0,1]$, let the set $\mathcal{G}$ be given by (114), shown at the bottom of the page. Then

$$\left(R_1 + R_2 < -\frac{1}{2}\log_2\left((1-\Theta^2)(1-\Delta^2)\right)\right) \Rightarrow \left(\lim_{n \to \infty} \Pr\left[\mathcal{G} | \mathcal{E}^c_{\mathbf{X}_1} \cap \mathcal{E}^c_{\mathbf{X}_2}\right] = 0, \quad \epsilon > 0\right). \tag{115}$$

Proof: The proof follows from upper bounding in every point on $\mathcal{S}_i$, $i \in \{1,2\}$, the density of every $\mathbf{u}_i(j) \in \mathcal{C}_i \setminus \{\mathbf{u}^*_i\}$ and then using a standard argument from sphere-packing. The proof is given in Appendix D-E3 □

Proof of Lemma D.5: We first prove (89)

$$\begin{aligned} \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}_1} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right] &\overset{a)}{\leq} \Pr\left[\mathcal{E}'_{\hat{\mathbf{U}}_1} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right] \\ &\overset{b)}{\leq} \Pr\left[\mathcal{E}'_{\hat{\mathbf{U}}_1} \middle| \mathcal{E}^c_{\mathbf{X}_1}\right] \end{aligned} \tag{116}$$

where $a)$ follows by Lemma D.6 and $b)$ follows because $\mathcal{E}^c_{\mathbf{X}} \subseteq \mathcal{E}^c_{\mathbf{X}_1}$. The proof of (89) is now completed by combining (116) with Lemma D.7. This gives that for every $\delta > 0$ and every $\epsilon > 0$ there exists some $n'_{41}(\delta, \epsilon)$ such that for all $n > n'_{41}(\delta, \epsilon)$ we have $\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}_1} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right] < \delta$ whenever

$$R_1 < -\frac{1}{2}\log_2\left(\frac{N(1-\tilde{\rho}^2)}{P_1(1-\tilde{\rho}^2) + N} + \kappa'' \epsilon\right)$$

$$\left(\left|\tilde{\rho} - \left\langle \frac{\mathbf{u}_1(j)}{\|\mathbf{u}_1(j)\|}, \frac{\mathbf{u}_2(\ell)}{\|\mathbf{u}_2(\ell)\|} \right\rangle\right| < 7\epsilon\right) \Rightarrow \left(\|\alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell)\|^2 \leq n\left(P_1 + 2\tilde{\rho}\sqrt{P_1 P_2} + P_2 + \xi_3 \epsilon\right)\right). \tag{112}$$

$$\left(|\tilde{\rho} - \cos \sphericalangle(\mathbf{u}_1(j), \mathbf{u}_2(\ell))| < 7\epsilon \text{ and } \|\mathbf{y} - (\alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell))\|^2 \leq \|\mathbf{y} - (\alpha_1 \mathbf{u}^*_1 + \alpha_2 \mathbf{u}^*_2)\|^2\right) \Rightarrow \left(\cos \sphericalangle(\mathbf{y}, \alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell)) \geq \Lambda(\epsilon)\right). \tag{113}$$

$$\mathcal{G} = \Big\{(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) : \exists \mathbf{u}_1(j) \in \mathcal{C}_1 \setminus \{\mathbf{u}^*_1\}, \mathbf{u}_2(\ell) \in \mathcal{C}_2 \setminus \{\mathbf{u}^*_2\} \text{ s.t. } \cos \sphericalangle(\mathbf{u}_1(j), \mathbf{u}_2(\ell)) \geq \Theta, \cos \sphericalangle(\mathbf{y}, \alpha_1 \mathbf{u}_1(j) + \alpha_2 \mathbf{u}_2(\ell)) \geq \Delta\Big\}. \tag{114}$$

$$\leq \frac{1}{2}\log_2\left(\frac{P_1(1-\tilde{\rho}^2)+N}{N(1-\tilde{\rho}^2)} - \kappa_1\epsilon\right)$$

where κ_1 is a positive constant determined by P_1, P_2, N, ζ_1 and ζ_2. A similar argument establishes (90).

We turn to the proof of (91)

$$\begin{aligned}\Pr&\left[\mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right]\\ &\overset{a)}{\leq}\Pr\left[\mathcal{E}'_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right]\\ &\overset{b)}{\leq}\Pr\left[\mathcal{E}'_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} | \mathcal{E}^c_{\mathbf{X}_1} \cap \mathcal{E}^c_{\mathbf{X}_2}\right] \quad (117)\end{aligned}$$

where $a)$ follows by Lemma D.8 and $b)$ follows because $\mathcal{E}^c_{\mathbf{X}} \subseteq \left(\mathcal{E}^c_{\mathbf{X}_1} \cap \mathcal{E}^c_{\mathbf{X}_2}\right)$. The proof of (91) is now completed by combining (117) with Lemma D.9, which gives that for every $\delta > 0$ and every $\epsilon > 0$ there exists some $n'_{43}(\delta,\epsilon)$ such that for all $n > n'_{43}(\delta,\epsilon)$ we have $\Pr\left[\mathcal{E}_{(\hat{\mathbf{U}}_1,\hat{\mathbf{U}}_2)} \cap \mathcal{E}^c_{\mathbf{Z}} \cap \mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\mathbf{X}}\right] < \delta$ whenever

$$\begin{aligned}R_1 + R_2 &< \frac{1}{2}\log_2\left(\frac{P_1+P_2+2\tilde{\rho}\sqrt{P_1P_2}+N+\xi_2\epsilon}{(N+(\xi'+\xi_2)\epsilon)(1-\tilde{\rho}^2+\xi'\epsilon)}\right)\\ &\leq \frac{1}{2}\log_2\left(\frac{P_1+P_2+2\tilde{\rho}\sqrt{P_1P_2}+N}{N(1-\tilde{\rho}^2)} - \kappa_3\epsilon\right)\end{aligned}$$

where κ_3 is a positive constant determined by P_1, P_2 and N. □

The proof of Lemma D.1 now follows straight forwardly.

Proof of Lemma D.1: Combining (88) with Lemma D.2, Lemma D.3, Lemma D.4, and Lemma D.5, yields that for every $\delta > 0$ and $0.3 > \epsilon > 0$ there exists some $n'_4(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n'_4(\delta,\epsilon)$

$$\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \leq 11\delta, \quad \text{if } (R_1,R_2) \in \mathcal{R}(\epsilon).$$

□

2) Concluding the Proof of Proposition D.1: We start with four lemmas. The first lemma upper bounds the impact of atypical source outputs on the expected distortion.

Lemma D.10: For every $\epsilon > 0$

$$\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2\middle|\mathcal{E}_{\mathbf{S}}\right]\Pr\left[\mathcal{E}_{\mathbf{S}}\right] \leq \sigma^2\left(\epsilon + \Pr\left[\mathcal{E}_{\mathbf{S}}\right]\right).$$

Proof:

$$\begin{aligned}\frac{1}{n}&\mathsf{E}\left[\|\mathbf{S}_1\|^2\middle|\mathcal{E}_{\mathbf{S}}\right]\Pr\left[\mathcal{E}_{\mathbf{S}}\right]\\ &=\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2\right] - \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2\middle|\mathcal{E}^c_{\mathbf{S}}\right]\Pr\left[\mathcal{E}^c_{\mathbf{S}}\right]\\ &\leq\sigma^2 - \sigma^2(1-\epsilon)\Pr\left[\mathcal{E}^c_{\mathbf{S}}\right]\\ &=\sigma^2 - \sigma^2(1-\epsilon)\left(1-\Pr\left[\mathcal{E}_{\mathbf{S}}\right]\right)\\ &=\sigma^2\epsilon + \sigma^2(1-\epsilon)\Pr\left[\mathcal{E}_{\mathbf{S}}\right]\\ &=\sigma^2\left(\epsilon + \Pr\left[\mathcal{E}_{\mathbf{S}}\right]\right).\end{aligned}$$

□

The second lemma gives upper bounds on norms related to the reconstructions $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}^{\mathrm{G}}_1$.

Lemma D.11: Let the reconstructions $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}^{\mathrm{G}}_1$ be as defined in (73) and (78). Then

$$\|\hat{\mathbf{s}}_1\|^2 \leq 4n\sigma^2, \quad \|\hat{\mathbf{s}}^{\mathrm{G}}_1\|^2 \leq 4n\sigma^2, \quad \|\hat{\mathbf{s}}^{\mathrm{G}}_1 - \hat{\mathbf{s}}_1\|^2 \leq 16n\sigma^2.$$

Proof: We start by upper bounding the squared norm of $\hat{\mathbf{s}}_1$

$$\begin{aligned}\|\hat{\mathbf{s}}_1\|^2 &= \|\gamma_{11}\hat{\mathbf{u}}_1 + \gamma_{12}\hat{\mathbf{u}}_2\|^2\\ &\leq \left(\gamma_{11}\|\hat{\mathbf{u}}_1\| + \gamma_{12}\|\hat{\mathbf{u}}_2\|\right)^2\\ &\overset{a)}{\leq} n\sigma^2(1+\rho)^2\\ &\leq 4n\sigma^2\end{aligned}$$

where in $a)$ we have used (77), i.e., that $\gamma_{11} < 1$ and $\gamma_{12} < \rho$, and that $\|\hat{\mathbf{u}}_i\| \leq \sqrt{n\sigma^2}$, $i \in \{1,2\}$. The upper bound on the squared norm of $\hat{\mathbf{s}}^{\mathrm{G}}_1$ is obtained similarly. Its proof is, therefore, omitted. The upper bound on the squared norm of the difference between $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}^{\mathrm{G}}_1$ now follows easily:

$$\begin{aligned}\left\|\hat{\mathbf{s}}^{\mathrm{G}}_1 - \hat{\mathbf{s}}_1\right\|^2 &\leq \|\hat{\mathbf{s}}^{\mathrm{G}}_1\|^2 + 2\|\hat{\mathbf{s}}^{\mathrm{G}}_1\|\|\hat{\mathbf{s}}_1\| + \|\hat{\mathbf{s}}_1\|^2\\ &= \left(\|\hat{\mathbf{s}}^{\mathrm{G}}_1\| + \|\hat{\mathbf{s}}_1\|\right)^2\\ &\leq 16n\sigma^2.\end{aligned}$$

□

The next two lemmas are used in the upcoming proof of Proposition D.1. They rely on Lemma D.10 and Lemma D.11.

Lemma D.12:

$$\frac{1}{n}\mathsf{E}\left[\left\langle\mathbf{S}_1, \hat{\mathbf{S}}^{\mathrm{G}}_1 - \hat{\mathbf{S}}_1\right\rangle\right] \leq \sigma^2\Big(\epsilon + 17\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + 4\sqrt{1+\epsilon}\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\Big).$$

Proof:

$$\begin{aligned}\frac{1}{n}&\mathsf{E}\left[\left\langle\mathbf{S}_1, \hat{\mathbf{S}}^{\mathrm{G}}_1 - \hat{\mathbf{S}}_1\right\rangle\right]\\ &=\frac{1}{n}\mathsf{E}\left[\left\langle\mathbf{S}_1, \hat{\mathbf{S}}^{\mathrm{G}}_1 - \hat{\mathbf{S}}_1\right\rangle\middle|\mathcal{E}_{\mathbf{S}}\right]\Pr\left[\mathcal{E}_{\mathbf{S}}\right]\\ &\quad+\frac{1}{n}\mathsf{E}\left[\left\langle\mathbf{S}_1, \hat{\mathbf{S}}^{\mathrm{G}}_1 - \hat{\mathbf{S}}_1\right\rangle\middle|\mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}_{\hat{\mathbf{U}}}\right]\\ &\quad+\frac{1}{n}\underbrace{\mathsf{E}\left[\left\langle\mathbf{S}_1, \hat{\mathbf{S}}^{\mathrm{G}}_1 - \hat{\mathbf{S}}_1\right\rangle\middle|\mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\hat{\mathbf{U}}}\right]}_{=0}\Pr\left[\mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}^c_{\hat{\mathbf{U}}}\right]\\ &\overset{a)}{\leq}\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 + \|\hat{\mathbf{S}}^{\mathrm{G}}_1 - \hat{\mathbf{S}}_1\|^2\middle|\mathcal{E}_{\mathbf{S}}\right]\Pr\left[\mathcal{E}_{\mathbf{S}}\right]\\ &\quad+\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|\|\hat{\mathbf{S}}^{\mathrm{G}}_1 - \hat{\mathbf{S}}_1\|\middle|\mathcal{E}^c_{\mathbf{S}} \cap \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\\ &\overset{b)}{\leq}\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2\middle|\mathcal{E}_{\mathbf{S}}\right]\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + 16\sigma^2\Pr\left[\mathcal{E}_{\mathbf{S}}\right]\\ &\quad+\sqrt{\sigma^2(1+\epsilon)}\sqrt{16\sigma^2}\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\\ &\overset{c)}{\leq}\sigma^2(\epsilon + \Pr\left[\mathcal{E}_{\mathbf{S}}\right]) + 16\sigma^2\Pr\left[\mathcal{E}_{\mathbf{S}}\right]\\ &\quad+4\sigma^2\sqrt{1+\epsilon}\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\\ &\leq\sigma^2\left(\epsilon + 17\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + 4\sqrt{1+\epsilon}\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\right). \quad (118)\end{aligned}$$

In the first equality the third expectation equals zero because under $\mathcal{E}^c_{\hat{\mathbf{U}}}$ we have $\|\hat{\mathbf{S}}^{\mathrm{G}}_1 - \hat{\mathbf{S}}_1\| = 0$. In $a)$ we have used two inequalities: in the first term, the inner product is upper bounded using the inequality

$$|\langle \mathbf{v}, \mathbf{w}\rangle| \le \|\mathbf{v}\| \cdot \|\mathbf{w}\| \le \frac{1}{2}\left(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2\right) \le \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2, \qquad \mathbf{v}, \mathbf{w} \in \mathbb{R}^n. \tag{119}$$

The second term is upper bounded by the Cauchy–Schwarz inequality and by $\Pr\left[\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\hat{\mathbf{U}}}\right] \le \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]$. In $b)$ we have used Lemma D.11 and in $c)$ we have used Lemma D.10. □

Lemma D.13:

$$\frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] \le 8\sigma^2 \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right].$$

Proof:

$$\begin{aligned}\frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] &= \frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \big| \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\ &\quad + \frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \big| \mathcal{E}_{\hat{\mathbf{U}}}^c\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}^c\right] \\ &\overset{a)}{\le} \frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 + \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \big| \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\ &\overset{a)}{\le} 8\sigma^2 \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\end{aligned}$$

where $a)$ follows since conditional on $\mathcal{E}_{\hat{\mathbf{U}}}^c$ we have $\hat{\mathbf{s}}_1 = \hat{\mathbf{s}}_1^{\mathrm{G}}$ and, therefore, $\|\hat{\mathbf{s}}_1\|^2 - \|\hat{\mathbf{s}}_1^{\mathrm{G}}\|^2 = 0$, and where $b)$ follows by Lemma D.11. □

Proof of Proposition D.1: We show that the asymptotic normalized distortion resulting from the proposed vector-quantizer scheme is the same as the asymptotic normalized distortion resulting from the genie-aided version of this scheme

$$\begin{aligned}&\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1\|^2\right] - \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] \\ &= \frac{1}{n}\Big(\mathsf{E}\left[\|\mathbf{S}_1\|^2\right] - 2\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1\right\rangle\right] + \mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2\right] \\ &\quad - \mathsf{E}\left[\|\mathbf{S}_1\|^2\right] + 2\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1^{\mathrm{G}}\right\rangle\right] - \mathsf{E}\left[\|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right]\Big) \\ &= 2\frac{1}{n}\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1^{\mathrm{G}} - \hat{\mathbf{S}}_1\right\rangle\right] + \frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] \\ &\overset{a)}{\le} 2\sigma^2\left(\epsilon + 17\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + 4\sqrt{1+\epsilon}\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\right) + 8\sigma^2 \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\ &= 2\sigma^2\left(\epsilon + 17\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + 4\left(\sqrt{1+\epsilon} + 1\right)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\right)\end{aligned} \tag{120}$$

where in step $a)$ we have used Lemma D.12 and Lemma D.13. Combining (120) with Lemma D.2 and Lemma D.1 gives that for every $\delta > 0$ and $0.3 > \epsilon > 0$, there exists an $n'(\delta, \epsilon) > 0$ such that for all $(R_1, R_2) \in \mathcal{R}(\epsilon)$ and $n > n'(\delta, \epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1\|^2\right] - \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] < 2\sigma^2\Big(\epsilon + \left(44\sqrt{1+\epsilon} + 61\right)\delta\Big).$$

□

D. Upper Bound on Expected Distortion

We now derive an upper bound on the achievable distortion for the proposed vector-quantizer scheme. By Corollary D.1, it suffices to analyze the genie-aided scheme. Using that $\hat{\mathbf{S}}_1^{\mathrm{G}} = \gamma_{11}\mathbf{U}_1^* + \gamma_{12}\mathbf{U}_2^*$

$$\begin{aligned}&\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] \\ &= \frac{1}{n}\Big(\mathsf{E}\left[\|\mathbf{S}_1\|^2\right] - 2\gamma_{11}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_1^*\rangle\right] \\ &\quad - 2\gamma_{12}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] + \gamma_{11}^2\mathsf{E}\left[\|\mathbf{U}_1^*\|^2\right] \\ &\quad + 2\gamma_{11}\gamma_{12}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle\right] + \gamma_{12}^2\mathsf{E}\left[\|\mathbf{U}_2^*\|^2\right]\Big) \\ &= \sigma^2 - 2\gamma_{11}\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_1^*\rangle\right] - 2\gamma_{12}\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] \\ &\quad + \gamma_{11}^2\sigma^2(1 - 2^{-2R_1}) + 2\gamma_{11}\gamma_{12}\frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle\right] \\ &\quad + \gamma_{12}^2\sigma^2(1 - 2^{-2R_2})\end{aligned} \tag{121}$$

where in the last equality all expected squared norms have been replaced by their explicit values, i.e., $\mathsf{E}\left[\|\mathbf{S}_1\|^2\right] = n\sigma^2$ and $\mathsf{E}\left[\|\mathbf{U}_i\|^2\right] = n\sigma^2(1 - 2^{-2R_i})$ for $i \in \{1, 2\}$. The remaining expectations of the inner products are bounded in the following three lemmas.

Lemma D.14: For every $\delta > 0$ and $0.3 > \epsilon > 0$ and every positive integer n

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_1^*\rangle\right] \ge \sigma^2(1 - 2^{-2R_1})(1 - 2\epsilon)(1 - 7\delta). \tag{122}$$

Proof:

$$\begin{aligned}&\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_1^*\rangle\right] \\ &= \frac{1}{n}\underbrace{\mathsf{E}\left[\|\mathbf{S}_1\|\|\mathbf{U}_1^*\|\cos\sphericalangle(\mathbf{S}_1, \mathbf{U}_1^*)\big|\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}}\right]}_{\ge 0}\cdot\Pr\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}}\right] \\ &\quad + \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|\|\mathbf{U}_1^*\|\cos\sphericalangle(\mathbf{S}_1, \mathbf{U}_1^*)\big|\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c\right]\cdot\Pr\left[\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c\right] \\ &\ge \sqrt{\sigma^2(1-\epsilon)\sigma^2(1-2^{-2R_1})(1-2^{-2R_1})}(1-\epsilon)\Pr\left[\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}^c\right] \\ &\ge \sigma^2(1-2^{-2R_1})(1-\epsilon)^2\left(1 - \Pr\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}}\right]\right) \\ &\ge \sigma^2(1 - 2^{-2R_1})(1 - 2\epsilon)\left(1 - \Pr\left[\mathcal{E}_{\mathbf{S}}\right] - \Pr\left[\mathcal{E}_{\mathbf{X}}\right]\right)\end{aligned}$$

where in the first equality the first expectation term is non-negative because conditioned on $\mathcal{E}_{\mathbf{X}}$ either $\mathbf{U}_1^* = \mathbf{0}$ or, if $\mathbf{U}_1^* \ne \mathbf{0}$, Then $\cos\left(\sphericalangle(\mathbf{S}_1, \mathbf{U}_1^*)\right) > 0$.

By Lemma D.2 and Lemma D.4, it now follows that for every $\delta > 0$ and $0.3 > \epsilon > 0$ there exists an $n'(\delta, \epsilon) \in \mathbb{N}$ such that for all $n > n'(\delta, \epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_1^*\rangle\right] \ge \sigma^2(1 - 2^{-2R_1})(1 - 2\epsilon)(1 - 7\delta).$$

□

Lemma D.15: For every $\delta > 0$ and $0.3 > \epsilon > 0$, there exists an $n_2'(\delta, \epsilon) \in \mathbb{N}$ such that for all $n > n_2'(\delta, \epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle\right] \le \sigma^2 6\delta + \sigma^2\rho(1 - 2^{-2R_1})(1 - 2^{-2R_2})(1 + 7\epsilon).$$

Proof:

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle\right] \\
&= \frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle \,\middle|\mathcal{E}_{\mathbf{X}}\right]\Pr\left[\mathcal{E}_{\mathbf{X}}\right] + \frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle \,\middle|\mathcal{E}_{\mathbf{X}}^c\right]\Pr\left[\mathcal{E}_{\mathbf{X}}^c\right] \\
&\le \frac{1}{n}\mathsf{E}\left[\|\mathbf{U}_1^*\|\|\mathbf{U}_2^*\|\middle|\mathcal{E}_{\mathbf{X}}\right]\Pr\left[\mathcal{E}_{\mathbf{X}}\right] + \frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle \,\middle|\mathcal{E}_{\mathbf{X}}^c\right] \\
&\le \sigma^2\sqrt{(1-2^{-2R_1})(1-2^{-2R_2})}\Pr\left[\mathcal{E}_{\mathbf{X}}\right] \\
&\quad + \frac{1}{n}\mathsf{E}\left[\tilde{\rho}(1+7\epsilon)\sqrt{n\sigma^2(1-2^{-2R_1})}\sqrt{n\sigma^2(1-2^{-2R_2})}\middle|\mathcal{E}_{\mathbf{X}}^c\right] \\
&\le \sigma^2\Pr\left[\mathcal{E}_{\mathbf{X}}\right] + \sigma^2\rho(1-2^{-2R_1})(1-2^{-2R_2})(1+7\epsilon).
\end{aligned}$$

Thus, it follows by Lemma D.4 that for every $\delta > 0$ and $0.3 > \epsilon > 0$ there exists an $n_2'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n_2'(\delta,\epsilon)$

$$\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle\right] \le \sigma^2 6\delta + \sigma^2\rho(1-2^{-2R_1})(1-2^{-2R_2})(1+7\epsilon).$$

□

Lemma D.16: For every $\delta > 0$ and $0.3 > \epsilon > 0$, there exists an $n'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n'(\delta,\epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] \ge \sigma^2\rho(1-2^{-2R_2})(1-\epsilon)^3 - \sigma^2\left(\epsilon + 21\delta + 6\delta\epsilon\right).$$

Proof: We begin with the following decomposition:

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] \\
&= \frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle \,\middle|\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}_2}\right]\Pr\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}_2}\right] \\
&\quad + \frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle \,\middle|\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right]\Pr\left[\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right].
\end{aligned} \tag{123}$$

The first term on the RHS of (123) is lower bounded as shown in (124) at the bottom of the page, where in a) we have used (119), in b) we have used that $\mathcal{E}_{\mathbf{X}} \supseteq \mathcal{E}_{\mathbf{X}_2}$, and in c) we have used Lemma D.10.

We now turn to lower bounding the second term on the RHS of (123). The probability term is lower bounded as follows:

$$\begin{aligned}
\Pr\left[\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right] &= 1 - \Pr\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}_2}\right] \\
&\ge 1 - \left(\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + \Pr\left[\mathcal{E}_{\mathbf{X}}\right]\right).
\end{aligned} \tag{125}$$

To lower bound the expectation term, we represent $\mathbf{u}_i^*$ as a scaled version of $\mathbf{s}_i$ corrupted by an additive "quantization noise" $\mathbf{v}_i$. More precisely

$$\mathbf{u}_i^* = \nu_i\mathbf{s}_i + \mathbf{v}_i \quad \text{where} \quad \nu_i = \frac{\|\mathbf{u}_i^*\|}{\|\mathbf{s}_i\|}\cos\sphericalangle(\mathbf{s}_i, \mathbf{u}_i^*) \tag{126}$$

for $i \in \{1,2\}$. With this choice of ν_i, the vector $\mathbf{v}_i$ is always orthogonal to $\mathbf{s}_i$. By (126), the inner product $\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle$ can now be rewritten as $\nu_2\langle \mathbf{S}_1, \mathbf{S}_2\rangle + \langle \mathbf{S}_1, \mathbf{V}_2\rangle$. This leads to (127), shown at the bottom of the page, in which we have denoted by $\mathfrak{C}_i$ the

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle \,\middle|\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}_2}\right]\Pr\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}_2}\right] \\
&\overset{a)}{\ge} -\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 + \|\mathbf{U}_2^*\|^2\middle|\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}_2}\right]\Pr\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\mathbf{X}_2}\right] \\
&\overset{b)}{\ge} -\frac{1}{n}\Big(\mathsf{E}\left[\|\mathbf{S}_1\|^2\middle|\mathcal{E}_{\mathbf{S}}\right]\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + \mathsf{E}\left[\|\mathbf{S}_1\|^2\middle|\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_2}\right]\Pr\left[\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}}\right] + \|\mathbf{U}_2^*\|^2(\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + \Pr\left[\mathcal{E}_{\mathbf{X}}\right])\Big) \\
&\overset{c)}{\ge} -\Big(\sigma^2\left(\epsilon + \Pr\left[\mathcal{E}_{\mathbf{S}}\right]\right) + \sigma^2(1+\epsilon)\Pr\left[\mathcal{E}_{\mathbf{X}}\right] + \sigma^2(1-2^{-2R_2})\left(\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + \Pr\left[\mathcal{E}_{\mathbf{X}}\right]\right)\Big) \\
&\ge -\sigma^2\big(\epsilon + 2\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + (2+\epsilon)\Pr\left[\mathcal{E}_{\mathbf{X}}\right]\big)
\end{aligned} \tag{124}$$

$$\begin{aligned}
&\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle \,\middle|\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right] \\
&\overset{a)}{=} \mathsf{E}_{\mathbf{S}_1,\mathbf{S}_2}\left[\mathsf{E}_{\mathfrak{C}_1,\mathfrak{C}_2}\left[\nu_2\langle \mathbf{s}_1, \mathbf{s}_2\rangle \,\middle|(\mathbf{S}_1,\mathbf{S}_2) = (\mathbf{s}_1,\mathbf{s}_2), \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right] + \underbrace{\mathsf{E}_{\mathfrak{C}_1,\mathfrak{C}_2}\left[\langle \mathbf{s}_1, \mathbf{V}_2\rangle \,\middle|(\mathbf{S}_1,\mathbf{S}_2) = (\mathbf{s}_1,\mathbf{s}_2), \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right]}_{=0}\right] \\
&= \mathsf{E}_{\mathbf{S}_1,\mathbf{S}_2}\left[\frac{\|\mathbf{U}_2^*\|}{\|\mathbf{S}_2\|}\langle \mathbf{S}_1, \mathbf{S}_2\rangle\,\mathsf{E}_{\mathfrak{C}_1,\mathfrak{C}_2}\left[\cos\sphericalangle(\mathbf{s}_2, \mathbf{U}_2^*)\middle|(\mathbf{S}_1,\mathbf{S}_2) = (\mathbf{s}_1,\mathbf{s}_2), \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right]\right] \\
&\overset{b)}{\ge} \mathsf{E}_{\mathbf{S}_1,\mathbf{S}_2}\left[\|\mathbf{U}_2^*\|\|\mathbf{S}_1\|\cos\left(\sphericalangle(\mathbf{S}_1,\mathbf{S}_2)\right)\sqrt{1-2^{-2R_2}}(1-\epsilon)\middle|\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right] \\
&\overset{c)}{\ge} n\sqrt{\sigma^2(1-2^{-2R_2})\sigma^2(1-\epsilon)}\rho(1-\epsilon)\sqrt{1-2^{-2R_2}}(1-\epsilon) \\
&\ge n\rho\sigma^2(1-2^{-2R_2})(1-\epsilon)^3
\end{aligned} \tag{127}$$

random codebook of user $i \in \{1,2\}$, and where in $a)$ the second expectation term is zero because for every $(\mathbf{s}_1, \mathbf{s}_2) \in \mathcal{E}_{\mathbf{S}}^c$

$$\mathsf{E}_{\mathfrak{C}_2}\left[\langle \mathbf{s}_1, \mathbf{V}_2\rangle \middle| (\mathbf{S}_1,\mathbf{S}_2) = (\mathbf{s}_1,\mathbf{s}_2), \mathcal{E}_{\mathbf{X}_2}^c\right] = 0.$$

This holds since in the expectation over the codebooks $\mathfrak{C}_2$ with conditioning on $\mathcal{E}_{\mathbf{X}_2}^c$, for every $\mathbf{v}_2 \in \mathbb{R}^n$ the sequences $\mathbf{v}_2$ and $-\mathbf{v}_2$ are equiprobable and, thus, their inner products with $\mathbf{s}_1$ cancel off each other. Inequality b) follows from lower bounding $\cos\sphericalangle(\mathbf{s}_2, \mathbf{U}_2^*)$ conditioned on $\mathcal{E}_{\mathbf{X}}^c$ combined with the fact that conditioned on $\mathcal{E}_{\mathbf{S}}^c$ the term $\cos\sphericalangle(\mathbf{S}_1,\mathbf{S}_2)$ is positive. Inequality c) follows from lower bounding $\|\mathbf{S}_1\|$ and $\cos\sphericalangle(\mathbf{S}_1,\mathbf{S}_2)$ conditioned on $\mathcal{E}_{\mathbf{S}}^c$.

Combining (123) with (124), (125), and (127) gives

$$\begin{aligned}\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] &\geq -\sigma^2\left(\epsilon + 2\mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}\right] + (2+\epsilon)\mathrm{Pr}\left[\mathcal{E}_{\mathbf{X}}\right]\right)\\ &\quad + \sigma^2\rho(1-2^{-2R_2})(1-\epsilon)^3\left(1 - \left(\mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}\right] + \mathrm{Pr}\left[\mathcal{E}_{\mathbf{X}}\right]\right)\right)\\ &\geq \sigma^2\rho(1-2^{-2R_2})(1-\epsilon)^3\\ &\quad - \sigma^2\left(\epsilon + 3\mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}\right] + (3+\epsilon)\mathrm{Pr}\left[\mathcal{E}_{\mathbf{X}}\right]\right).\end{aligned}$$

Thus, by Lemma D.2 and Lemma D.4, it follows that for every $\delta > 0$ and $0.3 > \epsilon > 0$ there exists an $n'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n'(\delta,\epsilon)$

$$\begin{aligned}\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] &\geq \sigma^2\rho(1-2^{-2R_2})(1-\epsilon)^3 - \sigma^2\left(\epsilon + 21\delta + 6\delta\epsilon\right).\end{aligned}$$ □

The distortion of the genie-aided scheme is now upper bounded as follows:

$$\begin{aligned}\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] &= \sigma^2 - 2\gamma_{11}\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_1^*\rangle\right] - 2\gamma_{12}\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right]\\ &\quad + \gamma_{11}^2\sigma^2(1-2^{-2R_1}) + 2\gamma_{11}\gamma_{12}\frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle\right]\\ &\quad + \gamma_{12}^2\sigma^2(1-2^{-2R_2})\\ &\stackrel{a)}{\leq} \sigma^2 2^{-2R_1}\frac{1-\rho^2(1-2^{-2R_2})}{1-\tilde{\rho}^2} + \xi'(\delta,\epsilon)\end{aligned}$$

where in $a)$ we have used Lemma D.14, Lemma D.15, and Lemma D.16, and where

$$\lim_{\delta,\epsilon \to 0}\xi'(\delta,\epsilon) = 0.$$

E. Proofs of Lemma D.4, Lemma D.7, and Lemma D.9

The proofs in this section rely on bounds from the geometry of sphere packing. To this end, we denote by $C_n(\varphi)$ the surface area of a polar cap of half angle φ on an $\mathbb{R}^n$-sphere of unit radius. An illustration of $C_n(\varphi)$ is given in Fig. 11. Upper and lower bounds on the surface area $C_n(\varphi)$ are given in the following lemma.

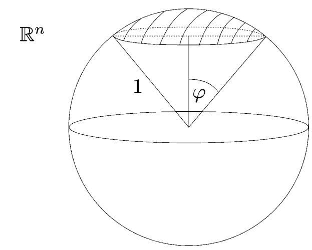

Fig. 11. Polar cap of half angle φ.

Lemma D.17: For any $\varphi \in [0, \pi/2]$

$$\frac{\Gamma\left(\frac{n}{2}+1\right)\sin^{(n-1)}\varphi}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}\cos\varphi}\left(1 - \frac{1}{n}\tan^2\varphi\right) \leq \frac{C_n(\varphi)}{C_n(\pi)} \leq \frac{\Gamma\left(\frac{n}{2}+1\right)\sin^{(n-1)}\varphi}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}\cos\varphi}.$$

Proof: See [30, Inequality (27)]. □

The ratio of the two gamma functions that appears in the upper bound and the lower bound of Lemma D.17 has the following asymptotic series.

Lemma D.18:

$$\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} = \sqrt{x}\left(1 - \frac{1}{8x} + \frac{1}{128x^2} + \frac{5}{1024x^3} - \frac{21}{32768x^4} + \cdots\right)$$

and in particular

$$\lim_{x\to\infty}\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)\sqrt{x}} = 1.$$

Proof: We first note that

$$\begin{aligned}\frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} &= \frac{(2x-1)!!}{2^x(x-1)!}\sqrt{\pi}\\ &= \frac{x}{4^x}\binom{2x}{x}\sqrt{\pi}\end{aligned} \tag{128}$$

where $\xi!!$ denotes the double factorial of ξ. The proof now follows by combining (128) with

$$\binom{2x}{x} = \frac{4^x}{\sqrt{\pi x}} \times\left(1 - \frac{1}{8x} + \frac{1}{128x^2} + \frac{5}{1024x^3} - \frac{21}{32768x^4} + \cdots\right)$$

which is given in [31, Problem 9.60, p. 495]. □

Before starting with the proofs of this section, we give one more lemma. To this end, whenever the vector-quantizer of Encoder 1 does not produce the all-zero sequence, denote by $\varsigma_1(\mathbf{s}_1, \mathcal{C}_1)$ the index of $\mathbf{u}_1^*$ in its codebook $\mathcal{C}_1$, and whenever the vector-quantizer of Encoder 1 produces the all-zero sequence, let $\varsigma_1(\mathbf{s}_1, \mathcal{C}_1) = 0$. Further, let $\lambda_1(\cdot)$ denote the measure on the codeword sphere $\mathcal{S}_1$ induced by the uniform distribution, and let $f^{\lambda_1}(\cdot)$ denote the density on $\mathcal{S}_1$ with respect to $\lambda_1(\cdot)$. Similarly, for Encoder 2 define $\varsigma_2(\mathbf{s}_2, \mathcal{C}_2)$ and $f^{\lambda_2}(\cdot)$.

Lemma D.19: Conditional on $\varsigma_1(\mathbf{S}_1, \mathfrak{C}_1) = 1$, the density of $\mathbf{U}_1(j)$ is upper bounded for every $j \in \{2, 3, \ldots, 2^{nR_1}\}$ and at every point $\mathbf{u} \in \mathcal{S}_1$ by twice the uniform density

$$f^{\lambda_1}\left(\mathbf{U}_1(j) = \mathbf{u} | \varsigma_1(\mathbf{S}_1, \mathfrak{C}_1) = 1\right) \leq 2 \frac{1}{r_1^{n-1} C_n(\pi)}$$

and similarly for Encoder 2.

Proof: We first write the conditional density as an average over $\cos\sphericalangle(\mathbf{S}_1, \mathbf{U}_1(1))$. Since conditioned on $\varsigma_1(\mathbf{S}_1, \mathfrak{C}_1) = 1$ we have $\cos\sphericalangle(\mathbf{S}_1, \mathbf{U}_1(1)) \in [\sqrt{1-2^{-2R_1}}(1-\epsilon), \sqrt{1-2^{-2R_1}}(1+\epsilon)]$, this then yields (129), shown at the bottom of the page. The proof now follows by upper bounding the conditional density

$$f^{\lambda_1}\Big(\mathbf{U}_1(j) = \mathbf{u} \Big| \mathbf{S}_1 = \mathbf{s}_1, \varsigma_1(\mathbf{s}_1, \mathfrak{C}_1) = 1, \cos\sphericalangle(\mathbf{s}_1, \mathbf{U}_1(1)) = a\Big).$$

To this end, define for every

$$a \in \left[\sqrt{1-2^{-2R_1}}(1-\epsilon), \sqrt{1-2^{-2R_1}}(1+\epsilon)\right]$$

the set $\mathcal{D}_a(\mathbf{s}_1)$ given in (130), shown at the bottom of the page, and its complement $\mathcal{D}_a^c(\mathbf{s}_1)$ given in (131), shown at the bottom of the page. The conditional density can now be upper bounded by distinguishing between $\mathbf{u} \in \mathcal{D}_a(\mathbf{s}_1)$ and $\mathbf{u} \in \mathcal{D}_a^c(\mathbf{s}_1)$. If $\mathbf{u} \in \mathcal{D}_a(\mathbf{s}_1)$, then the conditional density is zero because the fact that $\varsigma_1(\mathbf{s}_1, \mathfrak{C}_1)$ is 1 implies that for all $j \in \{2, 3, \ldots, 2^{nR_1}\}$

$$\left|\cos\sphericalangle(\mathbf{s}_1, \mathbf{U}_1(j)) - \sqrt{1-2^{-2R_1}}\right| > \left|a - \sqrt{1-2^{-2R_1}}\right|$$

and if $\mathbf{u} \in \mathcal{D}_a^c(\mathbf{s}_1)$ the conditional density is uniform over $\mathcal{D}_a^c(\mathbf{s}_1)$, i.e.,

$$f^{\lambda_1}\Big(\mathbf{U}_1(j) = \mathbf{u} \Big| \mathbf{S}_1 = \mathbf{s}_1, \varsigma_1(\mathbf{s}_1, \mathfrak{C}_1) = 1 \; \cos\sphericalangle(\mathbf{s}_1, \mathbf{U}_1(1)) = a\Big) = \upsilon, \quad \mathbf{u} \in \mathcal{D}_a^c(\mathbf{s}_1)$$

for some $\upsilon > 0$. Thus, for all $\mathbf{u} \in \mathcal{S}_1$, $\mathbf{s}_1 \in \mathbb{R}^n$, and all $a \in \left[\sqrt{1-2^{-2R_1}}(1-\epsilon), \sqrt{1-2^{-2R_1}}(1+\epsilon)\right]$,

$$f^{\lambda_1}\Big(\mathbf{U}_1(j) = \mathbf{u} \Big| \mathbf{S}_1 = \mathbf{s}_1, \varsigma_1(\mathbf{s}_1, \mathfrak{C}_1) = 1, \cos\sphericalangle(\mathbf{s}_1, \mathbf{U}_1(1)) = a\Big) \leq \upsilon. \tag{132}$$

It now remains to upper bound υ. To this end, notice that the surface area of $\mathcal{D}_a(\mathbf{s}_1)$ never exceeds half the surface area of $\mathcal{S}_1$. This follows since $\sqrt{1-2^{-2R_1}}(1-\epsilon) > 0$, and, therefore, every $\mathbf{u} \in \mathcal{D}_a(\mathbf{s}_1)$ satisfies $|\sphericalangle(\mathbf{s}_1, \mathbf{u})| < \pi/2$. Hence, the surface area of $\mathcal{D}_a^c(\mathbf{s}_1)$ is always larger than half the surface area of $\mathcal{S}_1$ and, therefore

$$\upsilon \leq 2 \cdot \frac{1}{r_1^{n-1} C_n(\pi)}. \tag{133}$$

Combining (133) with (132) and (129) proves the lemma. □

1) Proof of Lemma D.4: We begin with the following decomposition

$$\begin{aligned}
\Pr[\mathcal{E}_{\mathbf{X}}] &= \Pr[\mathcal{E}_{\mathbf{X}} \cap \mathcal{E}_{\mathbf{S}}] + \Pr[\mathcal{E}_{\mathbf{X}} \cap \mathcal{E}_{\mathbf{S}}^c] \\
&\leq \Pr[\mathcal{E}_{\mathbf{S}}] + \Pr[\mathcal{E}_{\mathbf{X}_1} \cap \mathcal{E}_{\mathbf{S}}^c] + \Pr[\mathcal{E}_{\mathbf{X}_2} \cap \mathcal{E}_{\mathbf{S}}^c] \\
&\quad + \Pr\left[\mathcal{E}_{(\mathbf{X}_1, \mathbf{X}_2)} \cap \mathcal{E}_{\mathbf{X}_2}^c \cap \mathcal{E}_{\mathbf{X}_1}^c \cap \mathcal{E}_{\mathbf{S}}^c\right] \\
&\leq \Pr[\mathcal{E}_{\mathbf{S}}] + \Pr[\mathcal{E}_{\mathbf{X}_1}] + \Pr[\mathcal{E}_{\mathbf{X}_2}] \\
&\quad + \Pr\left[\mathcal{E}_{(\mathbf{X}_1, \mathbf{X}_2)} \cap \mathcal{E}_{\mathbf{X}_2}^c \cap \mathcal{E}_{\mathbf{X}_1}^c \cap \mathcal{E}_{\mathbf{S}}^c\right].
\end{aligned}$$

$$\begin{aligned}
f^{\lambda_1}\left(\mathbf{U}_1(j) = \mathbf{u} \middle| \varsigma_1(\mathbf{S}_1, \mathfrak{C}_1) = 1\right) = \int_{\mathbf{s}_1 \in \mathbb{R}^n} \int_{\sqrt{1-2^{-2R_1}}(1-\epsilon)}^{\sqrt{1-2^{-2R_1}}(1+\epsilon)} & f^{\lambda_1}\Big(\mathbf{U}_1(j) = \mathbf{u} \Big| \mathbf{S}_1 = \mathbf{s}_1, \varsigma_1(\mathbf{s}_1, \mathfrak{C}_1) = 1, \cos\sphericalangle(\mathbf{s}_1, \mathbf{U}_1(1)) = a\Big) \\
& \cdot f\Big(\mathbf{S}_1 = \mathbf{s}_1, \cos\sphericalangle(\mathbf{s}_1, \mathbf{U}_1(1)) = a \Big| \varsigma_1(\mathbf{S}_1, \mathfrak{C}_1) = 1\Big) \mathrm{d}a \, \mathrm{d}\mathbf{s}_1.
\end{aligned} \tag{129}$$

$$\mathcal{D}_a(\mathbf{s}_1) \triangleq \left\{\mathbf{u} \in \mathcal{S}_1 : \left|\cos\sphericalangle(\mathbf{s}_1, \mathbf{u}) - \sqrt{1-2^{-2R_1}}\right| \leq \left|a - \sqrt{1-2^{-2R_1}}\right|\right\} \tag{130}$$

$$\mathcal{D}_a^c(\mathbf{s}_1) \triangleq \left\{\mathbf{u} \in \mathcal{S}_1 : \left|\cos\sphericalangle(\mathbf{s}_1, \mathbf{u}) - \sqrt{1-2^{-2R_1}}\right| > \left|a - \sqrt{1-2^{-2R_1}}\right|\right\}. \tag{131}$$

The proof of Lemma D.4 now follows by showing that for every $\delta > 0$ and $0.3 > \epsilon > 0$ there exists an $n_2'(\delta, \epsilon) > 0$ such that for all $n > n_2'(\delta, \epsilon)$

$$\Pr[\mathcal{E}_{\mathbf{X}_i}] \leq \delta, \quad i \in \{1, 2\} \tag{134}$$

$$\Pr\left[\mathcal{E}_{(\mathbf{X}_1, \mathbf{X}_2)} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_1}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right] \leq 3\delta. \tag{135}$$

a) Proof of (134): We give the proof for $\mathcal{E}_{\mathbf{X}_1}$. Due to the symmetry the proof for $\mathcal{E}_{\mathbf{X}_2}$ then follows by similar arguments. Let $\mathcal{E}_{\mathbf{X}_1}(j)$ be the event that $\mathbf{U}_1(j)$ does not have a typical angle to $\mathbf{S}_1$, i.e.,

$$\mathcal{E}_{\mathbf{X}_1}(j) = \left\{(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2) : \left|\cos\sphericalangle(\mathbf{u}_1(j), \mathbf{s}_1) - \tau\right| > \epsilon\tau\right\}$$

where we have use the shorthand notation $\tau = \sqrt{1 - 2^{-2R_1}}$. Then

$$\begin{aligned}
\Pr[\mathcal{E}_{\mathbf{X}_1}] &= \Pr\left[\mathcal{E}_{\mathbf{X}_1} \middle| \mathbf{S}_1 = \mathbf{s}_1\right] \\
&= \Pr\left[\bigcap_{j=1}^{2^{nR_1}} \mathcal{E}_{\mathbf{X}_1}(j) \middle| \mathbf{S}_1 = \mathbf{s}_1\right] \\
&= \prod_{j=1}^{2^{nR_1}} \Pr\left[\mathcal{E}_{\mathbf{X}_1}(j) \middle| \mathbf{S}_1 = \mathbf{s}_1\right] \\
&\overset{a)}{=} \prod_{j=1}^{2^{nR_1}} \Pr[\mathcal{E}_{\mathbf{X}_1}(j)] \\
&\overset{b)}{=} \left(\Pr[\mathcal{E}_{\mathbf{X}_1}(1)]\right)^{2^{nR_1}} \\
&= \left(1 - \Pr\left[\mathcal{E}_{\mathbf{X}_1}^c(1)\right]\right)^{2^{nR_1}}
\end{aligned} \tag{136}$$

where in $a)$ we have used that the probability of $\mathcal{E}_{\mathbf{X}_1}(j)$ does not depend on $\mathbf{S}_1 = \mathbf{s}_1$, and in $b)$ we have used that all $\mathbf{U}_1(j)$ have the same distribution. To upper bound (136) we rewrite $\mathcal{E}_{\mathbf{X}_1}^c(1)$ as in (137), shown at the bottom of the page, where we have used the notation

$$\cos\theta_{1,\max} \triangleq \sqrt{1 - 2^{-2R_1}}(1 - \epsilon)$$

and

$$\cos\theta_{1,\min} \triangleq \sqrt{1 - 2^{-2R_1}}(1 + \epsilon).$$

Hence, since $\mathbf{U}_1(1)$ is generated independently of $\mathbf{S}_1$ and distributed uniformly on $\mathcal{S}_1$

$$\Pr\left[\mathcal{E}_{\mathbf{X}_1}^c(1)\right] = \frac{C_n(\theta_{1,\max}) - C_n(\theta_{1,\min})}{C_n(\pi)}. \tag{138}$$

Combining (138) with (136) then gives (139), shown at the bottom of the page, where in $a)$ we have used that $1 - x \leq \exp(-x)$, and in $b)$ we have lower bounded $C_n(\theta_{1,\max})/C_n(\pi)$ and upper bounded $C_n(\theta_{1,\min})/C_n(\pi)$ according to Lemma D.17. It now follows from sphere-packing and -covering, see, e.g., [32], that for every $\epsilon > 0$ we have $\Pr[\mathcal{E}_{\mathbf{X}_1}] \to 0$ as $n \to \infty$. More precisely, this holds since the exponent on the RHS of (139) grows exponentially in n. This follows since on the one hand for large n

$$\frac{\Gamma\left(\frac{n}{2} + 1\right)}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}} \approx \frac{1}{\sqrt{n2\pi}}$$

$$\begin{aligned}
\mathcal{E}_{\mathbf{X}_1}^c(1) &= \left\{(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2) : \left|\cos\sphericalangle(\mathbf{u}_1(1), \mathbf{s}_1) - \sqrt{1 - 2^{-2R_1}}\right| \leq \epsilon\sqrt{1 - 2^{-2R_1}}\right\} \\
&= \left\{(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2) : \sqrt{1 - 2^{-2R_1}}(1 - \epsilon) \leq \cos\sphericalangle(\mathbf{u}_1(1), \mathbf{s}_1) \leq \sqrt{1 - 2^{-2R_1}}(1 + \epsilon)\right\} \\
&= \left\{(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2) : \cos\theta_{1,\max} \leq \cos\sphericalangle(\mathbf{u}_1(1), \mathbf{s}_1) \leq \cos\theta_{1,\min}\right\}
\end{aligned} \tag{137}$$

$$\begin{aligned}
\Pr[\mathcal{E}_{\mathbf{X}_1}] &= \left(1 - \frac{C_n(\theta_{1,\max}) - C_n(\theta_{1,\min})}{C_n(\pi)}\right)^{2^{nR_1}} \\
&\overset{a)}{\leq} \left(\exp\left(-\frac{C_n(\theta_{1,\max}) - C_n(\theta_{1,\min})}{C_n(\pi)}\right)\right)^{2^{nR_1}} \\
&\overset{b)}{\leq} \exp\left[-2^{nR_1}\frac{\Gamma\left(\frac{n}{2} + 1\right)}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}\left(\frac{\sin^{(n-1)}\theta_{1,\max}}{\cos\theta_{1,\max}}\left(1 - \frac{1}{n}\tan^2\theta_{1,\max}\right) - \frac{\sin^{(n-1)}\theta_{1,\min}}{\cos\theta_{1,\min}}\right)\right] \\
&= \exp\left[-\frac{\Gamma\left(\frac{n}{2} + 1\right)}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}\left(\frac{2^{n(R_1 + \log_2(\sin\theta_{1,\max}))}}{\sin\theta_{1,\max}\cos\theta_{1,\max}}\left(1 - \frac{1}{n}\tan^2\theta_{1,\max}\right) - \frac{2^{n(R_1 + \log_2(\sin\theta_{1,\min}))}}{\sin\theta_{1,\min}\cos\theta_{1,\min}}\right)\right]
\end{aligned} \tag{139}$$

and on the other hand the term

$$\frac{2^{n(R_1+\log_2(\sin\theta_{1,\max}))}}{\sin\theta_{1,\max}\cos\theta_{1,\max}}\left(1-\frac{1}{n}\tan^2\theta_{1,\max}\right) - \frac{2^{n(R_1+\log_2(\sin\theta_{1,\min}))}}{\sin\theta_{1,\min}\cos\theta_{1,\min}} \quad (140)$$

grows exponentially in n. The latter holds since first of all

$$\left(1-\frac{1}{n}\tan^2\theta_{1,\max}\right)\approx 1 \quad \text{for large } n$$

second, the denominators of the fractions are independent of n, and third since

$$R_1+\log_2(\sin\theta_{1,\max}) \geq R_1+\log_2(\sin\theta_{1,\min})$$

with $R_1+\log_2(\sin\theta_{1,\max})>0$. That $R_1+\log_2(\sin\theta_{1,\max})>0$ can be seen as follows:

$$\begin{aligned}
&-\log_2(\sin\theta_{1,\max})\\
&= -\log_2\left(\sqrt{1-\cos^2\theta_{1,\max}}\right)\\
&\overset{a)}{=} -\frac{1}{2}\log_2\left(2^{-2R_1}+\epsilon(2-\epsilon)(1-2^{-2R_1})\right)\\
&< -\frac{1}{2}\log_2\left(2^{-2R_1}\right)\\
&= R_1
\end{aligned}$$

where in $a)$ we have used the definition of $\cos\theta_{1,\max}$. □

Proof of (135): By the notation in (126) we have

$$\begin{aligned}
\cos\sphericalangle(\mathbf{u}_1^*,\mathbf{u}_2^*) &= \frac{\langle\mathbf{u}_1^*,\mathbf{u}_2^*\rangle}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\\
&= \frac{1}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\Big(\nu_1\nu_2\langle\mathbf{s}_1,\mathbf{s}_2\rangle+\nu_1\langle\mathbf{s}_1,\mathbf{v}_2\rangle\\
&\qquad +\nu_2\langle\mathbf{v}_1,\mathbf{s}_2\rangle+\langle\mathbf{v}_1,\mathbf{v}_2\rangle\Big)
\end{aligned} \quad (141)$$

where we recall that ν_1 is a function of $\|\mathbf{s}_1\|$ and $\cos\sphericalangle(\mathbf{s}_1,\mathbf{u}_1^*)$ and similarly ν_2 is a function of $\|\mathbf{s}_2\|$ and $\cos\sphericalangle(\mathbf{s}_2,\mathbf{u}_2^*)$. Now, define the four events

$$\begin{aligned}
\mathcal{A}_1 &= \left\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2): \left|\tilde{\rho}-\frac{\nu_1\nu_2}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\langle\mathbf{s}_1,\mathbf{s}_2\rangle\right|>4\epsilon\right\}\\
\mathcal{A}_2 &= \left\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2): \left|\frac{\nu_1}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\langle\mathbf{s}_1,\mathbf{v}_2\rangle\right|>\epsilon\right\}\\
\mathcal{A}_3 &= \left\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2): \left|\frac{\nu_2}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\langle\mathbf{s}_2,\mathbf{v}_1\rangle\right|>\epsilon\right\}\\
\mathcal{A}_4 &= \left\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2): \left|\frac{1}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\langle\mathbf{v}_1,\mathbf{v}_2\rangle\right|>\epsilon\right\}.
\end{aligned}$$

Note that by (141), $\mathcal{E}_{(\mathbf{X}_1,\mathbf{X}_2)}\subset(\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_3\cup\mathcal{A}_4)$. Thus

$$\begin{aligned}
&\Pr\left[\mathcal{E}_{(\mathbf{X}_1,\mathbf{X}_2)}\cap\mathcal{E}_{\mathbf{S}}^c\cap\mathcal{E}_{\mathbf{X}_1}^c\cap\mathcal{E}_{\mathbf{X}_2}^c\right]\\
&\quad\leq \Pr\left[\mathcal{A}_1\cap\mathcal{E}_{\mathbf{S}}^c\cap\mathcal{E}_{\mathbf{X}_1}^c\cap\mathcal{E}_{\mathbf{X}_2}^c\right]+\Pr\left[\mathcal{A}_2\cap\mathcal{E}_{\mathbf{S}}^c\cap\mathcal{E}_{\mathbf{X}_1}^c\cap\mathcal{E}_{\mathbf{X}_2}^c\right]\\
&\qquad+\Pr\left[\mathcal{A}_3\cap\mathcal{E}_{\mathbf{S}}^c\cap\mathcal{E}_{\mathbf{X}_1}^c\cap\mathcal{E}_{\mathbf{X}_2}^c\right]\\
&\qquad+\Pr\left[\mathcal{A}_4\cap\mathcal{E}_{\mathbf{S}}^c\cap\mathcal{E}_{\mathbf{X}_1}^c\cap\mathcal{E}_{\mathbf{X}_2}^c\right]\\
&\quad\leq \Pr\left[\mathcal{A}_1|\mathcal{E}_{\mathbf{S}}^c\cap\mathcal{E}_{\mathbf{X}_1}^c\cap\mathcal{E}_{\mathbf{X}_2}^c\right]+\Pr\left[\mathcal{A}_2|\mathcal{E}_{\mathbf{S}}^c\right]\\
&\qquad+\Pr\left[\mathcal{A}_3|\mathcal{E}_{\mathbf{S}}^c\right]+\Pr\left[\mathcal{A}_4|\mathcal{E}_{\mathbf{S}}^c\right].
\end{aligned} \quad (142)$$

The four terms on the RHS of (142) are now bounded in the following two lemmas.

Lemma D.20: For $\epsilon\leq 0.3$

$$\Pr\left[\mathcal{A}_1|\mathcal{E}_{\mathbf{S}}^c\cap\mathcal{E}_{\mathbf{X}_1}^c\cap\mathcal{E}_{\mathbf{X}_2}^c\right]=0.$$

Proof: We first note that the term in the definition of $\mathcal{A}_1$ can be rewritten as

$$\frac{\nu_1\nu_2}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\langle\mathbf{s}_1,\mathbf{s}_2\rangle = \cos\sphericalangle(\mathbf{s}_1,\mathbf{u}_1^*)\cos\sphericalangle(\mathbf{s}_2,\mathbf{u}_2^*)\cos\sphericalangle(\mathbf{s}_1,\mathbf{s}_2). \quad (143)$$

We can now upper and lower bound the RHS of (143) for $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2)\in\mathcal{E}_{\mathbf{S}}^c\cap\mathcal{E}_{\mathbf{X}_1}^c\cap\mathcal{E}_{\mathbf{X}_2}^c$ by noticing that $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2)\in\mathcal{E}_{\mathbf{S}}^c$ implies

$$|\cos\sphericalangle(\mathbf{s}_1,\mathbf{s}_2)-\rho|<\rho\epsilon$$

that $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2)\in\mathcal{E}_{\mathbf{X}_1}^c$ implies

$$\left|\sqrt{1-2^{-2R_1}}-\cos\sphericalangle(\mathbf{s}_1,\mathbf{u}_1^*)\right|<\epsilon\sqrt{1-2^{-2R_1}}$$

and that $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2)\in\mathcal{E}_{\mathbf{X}_2}^c$ implies

$$\left|\sqrt{1-2^{-2R_2}}-\cos\sphericalangle(\mathbf{s}_2,\mathbf{u}_2^*)\right|<\epsilon\sqrt{1-2^{-2R_2}}.$$

Hence, combined with (143) this gives

$$\tilde{\rho}(1-\epsilon)^3\leq\frac{\nu_1\nu_2}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\langle\mathbf{s}_1,\mathbf{s}_2\rangle\leq\tilde{\rho}(1+\epsilon)^3$$

for all $(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2)\in\mathcal{E}_{\mathbf{S}}^c\cap\mathcal{E}_{\mathbf{X}_1}^c\cap\mathcal{E}_{\mathbf{X}_2}^c$. The LHS can be lower bounded by $\tilde{\rho}(1-3\epsilon)\leq\tilde{\rho}(1-\epsilon)^3$, and the RHS can be upper bounded by $\tilde{\rho}(1+\epsilon)^3\leq\tilde{\rho}(1+4\epsilon)$ whenever $\epsilon\leq 0.3$. Hence, for $\epsilon\leq 0.3$

$$\left|\tilde{\rho}-\frac{\nu_1\nu_2}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\langle\mathbf{s}_1,\mathbf{s}_2\rangle\right|\leq 4\tilde{\rho}\epsilon\leq 4\epsilon.$$

□

Lemma D.21: For every $\delta>0$ and $\epsilon>0$ there exists an $n'_{\mathcal{A}}(\delta,\epsilon)$ such that for all $n>n'_{\mathcal{A}}(\delta,\epsilon)$

$$\Pr\left[\mathcal{A}_2|\mathcal{E}_{\mathbf{S}}^c\right]<\delta,\quad \Pr\left[\mathcal{A}_3|\mathcal{E}_{\mathbf{S}}^c\right]<\delta,\quad \Pr\left[\mathcal{A}_4|\mathcal{E}_{\mathbf{S}}^c\right]<\delta.$$

Proof: We start with the derivation of the bound on $\mathcal{A}_2$. To this end, we first upper bound the inner product between $\mathbf{s}_1$ and

$\mathbf{v}_2$. Let $\mathbf{s}_{1,\mathrm{P}}$ denote the projection of $\mathbf{s}_1$ onto the subspace of $\mathbb{R}^n$ that is orthogonal to $\mathbf{s}_2$, and that, thus, contains $\mathbf{v}_2$. Hence

$$\begin{aligned}\left|\frac{\nu_1}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\langle\mathbf{s}_1,\mathbf{v}_2\rangle\right| &\overset{a)}{=} \left|\cos\sphericalangle(\mathbf{s}_1,\mathbf{u}_1^*)\left\langle\frac{\mathbf{s}_1}{\|\mathbf{s}_1\|},\frac{\mathbf{v}_2}{\|\mathbf{u}_2^*\|}\right\rangle\right| \\ &\overset{b)}{\le} |\cos\sphericalangle(\mathbf{s}_1,\mathbf{u}_1^*)|\left|\left\langle\frac{\mathbf{s}_1}{\|\mathbf{s}_1\|},\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}\right\rangle\right| \\ &\le \left|\left\langle\frac{\mathbf{s}_1}{\|\mathbf{s}_1\|},\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}\right\rangle\right| \\ &= \left|\left\langle\frac{\mathbf{s}_{1,\mathrm{P}}}{\|\mathbf{s}_1\|},\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}\right\rangle\right| \\ &\le \left|\left\langle\frac{\mathbf{s}_{1,P}}{\|\mathbf{s}_{1,\mathrm{P}}\|},\frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}\right\rangle\right| \\ &= |\cos\sphericalangle(\mathbf{s}_{1,\mathrm{P}},\mathbf{v}_2)| \end{aligned} \tag{144}$$

where $a)$ follows by the definition of ν_1 and $b)$ follows since by the definition of $\mathbf{v}_2$ we have $\|\mathbf{v}_2\| \le \|\mathbf{u}_2^*\|$. By (144), we obtain the inequality shown in the equation at the bottom of the page, where in the last line we have denoted by $\Pr_{\mathfrak{C}_1,\mathfrak{C}_2}(\cdot|\cdot)$ the conditional probability of the codebooks $\mathfrak{C}_1$ and $\mathfrak{C}_2$ being such that $|\pi/2 - \sphericalangle(\mathbf{s}_{1,\mathrm{P}},\mathbf{V}_2)| > \epsilon$ given $(\mathbf{S}_1,\mathbf{S}_2) = (\mathbf{s}_1,\mathbf{s}_2)$ and $(\mathbf{s}_1,\mathbf{s}_2) \in \mathcal{E}_{\mathbf{S}}^c$. To conclude our bound we now notice that conditioned on $(\mathbf{S}_1,\mathbf{S}_2) = (\mathbf{s}_1,\mathbf{s}_2)$, the random vector $\mathbf{V}_2/\|\mathbf{V}_2\|$ is distributed uniformly on the surface of the centered $\mathbb{R}^{n-1}$-sphere of unit radius that lies in the subspace that is orthogonal to $\mathbf{s}_2$. Hence

$$\begin{aligned}\Pr[\mathcal{A}_2|\mathcal{E}_{\mathbf{S}}^c] &\le \mathsf{E}_{\mathbf{S}_1,\mathbf{S}_2}\left[\frac{2C_{n-1}\left(\frac{\pi}{2}-\epsilon\right)}{C_{n-1}(\pi)}\middle|\mathcal{E}_{\mathbf{S}}^c\right] \\ &\le \frac{2C_{n-1}\left(\frac{\pi}{2}-\epsilon\right)}{C_{n-1}(\pi)} \\ &\le \frac{2\Gamma\left(\frac{n+1}{2}\right)}{(n-1)\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}}\frac{\sin^{(n-2)}\left(\frac{\pi}{2}-\epsilon\right)}{\cos\left(\frac{\pi}{2}-\epsilon\right)} \\ &\le \frac{2\Gamma\left(\frac{n+1}{2}\right)}{(n-1)\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}\cos\left(\frac{\pi}{2}-\epsilon\right)}.\end{aligned}$$

Upper bounding the ratio of Gamma functions by the asymptotic series of Lemma D.18, gives for every $\epsilon > 0$ that $\Pr[\mathcal{A}_2|\mathcal{E}_{\mathbf{S}}^c] \to 0$ as $n \to \infty$. By similar arguments it also follows that $\Pr[\mathcal{A}_3|\mathcal{E}_{\mathbf{S}}^c] \to 0$ as $n \to \infty$.

To conclude the proof of Lemma D.21, we derive the bound on $\mathcal{A}_4$. The derivations are similar to those for $\mathcal{A}_2$. First, define by $\mathbf{v}_{1,\mathrm{P}}$ the projection of $\mathbf{v}_1$ onto the subspace of $\mathbb{R}^n$ that is orthogonal to $\mathbf{s}_2$. As in (144) we can show that

$$\left|\frac{1}{\|\mathbf{u}_1^*\|\|\mathbf{u}_2^*\|}\langle\mathbf{v}_1,\mathbf{v}_2\rangle\right| \le |\cos\sphericalangle(\mathbf{v}_{1,\mathrm{P}},\mathbf{v}_2)| \tag{145}$$

from which, using $|\cos x| \le |\pi/2 - x|$, we then obtain (146), shown at the bottom of the page. The desired bound now follows from noticing that conditioned on $(\mathbf{S}_1,\mathbf{S}_2) = (\mathbf{s}_1,\mathbf{s}_2)$ and $\mathfrak{C}_1 = \mathcal{C}_1$, the random vector $\mathbf{V}_2/\|\mathbf{V}_2\|$ is distributed uniformly on the surface of the centered $\mathbb{R}^{n-1}$-sphere of unit radius that lies in the subspace that is orthogonal to $\mathbf{s}_2$. Hence, similarly as in the derivation for $\mathcal{A}_2$

$$\begin{aligned}\Pr[\mathcal{A}_4|\mathcal{E}_{\mathbf{S}}^c] &\le \mathsf{E}_{\mathbf{S}_1,\mathbf{S}_2,\mathfrak{C}_1}\left[\frac{2C_{n-1}\left(\frac{\pi}{2}-\epsilon\right)}{C_{n-1}(\pi)}\middle|\mathcal{E}_{\mathbf{S}}^c\right] \\ &\le \frac{2\Gamma\left(\frac{n+1}{2}\right)}{(n-1)\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}\cos\left(\frac{\pi}{2}-\epsilon\right)}.\end{aligned}$$

Upper bounding the ratio of Gamma functions by the asymptotic series of Lemma D.18, gives for every $\epsilon > 0$ that $\Pr[\mathcal{A}_4|\mathcal{E}_{\mathbf{S}}^c] \to 0$ as $n \to \infty$. □

Combining Lemma D.20 and Lemma D.21 with (142) gives that for every $\delta > 0$ and $0.3 > \epsilon > 0$ there exists an $n'_{\mathcal{A}}(\delta,\epsilon)$ such that for all $n > n'_{\mathcal{A}}(\delta,\epsilon)$

$$\Pr\left[\mathcal{E}_{(\mathbf{X}_1,\mathbf{X}_2)} \cap \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\mathbf{X}_1}^c \cap \mathcal{E}_{\mathbf{X}_2}^c\right] \le 3\delta. \qquad \square$$

2) Proof of Lemma D.7: The proof follows from upper bounding $\Pr\left[\mathcal{G}|\mathcal{E}_{\mathbf{X}_1}^c\right]$ as a function of R_1. First, note that

$$\begin{aligned}\Pr\left[\mathcal{G}|\mathcal{E}_{\mathbf{X}_1}^c\right] &= \Pr[\mathcal{G}|\mathbf{U}_1^* \ne \mathbf{0}] \\ &= \Pr[\mathcal{G}|\varsigma_1(\mathbf{S}_1,\mathfrak{C}_1) = 1]\end{aligned} \tag{147}$$

where the second equality holds because the conditional distribution of the codewords conditional on $\mathbf{u}_1^* \ne \mathbf{0}$ is invariant with respect to permutations of the indexing of the codewords. The desired upper bound is now obtained by decomposing $\mathcal{G}$ into sub-events $\mathcal{G}_j$, $j \in \{2,3,\ldots,2^{nR_1}\}$, where

$$\mathcal{G}_j \triangleq \Big\{(\mathbf{s}_1,\mathbf{s}_2,\mathcal{C}_1,\mathcal{C}_2,\mathbf{z}) : \cos\sphericalangle(\mathbf{w},\mathbf{u}_1(j)) \ge \Delta\Big\}.$$

$$\begin{aligned}\Pr[\mathcal{A}_2|\mathcal{E}_{\mathbf{S}}^c] &\le \Pr\left[(\mathbf{S}_1,\mathbf{S}_2,\mathfrak{C}_1,\mathfrak{C}_2) : |\cos\sphericalangle(\mathbf{S}_{1,\mathrm{P}},\mathbf{V}_2)| > \epsilon\middle|\mathcal{E}_{\mathbf{S}}^c\right] \\ &\le \Pr\left[(\mathbf{S}_1,\mathbf{S}_2,\mathfrak{C}_1,\mathfrak{C}_2) : \left|\frac{\pi}{2} - \sphericalangle(\mathbf{S}_{1,\mathrm{P}},\mathbf{V}_2)\right| > \epsilon\middle|\mathcal{E}_{\mathbf{S}}^c\right] \\ &= \mathsf{E}_{\mathbf{S}_1,\mathbf{S}_2}\left[\Pr_{\mathfrak{C}_1,\mathfrak{C}_2}\left(\left|\frac{\pi}{2} - \sphericalangle(\mathbf{s}_{1,\mathrm{P}},\mathbf{V}_2)\right| > \epsilon\middle|(\mathbf{S}_1,\mathbf{S}_2) = (\mathbf{s}_1,\mathbf{s}_2),\mathcal{E}_{\mathbf{S}}^c\right)\right]\end{aligned}$$

$$\Pr[\mathcal{A}_4|\mathcal{E}_{\mathbf{S}}^c] \le \mathsf{E}_{\mathbf{S}_1,\mathbf{S}_2,\mathfrak{C}_1}\left[\Pr_{\mathfrak{C}_2}\left(\left|\frac{\pi}{2} - \sphericalangle(\mathbf{v}_{1,\mathrm{P}},\mathbf{V}_2)\right| > \epsilon\middle|(\mathbf{S}_1,\mathbf{S}_2) = (\mathbf{s}_1,\mathbf{s}_2),\mathfrak{C}_1 = \mathcal{C}_1,\mathcal{E}_{\mathbf{S}}^c\right)\right]. \tag{146}$$

By (147) we now have

$$\begin{aligned}\Pr\left[\mathcal{G}|\mathcal{E}^c_{\mathbf{X}_1}\right] &= \Pr\left[\bigcup_{j=2}^{2^{nR_1}} \mathcal{G}_j \middle| \varsigma_1(\mathbf{S}_1, \mathfrak{C}_1) = 1\right] \\ &\leq \sum_{j=2}^{2^{nR_1}} \Pr\left[\mathcal{G}_j | \varsigma_1(\mathbf{S}_1, \mathfrak{C}_1) = 1\right] \\ &< 2^{nR_1} \Pr\left[\mathcal{G}_2 | \varsigma_1(\mathbf{S}_1, \mathfrak{C}_1) = 1\right] \\ &\leq 2^{nR_1} \cdot 2 \frac{C_n(\arccos \Delta)}{C_n(\pi)} \end{aligned} \tag{148}$$

where in the third step we have used that $\Pr\left[\mathcal{G}_j|\varsigma_i(\mathbf{s}_i, \mathfrak{C}_i) = 1\right]$ is the same for all $j \in \{2, 3, \ldots 2^{nR_1}\}$ because the conditional distribution of $\mathbf{u}_1(j)$ given $\varsigma_i(\mathbf{s}_i, \mathfrak{C}_i) = 1$ does not depend on $j \in \{2, 3, \ldots, 2^{nR_1}\}$ and where in the last step we have upper bounded the density of $\mathbf{U}_1(2)$, conditional on $\varsigma_1(\mathbf{S}_1, \mathfrak{C}_1) = 1$, by Lemma D.19. Thus, combining (148) with Lemma D.17 gives

$$\begin{aligned}\Pr\left[\mathcal{G}|\mathcal{E}^c_{\mathbf{X}_1}\right] &\leq 2^{nR_1} \cdot 2 \frac{\Gamma\left(\frac{n}{2}+1\right)(1-\Delta^2)^{(n-1)/2}}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}\Delta} \\ &= \frac{2\Gamma\left(\frac{n}{2}+1\right)}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}\Delta\sqrt{1-\Delta^2}} \\ &\quad \times 2^{n(R_1 + 1/2\log_2(1-\Delta^2))}.\end{aligned}$$

Replacing the ratio of the Gamma functions by the asymptotic series of Lemma D.18 establishes (106). □

3) Proof of Lemma D.9: The proof follows by upper bounding $\Pr\left[\mathcal{G}|\mathcal{E}^c_{\mathbf{X}_1} \cap \mathcal{E}^c_{\mathbf{X}_2}\right]$ as a function of $R_1 + R_2$. To this end, define

$$\tilde{\varsigma}(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2) \triangleq \left(\varsigma_1(\mathbf{s}_1, \mathcal{C}_1), \varsigma_2(\mathbf{s}_2, \mathcal{C}_2)\right).$$

By a symmetry argument, which is similar to the one in the proof of Lemma D.7, we obtain

$$\Pr\left[\mathcal{G}|\mathcal{E}^c_{\mathbf{X}_1} \cap \mathcal{E}^c_{\mathbf{X}_2}\right] = \Pr\left[\mathcal{G}|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)\right]. \tag{149}$$

The desired upper bound is now obtained by decomposing $\mathcal{G}$ into subevents $\mathcal{G}_{j,\ell}$, where

$$\mathcal{G}_{j,\ell} = \Big\{(\mathbf{s}_1, \mathbf{s}_2, \mathcal{C}_1, \mathcal{C}_2, \mathbf{z}) : \cos\sphericalangle(\mathbf{u}_1(j), \mathbf{u}_2(\ell)) \geq \Theta \\ \cos\sphericalangle(\mathbf{y}, \alpha_1\mathbf{u}_1(j) + \alpha_2\mathbf{u}_2(\ell)) \geq \Delta\Big\}$$

for $j \in \{2, 3, \ldots 2^{nR_1}\}$ and $\ell \in \{2, 3, \ldots 2^{nR_2}\}$. Hence, by (149)

$$\begin{aligned}&\Pr\left[\mathcal{G}|\mathcal{E}^c_{\mathbf{X}_1} \cap \mathcal{E}^c_{\mathbf{X}_2}\right] \\ &\quad= \Pr\left[\bigcup_{j=2}^{2^{nR_1}} \bigcup_{\ell=2}^{2^{nR_2}} \mathcal{G}_{j,\ell} \middle| \tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)\right] \\ &\quad\leq \sum_{j=2}^{2^{nR_1}} \sum_{\ell=2}^{2^{nR_2}} \Pr\left[\mathcal{G}_{j,\ell}\middle|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)\right] \\ &\quad\overset{a)}{<} 2^{n(R_1+R_2)} \Pr\left[\mathcal{G}_{2,2}|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)\right] \end{aligned} \tag{150}$$

where $a)$ follows since conditioned on $\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)$, the laws of $\sphericalangle(\mathbf{U}_1(j), \mathbf{U}_2(\ell))$ and $\sphericalangle(\mathbf{Y}, \alpha_1\mathbf{U}_1(j) + \alpha_2\mathbf{U}_2(\ell))$ do not depend on $j \in \{2, 3, \ldots, 2^{nR_1}\}$ or $\ell \in \{2, 3, \ldots, 2^{nR_2}\}$. We now rewrite the probability $\Pr\left[\mathcal{G}_{2,2}|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)\right]$ as in (151), shown at the bottom of the page, where in the last step we have used that the probability term does not depend on $\mathbf{z}$. To upper bound the integral we now upper bound this probability term as shown in (152) at the bottom of the next page, where in $a)$ we have used

$$\begin{aligned}&\Pr\left[\mathcal{G}_{2,2}|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)\right] \\ &\quad= \Pr\Big[(\mathbf{S}_1, \mathbf{S}_2, \mathbf{U}_1(1), \mathbf{U}_2(1), \mathbf{U}_1(2), \mathbf{U}_2(2), \mathbf{Z}) \text{ are such that } \mathcal{G}_{2,2} \text{ occurs } \Big|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)\Big] \\ &\quad= \int\limits_{\substack{(\mathbf{s}_1,\mathbf{s}_2)\in\mathbb{R}^n\times\mathbb{R}^n \\ (\mathbf{u}_1,\mathbf{u}_2)\in\mathcal{S}_1\times\mathcal{S}_2 \\ \mathbf{z}\in\mathbb{R}^n}} f\Big((\mathbf{S}_1, \mathbf{S}_2, \mathbf{U}_1(1), \mathbf{U}_2(1), \mathbf{Z}) = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1, \mathbf{u}_2, \mathbf{z})\Big|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)\Big) \\ &\qquad \cdot \Pr\Big[\cos\sphericalangle(\mathbf{U}_1(2), \mathbf{U}_2(2)) \geq \Theta, \cos\sphericalangle(\mathbf{y}, \alpha_1\mathbf{U}_1(2) + \alpha_2\mathbf{U}_2(2)) \geq \Delta\Big|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1) \\ &\qquad\qquad (\mathbf{S}_1, \mathbf{S}_2, \mathbf{U}_1(1), \mathbf{U}_2(1), \mathbf{Z}) = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1, \mathbf{u}_2, \mathbf{z})\Big] \mathrm{d}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1, \mathbf{u}_2, \mathbf{z}) \\ &\quad= \int\limits_{\substack{(\mathbf{s}_1,\mathbf{s}_2)\in\mathbb{R}^n\times\mathbb{R}^n \\ (\mathbf{u}_1,\mathbf{u}_2)\in\mathcal{S}_1\times\mathcal{S}_2}} f\Big((\mathbf{S}_1, \mathbf{S}_2, \mathbf{U}_1(1), \mathbf{U}_2(1)) = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1, \mathbf{u}_2)\Big|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1)\Big) \\ &\qquad \cdot \Pr\Big[\cos\sphericalangle(\mathbf{U}_1(2), \mathbf{U}_2(2)) \geq \Theta, \cos\sphericalangle(\mathbf{y}, \alpha_1\mathbf{U}_1(2) + \alpha_2\mathbf{U}_2(2)) \geq \Delta\Big|\tilde{\varsigma}(\mathbf{S}_1, \mathbf{S}_2, \mathfrak{C}_1, \mathfrak{C}_2) = (1,1) \\ &\qquad\qquad (\mathbf{S}_1, \mathbf{S}_2, \mathbf{U}_1(1), \mathbf{U}_2(1)) = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1, \mathbf{u}_2)\Big] \mathrm{d}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{u}_1, \mathbf{u}_2) \end{aligned} \tag{151}$$

Lemma D.19 and in b) we have used that under distributions of $\mathbf{U}_1(2)$ and $\mathbf{U}_2(2)$ that are independent of $\mathbf{y}$ and uniform over $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, the angles $\sphericalangle(\mathbf{U}_1(j), \mathbf{U}_2(\ell))$ and $\sphericalangle(\mathbf{y}, \alpha_1\mathbf{U}_1(j) + \alpha_2\mathbf{U}_2(\ell))$ are independent. Thus, combining (152) with (150) gives

$$\Pr\left[\mathcal{G}|\mathcal{E}^c_{\mathbf{X}_1} \cap \mathcal{E}^c_{\mathbf{X}_2}\right] < 2^{n(R_1+R_2)} \cdot 4\frac{C_n(\arccos\Theta)}{C_n(\pi)} \cdot \frac{C_n(\arccos\Delta)}{C_n(\pi)} \quad (153)$$

and combining (153) with Lemma D.17 gives

$$\begin{aligned}\Pr\left[\mathcal{G}|\mathcal{E}^c_{\mathbf{X}_1} \cap \mathcal{E}^c_{\mathbf{X}_2}\right] &< 2^{n(R_1+R_2)} \cdot 4\frac{\Gamma\left(\frac{n}{2}+1\right)(1-\Theta^2)^{(n-1)/2}}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}\Theta} \\ &\quad \cdot \frac{\Gamma\left(\frac{n}{2}+1\right)(1-\Delta^2)^{(n-1)/2}}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}\Delta} \\ &= 4\cdot\left(\frac{\Gamma\left(\frac{n}{2}+1\right)}{n\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}\right)^2 \frac{1}{\Theta\sqrt{1-\Theta^2}\Delta\sqrt{1-\Delta^2}} \\ &\quad \cdot 2^{n\left(R_1+R_2+\frac{1}{2}\log_2\left((1-\Theta^2)(1-\Delta^2)\right)\right)}.\end{aligned}$$

Replacing the ratios of the Gamma-functions by their asymptotic series in Lemma D.18 finally establishes (115). □

Appendix E
Proof of Theorem IV.5

The high-SNR asymptotics for the multiple-access problem without feedback can be obtained from the necessary condition for the achievability of a distortion pair (D_1, D_2) in Theorem IV.1, and from the sufficient conditions for the achievability of a distortion pair (D_1, D_2) deriving from the vector-quantizer scheme in Theorem IV.4.

By Theorem IV.4 it follows that any distortion pair $(\bar{D}_1, \bar{D}_2)$ satisfying $\bar{D}_1 \leq \sigma^2$, $\bar{D}_2 \leq \sigma^2$ and

$$\bar{D}_1 \geq \sigma^2\frac{N}{P_1} \quad (154)$$

$$\bar{D}_2 \geq \sigma^2\frac{N}{P_2} \quad (155)$$

$$\bar{D}_1\bar{D}_2 = \sigma^4\frac{N(1-\breve{\rho}^2)}{P_1+P_2+2\breve{\rho}\sqrt{P_1P_2}} \quad (156)$$

where

$$\breve{\rho} = \rho\sqrt{\left(1-\frac{\bar{D}_1}{\sigma^2}\right)\left(1-\frac{\bar{D}_2}{\sigma^2}\right)} \quad (157)$$

$$\begin{aligned}&\Pr\Big[\cos\sphericalangle(\mathbf{U}_1(2),\mathbf{U}_2(2)) \geq \Theta, \cos\sphericalangle(\mathbf{y},\alpha_1\mathbf{U}_1(2)+\alpha_2\mathbf{U}_2(2)) \geq \Delta \Big| \tilde{\varsigma}(\mathbf{S}_1,\mathbf{S}_2,\mathfrak{C}_1,\mathfrak{C}_2)=(1,1) \\ &\qquad (\mathbf{S}_1,\mathbf{S}_2,\mathbf{U}_1(1),\mathbf{U}_2(1)) = (\mathbf{s}_1,\mathbf{s}_2,\mathbf{u}_1,\mathbf{u}_2)\Big] \\ &= \int_{\substack{(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)\in\mathcal{S}_1\times\mathcal{S}_2:\\ \cos\sphericalangle(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)\geq\Theta,\\ \cos\sphericalangle(\mathbf{y},\alpha_1\tilde{\mathbf{u}}_1+\alpha_2\tilde{\mathbf{u}}_2)\geq\Delta}} \cdot f^{\lambda_1\times\lambda_2}\Big((\mathbf{U}_1(2),\mathbf{U}_2(2)) = (\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)\Big|\tilde{\varsigma}(\mathbf{S}_1,\mathbf{S}_2,\mathfrak{C}_1,\mathfrak{C}_2)=(1,1) \\ &\qquad (\mathbf{S}_1,\mathbf{S}_2,\mathbf{U}_1(1),\mathbf{U}_2(1)) = (\mathbf{s}_1,\mathbf{s}_2,\mathbf{u}_1,\mathbf{u}_2)\Big)\mathrm{d}(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2) \\ &= \int_{\substack{(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)\in\mathcal{S}_1\times\mathcal{S}_2:\\ \cos\sphericalangle(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)\geq\Theta,\\ \cos\sphericalangle(\mathbf{y},\alpha_1\tilde{\mathbf{u}}_1+\alpha_2\tilde{\mathbf{u}}_2)\geq\Delta}} f^{\lambda_1}\Big(\mathbf{U}_1(2)=\tilde{\mathbf{u}}_1\Big|\tilde{\varsigma}(\mathbf{S}_1,\mathbf{S}_2,\mathfrak{C}_1,\mathfrak{C}_2)=(1,1),(\mathbf{S}_1,\mathbf{S}_2,\mathbf{U}_1(1),\mathbf{U}_2(1)) = (\mathbf{s}_1,\mathbf{s}_2,\mathbf{u}_1,\mathbf{u}_2)\Big) \\ &\qquad \cdot f^{\lambda_2}\Big(\mathbf{U}_2(2)=\tilde{\mathbf{u}}_2\Big|\tilde{\varsigma}(\mathbf{S}_1,\mathbf{S}_2,\mathfrak{C}_1,\mathfrak{C}_2)=(1,1)\ (\mathbf{S}_1,\mathbf{S}_2,\mathbf{U}_1(1),\mathbf{U}_2(1)) = (\mathbf{s}_1,\mathbf{s}_2,\mathbf{u}_1,\mathbf{u}_2)\Big)\mathrm{d}(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2) \\ &= \int_{\substack{(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)\in\mathcal{S}_1\times\mathcal{S}_2:\\ \cos\sphericalangle(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)\geq\Theta,\\ \cos\sphericalangle(\mathbf{y},\alpha_1\tilde{\mathbf{u}}_1+\alpha_2\tilde{\mathbf{u}}_2)\geq\Delta}} f^{\lambda_1}\Big(\mathbf{U}_1(2)=\tilde{\mathbf{u}}_1\Big|\varsigma_1(\mathbf{S}_1,\mathfrak{C}_1)=1,(\mathbf{S}_1,\mathbf{U}_1(1))=(\mathbf{s}_1,\mathbf{u}_1)\Big) \\ &\qquad \cdot f^{\lambda_2}\Big(\mathbf{U}_2(2)=\tilde{\mathbf{u}}_2\Big|\varsigma_2(\mathbf{S}_2,\mathfrak{C}_2)=1,(\mathbf{S}_2,\mathbf{U}_2(1))=(\mathbf{s}_2,\mathbf{u}_2)\Big)\mathrm{d}(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2) \\ &\overset{a)}{\leq} \int_{\substack{(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)\in\mathcal{S}_1\times\mathcal{S}_2:\\ \cos\sphericalangle(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2)\geq\Theta,\\ \cos\sphericalangle(\mathbf{y},\alpha_1\tilde{\mathbf{u}}_1+\alpha_2\tilde{\mathbf{u}}_2)\geq\Delta}} \frac{2}{r_1^{n-1}C_n(\pi)}\cdot\frac{2}{r_2^{n-1}C_n(\pi)}\mathrm{d}(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2) \\ &\overset{b)}{=} 4\frac{C_n(\arccos\Theta)}{C_n(\pi)}\cdot\frac{C_n(\arccos\Delta)}{C_n(\pi)}\end{aligned} \quad (152)$$

is achievable. If

$$\lim_{N \longrightarrow 0} \frac{N}{P_1 \bar{D}_1} = 0 \quad \text{and} \quad \lim_{N \longrightarrow 0} \frac{N}{P_2 \bar{D}_2} = 0 \tag{158}$$

then for N sufficiently small, (154) and (155) are satisfied. Consequently, for N sufficiently small any pair satisfying (156) and (158) is achievable. We next show that if the pair $(\bar{D}_1, \bar{D}_2)$ satisfies (156) and (158), then $\breve{\rho} \longrightarrow \rho$ as $N \longrightarrow 0$. To show this, we note that if $(\bar{D}_1, \bar{D}_2)$ satisfies (156) then

$$\bar{D}_2 \leq \sigma^4 \frac{N}{P_1 \bar{D}_1}, \quad \text{and} \quad \bar{D}_1 \leq \sigma^4 \frac{N}{P_2 \bar{D}_2}. \tag{159}$$

Combining (159) with (157) gives that if in addition to (156) the pair $(\bar{D}_1, \bar{D}_2)$ also satisfies (159), then $\breve{\rho} \longrightarrow \rho$ as $N \longrightarrow 0$. Thus, if $(\bar{D}_1, \bar{D}_2)$ satisfies (156) and (158), then

$$\lim_{N \longrightarrow 0} \frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2}}{N} \bar{D}_1 \bar{D}_2 \leq \sigma^4(1-\rho^2). \tag{160}$$

Now, let $(D_1^*(\sigma^2, \rho, P_1, P_2, N), D_2^*(\sigma^2, \rho, P_1, P_2, N))$ be a distortion pair resulting from an optimal scheme and let (D_1^*, D_2^*) be the shorthand notation for this distortion pair. By Theorem IV.1, we have that

$$R_{S_1,S_2}(D_1^*, D_2^*) \leq \frac{1}{2}\log_2\left(1 + \frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2}}{N}\right). \tag{161}$$

If (D_1^*, D_2^*) satisfies

$$\lim_{N \longrightarrow 0} \frac{N}{P_1 D_1^*} = 0 \quad \text{and} \quad \lim_{N \longrightarrow 0} \frac{N}{P_2 D_2^*} = 0 \tag{162}$$

then for N sufficiently small

$$R_{S_1,S_2}(D_1^*, D_2^*) = \frac{1}{2}\log_2^+\left(\frac{\sigma^4(1-\rho^2)}{D_1^* D_2^*}\right) \tag{163}$$

by Theorem III.1 and because $(D_1^*, D_2^*) \in \mathcal{D}_2$. From (161) and (163), we, thus, get that if (D_1^*, D_2^*) satisfies (162), then

$$\lim_{N \longrightarrow 0} \frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2}}{N} D_1^* D_2^* \geq \sigma^4(1-\rho^2). \tag{164}$$

Combining (160) with (164) yields Theorem IV.5. □

APPENDIX F
PROOF OF THEOREM IV.6

Our analysis of the expected distortion for the superimposed scheme is based on a genie-aided argument, similar as in the analysis of the vector-quantizer scheme. This argument is described more precisely now.

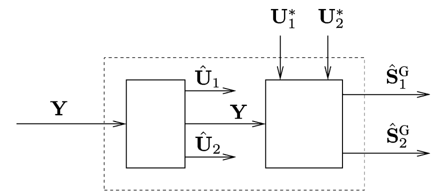

Fig. 12. Genie-aided decoder.

A) Genie-Aided Scheme: In our genie-aided argument, the genie assists the decoder. An illustration of this decoder is given in Fig. 12. In addition to the channel output $\mathbf{Y}$ that is observed originally, the decoder is now also provided with the transmitted codeword pair $(\mathbf{U}_1^*, \mathbf{U}_2^*)$. Based on $(\mathbf{U}_1^*, \mathbf{U}_2^*)$ and $\mathbf{Y}$, the decoder then estimates the source pair $(\mathbf{S}_1, \mathbf{S}_2)$ and thereby ignores the guess $(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2)$ produced in the first step of the original decoder. The estimate of this genie-aided decoder is denoted by $(\hat{\mathbf{S}}_1^{\mathrm{G}}, \hat{\mathbf{S}}_2^{\mathrm{G}})$ and is given by

$$\hat{\mathbf{S}}_1^{\mathrm{G}} = \gamma_{11}\mathbf{U}_1^* + \gamma_{12}\mathbf{U}_2^* + \gamma_{13}\mathbf{Y} \tag{165}$$
$$\hat{\mathbf{S}}_2^{\mathrm{G}} = \gamma_{21}\mathbf{U}_2^* + \gamma_{22}\mathbf{U}_1^* + \gamma_{23}\mathbf{Y} \tag{166}$$

where the coefficients γ_{ij} are as defined in (44). We now show that under certain rate constraints, the normalized asymptotic distortion of this genie-aided scheme is the same as for the originally proposed scheme. The key argument is stated in the following proposition.

Proposition F.1: For every $\delta > 0$ and $0 < \epsilon < 0.3$ there exists an $n'(\delta, \epsilon) > 0$ such that for all $n > n'(\delta, \epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1\|^2\right] \leq \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] + \xi_1'\delta + \xi_2'\epsilon \tag{167}$$

whenever (R_1, R_2) is in the rate region $\mathcal{R}'(\epsilon)$ given in (168), shown at the bottom of the page, where in (167) ξ_1' and ξ_2' depend only on $\sigma^2, \gamma_{13}, P_1, P_2$ and N, and where in the expression of $\mathcal{R}'(\epsilon)$ the terms κ_1, κ_2 and κ_3 depend only on P_1, P_2, N', and $\tilde{\rho}$, and where N' and β_1', β_2' are as given in (45), (46) and (47) respectively.

Proof: See Appendix F-B. □

From Proposition F.1, it now follows easily that the expected distortion asymptotically achievable by the genie-aided scheme is the same as the expected distortion achievable by the original scheme.

$$\mathcal{R}'(\epsilon) = \Bigg\{ R_1 \leq \frac{1}{2}\log_2\left(\frac{\beta_1'^2\|\mathbf{U}_1\|^2(1-\tilde{\rho}^2) + N'}{N'(1-\tilde{\rho}^2)} - \kappa_1\epsilon\right)$$
$$R_2 \leq \frac{1}{2}\log_2\left(\frac{\beta_2'^2\|\mathbf{U}_2\|^2(1-\tilde{\rho}^2) + N'}{N'(1-\tilde{\rho}^2)} - \kappa_2\epsilon\right)$$
$$R_1 + R_2 \leq \frac{1}{2}\log_2\left(\frac{\beta_1'^2\|\mathbf{U}_1\|^2 + \beta_1'^2\|\mathbf{U}_1\|^2 + 2\tilde{\rho}\beta_1'\beta_2'\|\mathbf{U}_1\|\|\mathbf{U}_2\| + N'}{N'(1-\tilde{\rho}^2)} - \kappa_3\epsilon\right) \Bigg\} \tag{168}$$

Corollary F.1: If (R_1, R_2) satisfy (169)–(171), shown at the bottom of the page, then

$$\overline{\lim_{n\to\infty}} \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1\|^2\right] \le \overline{\lim_{n\to\infty}} \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right].$$

Proof: Corollary F.1 follows from Proposition F.1 by first letting $n \to \infty$ and then $\epsilon \to 0$ and $\delta \to 0$. □

It follows by Corollary F.1 that to analyze the distortion achievable by our scheme it suffices to analyze the genie-aided scheme. This is done in Appendix F-C.

B) Proof of Proposition F.1: The proof of Proposition F.1 consists of upper bounding the difference between $\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1\|^2\right]$ and $\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right]$. Since the two estimates $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_1^{\mathrm{G}}$ differ only if $(\hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2) \neq (\mathbf{U}_1^*, \mathbf{U}_2^*)$, the main step is to upper bound the probability of a decoding error. This is what we do now.

Let the error event $\mathcal{E}_{\hat{\mathbf{U}}}$ be as defined in (80)–(82) for the vector-quantizer scheme. The probability of $\mathcal{E}_{\hat{\mathbf{U}}}$ is upper bounded in the following lemma.

Lemma F.1: For every $\delta > 0$ and $0 < \epsilon < 0.3$, there exists an $n_4'(\delta, \epsilon) \in \mathbb{N}$ such that for all $n > n_4'(\delta, \epsilon)$

$$\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] < 11\delta \quad \text{whenever } (R_1, R_2) \in \mathcal{R}'(\epsilon).$$

Proof: The proof follows from restating the decoding problem for the superimposed scheme in the form of the decoding problem for the vector-quantizer scheme. That is, we seek to rewrite the channel output in the form

$$\mathbf{Y} = \beta_1'\mathbf{U}_1^* + \beta_2'\mathbf{U}_2^* + \mathbf{Z}' \tag{172}$$

with an additive noise sequence $\mathbf{Z}'$ that satisfies the properties needed for the analysis of the vector-quantizer scheme. This representation is obtained by first rewriting the source sequences as

$$\mathbf{S}_1 = (1 - a_1\tilde{\rho})\mathbf{U}_1^* + a_1\mathbf{U}_2^* + \mathbf{W}_1 \tag{173}$$
$$\mathbf{S}_2 = (1 - a_2\tilde{\rho})\mathbf{U}_2^* + a_2\mathbf{U}_1^* + \mathbf{W}_2 \tag{174}$$

where a_1 and a_2 are defined in (48), and $\tilde{\rho}$ is defined in (37). Combining (173) and (174) with the expressions for $\mathbf{X}_1$ and $\mathbf{X}_2$ in (40) and with $\mathbf{Y} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{Z}$ yields the desired form of (172) with

$$\beta_1' = (\alpha_1(1 - a_1\tilde{\rho}) + \beta_1 + \alpha_2 a_2)$$
$$\beta_2' = (\alpha_2(1 - a_2\tilde{\rho}) + \beta_2 + \alpha_1 a_1)$$

and with

$$\mathbf{Z}' = \alpha_1\mathbf{W}_1 + \alpha_2\mathbf{W}_2 + \mathbf{Z}.$$

For the additive noise sequence $\mathbf{Z}'$ it can now be verified that for every $\delta > 0$ and $\epsilon > 0$ there exists an $n'(\delta, \epsilon) > 0$, such that for N' as in (45) and for all $n > n'(\delta, \epsilon)$ we have that

$$\Pr\left[\left|\frac{1}{n}\|\mathbf{Z}'\|^2 - N'\right| \le N'\epsilon\right] > 1 - \delta \tag{175}$$

and that for $i \in \{1, 2\}$

$$\Pr\left[\left|\langle \mathbf{U}_i^*, \mathbf{Z}' \rangle\right| \le n\sqrt{\sigma^2(1 - 2^{-2R_i})N'\epsilon}\right] > 1 - \delta. \tag{176}$$

Condition (176) follows since for a_1 and a_2, given in (48), for sufficiently large n, we have with high probability that

$$\langle \mathbf{U}_i^*, \mathbf{W}_j \rangle \approx 0 \qquad \forall i, j \in \{1, 2\}.$$

Conditions (175) and (176) are precisely those needed in the proof of the achievable rates for the vector-quantizer scheme. Hence, the upper bound on the probability of a decoding error in the vector-quantizer scheme given in Lemma D.1 can be adopted to the superimposed scheme. This yields Lemma F.1. □

To ease the upper bounding of the difference between $\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1\|^2\right]$ and $\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right]$ we now state three more lemmas which upper bound different norms and inner products involving $\mathbf{S}_1$, $\hat{\mathbf{S}}_1$, and $\hat{\mathbf{S}}_1^{\mathrm{G}}$. The first lemma gives an upper bound on the squared norm of $\hat{\mathbf{S}}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}$.

Lemma F.2: Let the reconstructions $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_1^{\mathrm{G}}$ be as defined in (42) and (165). Then, with probability one

$$\|\hat{\mathbf{S}}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \le 16n\sigma^2.$$

Proof:

$$\begin{aligned}
&\|\hat{\mathbf{S}}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \\
&\quad = \|\gamma_{11}(\hat{\mathbf{U}}_1 - \mathbf{U}_1^*) + \gamma_{12}(\hat{\mathbf{U}}_2 - \mathbf{U}_2^*)\|^2 \\
&\quad = \gamma_{11}^2\|\hat{\mathbf{U}}_1 - \mathbf{U}_1^*\|^2 + 2\gamma_{11}\gamma_{12}\left\langle \hat{\mathbf{U}}_1 - \mathbf{U}_1^*, \hat{\mathbf{U}}_2 - \mathbf{U}_2^* \right\rangle \\
&\qquad + \gamma_{12}^2\|\hat{\mathbf{U}}_2 - \mathbf{U}_2^*\|^2 \\
&\quad \le \gamma_{11}^2 \underbrace{\|\hat{\mathbf{U}}_1 - \mathbf{U}_1^*\|^2}_{\le 4n\sigma^2} + 2\gamma_{11}\gamma_{12} \underbrace{\|\hat{\mathbf{U}}_1 - \mathbf{U}_1^*\|\|\hat{\mathbf{U}}_2 - \mathbf{U}_2^*\|}_{\le 4n\sigma^2} \\
&\qquad + \gamma_{12}^2 \underbrace{\|\hat{\mathbf{U}}_2 - \mathbf{U}_2^*\|^2}_{\le 4n\sigma^2}
\end{aligned}$$

$$R_1 < \frac{1}{2}\log_2\left(\frac{\beta_1'^2\|\mathbf{U}_1\|^2(1 - \tilde{\rho}^2) + N'}{N'(1 - \tilde{\rho}^2)}\right) \tag{169}$$

$$R_2 < \frac{1}{2}\log_2\left(\frac{\beta_2'^2\|\mathbf{U}_2\|^2(1 - \tilde{\rho}^2) + N'}{N'(1 - \tilde{\rho}^2)}\right) \tag{170}$$

$$R_1 + R_2 < \frac{1}{2}\log_2\left(\frac{\beta_1'^2\|\mathbf{U}_1\|^2 + \beta_1'^2\|\mathbf{U}_1\|^2 + 2\tilde{\rho}\beta_1'\beta_2'\|\mathbf{U}_1\|\|\mathbf{U}_2\| + N'}{N'(1 - \tilde{\rho}^2)}\right) \tag{171}$$

$$\leq 4n\sigma^2(\gamma_{11}+\gamma_{12})^2$$
$$\leq 16n\sigma^2$$

where in the last step we have used that $0 \leq \gamma_{11},\ \gamma_{12} \leq 1$. □

For the next two lemmas, we reuse the two error events $\mathcal{E}_{\mathbf{S}}$ and $\mathcal{E}_{\mathbf{Z}}$ which were defined in (83) and (84) for the proof of the vector-quantizer scheme. We then have:

Lemma F.3: For every $\epsilon > 0$

$$\frac{1}{n}\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1^{\mathrm{G}} - \hat{\mathbf{S}}_1 \right\rangle\right] \leq \sigma^2\big(\epsilon + 17\mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}\right] + (17+\epsilon)\mathrm{Pr}\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\big).$$

Proof:

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1^{\mathrm{G}} - \hat{\mathbf{S}}_1 \right\rangle\right] \\
&= \frac{1}{n}\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1^{\mathrm{G}} - \hat{\mathbf{S}}_1 \right\rangle \middle| \mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\hat{\mathbf{U}}}\right] \mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\hat{\mathbf{U}}}\right] \\
&\quad + \frac{1}{n}\underbrace{\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1^{\mathrm{G}} - \hat{\mathbf{S}}_1 \right\rangle \middle| \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\hat{\mathbf{U}}}^c\right]}_{=0} \mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\hat{\mathbf{U}}}^c\right] \\
&\overset{a)}{\leq} \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 + \|\hat{\mathbf{S}}_1^{\mathrm{G}} - \hat{\mathbf{S}}_1\|^2 \middle| \mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\hat{\mathbf{U}}}\right] \mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\hat{\mathbf{U}}}\right] \\
&= \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 \middle| \mathcal{E}_{\mathbf{S}}\right] \mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}\right] \\
&\quad + \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 \middle| \mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\hat{\mathbf{U}}}\right] \mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}^c \cap \mathcal{E}_{\hat{\mathbf{U}}}\right] \\
&\quad + \frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1^{\mathrm{G}} - \hat{\mathbf{S}}_1\|^2 \middle| \mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\hat{\mathbf{U}}}\right] \mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}} \cup \mathcal{E}_{\hat{\mathbf{U}}}\right] \\
&\overset{b)}{\leq} \sigma^2(\epsilon + \mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}\right]) + \sigma^2(1+\epsilon)\mathrm{Pr}\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\
&\quad + 16\sigma^2\left(\mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}\right] + \mathrm{Pr}\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\right) \\
&\leq \sigma^2\big(\epsilon + 17\mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}\right] + (17+\epsilon)\mathrm{Pr}\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\big)
\end{aligned} \tag{177}$$

where in the first equality, the second expectation term equals zero because by $\mathcal{E}_{\hat{\mathbf{U}}}^c$ we have $\|\hat{\mathbf{S}}_1^{\mathrm{G}} - \hat{\mathbf{S}}_1\| = 0$ and by $\mathcal{E}_{\mathbf{S}}^c$ the norm $\|\mathbf{S}_1\|$ is bounded. In $a)$ we have used (119), and in $b)$ we have used Lemma D.10, Lemma F.2, and the fact that conditioned on $\mathcal{E}_{\mathbf{S}}^c$ we have $\|\mathbf{S}_1\|^2 \leq n\sigma^2(1+\epsilon)$. □

Lemma F.4: For every $\epsilon > 0$

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] \\
&\leq 4\Big(\sigma^2(1+4\gamma_{13}) + 9\gamma_{13}(P_1+P_2+N)(1+\epsilon)\Big)\mathrm{Pr}\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\
&\quad + 36\gamma_{13}\Big((P_1+P_2)\mathrm{Pr}\left[\mathcal{E}_{\mathbf{S}}\right] + N\mathrm{Pr}\left[\mathcal{E}_{\mathbf{Z}}\right] + (P_1+P_2+N)\epsilon\Big).
\end{aligned}$$

Proof:

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] \\
&= \frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \middle| \mathcal{E}_{\hat{\mathbf{U}}}\right] \mathrm{Pr}\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\
&\quad + \frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - |\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \| \mathcal{E}_{\hat{\mathbf{U}}}^c\right] \mathrm{Pr}\left[\mathcal{E}_{\hat{\mathbf{U}}}^c\right] \\
&\leq \frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - |\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \| \mathcal{E}_{\hat{\mathbf{U}}}\right] \mathrm{Pr}\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]
\end{aligned} \tag{178}$$

where the last inequality follows since conditional on $\mathcal{E}_{\hat{\mathbf{U}}}^c$ we have $\hat{\mathbf{S}}_1 = \hat{\mathbf{S}}_1^{\mathrm{G}}$ and, therefore, $\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 = 0$. To upper bound the RHS of (178), we now upper bound the difference $\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2$:

$$\begin{aligned}
&\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \\
&= \gamma_{11}^2\|\hat{\mathbf{U}}_1\|^2 + 2\gamma_{11}\gamma_{12}\left\langle \hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2 \right\rangle + 2\gamma_{11}\gamma_{13}\left\langle \hat{\mathbf{U}}_1, \mathbf{Y} \right\rangle \\
&\quad + \gamma_{12}^2\|\hat{\mathbf{U}}_2\|^2 + 2\gamma_{12}\gamma_{13}\left\langle \hat{\mathbf{U}}_2, \mathbf{Y} \right\rangle + \gamma_{13}^2\|\mathbf{Y}\|^2 \\
&\quad - \gamma_{11}^2\|\mathbf{U}_1^*\|^2 - 2\gamma_{11}\gamma_{12}\left\langle \mathbf{U}_1^*, \mathbf{U}_2^* \right\rangle - 2\gamma_{11}\gamma_{13}\left\langle \mathbf{U}_1^*, \mathbf{Y} \right\rangle \\
&\quad - \gamma_{12}^2\|\mathbf{U}_2^*\|^2 - 2\gamma_{12}\gamma_{13}\left\langle \mathbf{U}_2^*, \mathbf{Y} \right\rangle - \gamma_{13}^2\|\mathbf{Y}\|^2 \\
&= 2\gamma_{11}\gamma_{12}\underbrace{\left\langle \hat{\mathbf{U}}_1, \hat{\mathbf{U}}_2 \right\rangle}_{\leq n\sigma^2} + 2\gamma_{11}\gamma_{13}\left\langle \hat{\mathbf{U}}_1, \mathbf{Y} \right\rangle \\
&\quad + 2\gamma_{12}\gamma_{13}\left\langle \hat{\mathbf{U}}_2, \mathbf{Y} \right\rangle - 2\gamma_{11}\gamma_{12}\underbrace{\left\langle \mathbf{U}_1^*, \mathbf{U}_2^* \right\rangle}_{\geq -n\sigma^2} \\
&\quad - 2\gamma_{11}\gamma_{13}\left\langle \mathbf{U}_1^*, \mathbf{Y} \right\rangle - 2\gamma_{12}\gamma_{13}\left\langle \mathbf{U}_2^*, \mathbf{Y} \right\rangle \\
&\leq 4\gamma_{11}\gamma_{12}n\sigma^2 + 2\gamma_{11}\gamma_{13}\left\langle \hat{\mathbf{U}}_1 - \mathbf{U}_1^*, \mathbf{Y} \right\rangle \\
&\quad + 2\gamma_{12}\gamma_{13}\left\langle \hat{\mathbf{U}}_2 - \mathbf{U}_2^*, \mathbf{Y} \right\rangle \\
&\overset{(a)}{\leq} 4\gamma_{11}\gamma_{12}n\sigma^2 + 2\gamma_{11}\gamma_{13}\big(\underbrace{\|\hat{\mathbf{U}}_1 - \mathbf{U}_1^*\|^2}_{\leq 4n\sigma^2} + \|\mathbf{Y}\|^2\big) \\
&\quad + 2\gamma_{12}\gamma_{13}\big(\underbrace{\|\hat{\mathbf{U}}_2 - \mathbf{U}_2^*\|^2}_{\leq 4n\sigma^2} + \|\mathbf{Y}\|^2\big) \\
&\leq 4\gamma_{11}\gamma_{12}n\sigma^2 + 2\gamma_{13}(\gamma_{11}+\gamma_{12})\big(4n\sigma^2 + \|\mathbf{Y}\|^2\big) \\
&\leq 4n\sigma^2 + 4\gamma_{13}\big(4n\sigma^2 + \|\mathbf{Y}\|^2\big)
\end{aligned} \tag{179}$$

where in $a)$ we have used (119), and in the last inequality we have used that $0 \leq \gamma_{11}, \gamma_{12} \leq 1$. We now upper bound the squared norm of $\mathbf{Y}$ on the RHS of (179) in terms of $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{U}_1^*$, $\mathbf{U}_2^*$ and $\mathbf{Z}$

$$\begin{aligned}
&\|\mathbf{Y}\|^2 \\
&\leq \alpha_1^2\|\mathbf{S}_1\|^2 + 2\left\langle \alpha_1\mathbf{S}_1, \beta_1\mathbf{U}_1^* \right\rangle + 2\left\langle \alpha_1\mathbf{S}_1, \alpha_2\mathbf{S}_2 \right\rangle \\
&\quad + 2\left\langle \alpha_1\mathbf{S}_1, \beta_2\mathbf{U}_2^* \right\rangle + 2\left\langle \alpha_1\mathbf{S}_1, \mathbf{Z} \right\rangle \\
&\quad + \beta_1^2\|\mathbf{U}_1^*\|^2 + 2\left\langle \beta_1\mathbf{U}_1^*, \alpha_2\mathbf{S}_2 \right\rangle + 2\left\langle \beta_1\mathbf{U}_1^*, \beta_2\mathbf{U}_2^* \right\rangle \\
&\quad + 2\left\langle \beta_1\mathbf{U}_1^*, \mathbf{Z} \right\rangle + \alpha_2^2\|\mathbf{S}_2\|^2 + 2\left\langle \alpha_2\mathbf{S}_2, \beta_2\mathbf{U}_2^* \right\rangle \\
&\quad + 2\left\langle \alpha_2\mathbf{S}_2, \mathbf{Z} \right\rangle + \beta_2^2\|\mathbf{U}_2^*\|^2 + 2\left\langle \beta_2\mathbf{U}_2^*, \mathbf{Z} \right\rangle + \|\mathbf{Z}\|^2 \\
&\overset{a)}{\leq} 9\left(\alpha_1^2\|\mathbf{S}_1\|^2 + \alpha_2^2\|\mathbf{S}_2\|^2 + \beta_1^2\|\mathbf{U}_1^*\|^2 + \beta_2^2\|\mathbf{U}_2^*\|^2 + \|\mathbf{Z}\|^2\right) \\
&\leq 9\left(\alpha_1^2\|\mathbf{S}_1\|^2 + \alpha_2^2\|\mathbf{S}_2\|^2 + nP_1 + nP_2 + \|\mathbf{Z}\|^2\right).
\end{aligned} \tag{180}$$

where $a)$ follows from upper bounding all inner products by (119). Thus, Combining (180) with (179) gives

$$\begin{aligned}
&\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2 \\
&\quad \leq 4n\sigma^2 + 16\gamma_{13}n\sigma^2 + 36n\gamma_{13}(P_1+P_2) \\
&\qquad + 36\gamma_{13}\left(\alpha_1^2\|\mathbf{S}_1\|^2 + \alpha_2^2\|\mathbf{S}_2\|^2 + \|\mathbf{Z}\|^2\right)
\end{aligned} \tag{181}$$

and combining (181) with (178) gives

$$\frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right]$$

$$\leq 4\sigma^2 \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] + 16\gamma_{13}\sigma^2 \Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] + 36\gamma_{13}(P_1+P_2)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] + 36\gamma_{13}\Big(\alpha_1^2 \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 \middle| \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] + \alpha_2^2 \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_2\|^2 \middle| \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] + \frac{1}{n}\mathsf{E}\left[\|\mathbf{Z}\|^2 \middle| \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\Big). \tag{182}$$

It now remains to upper bound the expectations on $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{Z}$ on the RHS of (182). Since $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{Z}$ are each Gaussian, their corresponding terms can be bounded in similar ways. We show here the derivation for $\mathbf{S}_1$

$$\begin{aligned}\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 \middle| \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] &= \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 \middle| \mathcal{E}_{\hat{\mathbf{U}}}\cap\mathcal{E}_{\mathbf{S}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\cap\mathcal{E}_{\mathbf{S}}\right] + \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 \middle| \mathcal{E}_{\hat{\mathbf{U}}}\cap\mathcal{E}_{\mathbf{S}}^c\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\cap\mathcal{E}_{\mathbf{S}}^c\right] \\ &\leq \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1\|^2 \middle| \mathcal{E}_{\mathbf{S}}\right]\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + \sigma^2(1+\epsilon)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\ &\leq \sigma^2(\epsilon + \Pr\left[\mathcal{E}_{\mathbf{S}}\right]) + \sigma^2(1+\epsilon)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \end{aligned} \tag{183}$$

where in the last step we have used Lemma D.10. For the expectations on $\mathbf{S}_2$ and $\mathbf{Z}$, we similarly obtain

$$\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_2\|^2 \middle| \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \leq \sigma^2(\epsilon + \Pr\left[\mathcal{E}_{\mathbf{S}}\right]) + \sigma^2(1+\epsilon)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \tag{184}$$

and

$$\frac{1}{n}\mathsf{E}\left[\|\mathbf{Z}\|^2 \middle| \mathcal{E}_{\hat{\mathbf{U}}}\right]\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \leq N(\epsilon + \Pr\left[\mathcal{E}_{\mathbf{Z}}\right]) + N(1+\epsilon)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]. \tag{185}$$

Thus, combining (183)–(185) with (182) gives the inequality shown at the bottom of the page. □

Based on Lemma F.3 and Lemma F.4, the proof of Proposition F.1 now follows easily.

Proof of Proposition F.1:

$$\begin{aligned}\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1-\hat{\mathbf{S}}_1\|^2\right] - \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1-\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] &= \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1-\hat{\mathbf{S}}_1\|^2 - \|\mathbf{S}_1-\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] \\ &= \frac{1}{n}\Big(\mathsf{E}\left[\|\mathbf{S}_1\|^2\right] - 2\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1\right\rangle\right] + \mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2\right] - \mathsf{E}\left[\|\mathbf{S}_1\|^2\right] + 2\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1^{\mathrm{G}}\right\rangle\right] - \mathsf{E}\left[\|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right]\Big) \\ &= 2\frac{1}{n}\mathsf{E}\left[\left\langle \mathbf{S}_1, \hat{\mathbf{S}}_1^{\mathrm{G}} - \hat{\mathbf{S}}_1\right\rangle\right] + \frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] \\ &\overset{a)}{\leq} 2\sigma^2\left(\epsilon + 17\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + (17+\epsilon)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right]\right) + 4\big(\sigma^2(1+4\gamma_{13}) + 9\gamma_{13}(P_1+P_2+N)(1+\epsilon)\big)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\ &\quad + 36\gamma_{13}\big((P_1+P_2)\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + N\Pr\left[\mathcal{E}_{\mathbf{Z}}\right] + (P_1+P_2+N)\epsilon\big) \\ &= \xi_1\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] + \xi_2\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + \xi_3\Pr\left[\mathcal{E}_{\mathbf{Z}}\right] + \xi_4\epsilon \end{aligned} \tag{186}$$

where in step $a)$ we have used Lemma F.3 and Lemma F.4, and where ξ_ℓ, $\ell \in \{1,2,3,4\}$, depend only on σ^2, γ_{13}, P_1, P_2, and N. Combining (186) with Lemma D.1, Lemma D.2, and Lemma D.3 gives that for every $\delta > 0$ and $0.3 > \epsilon > 0$, there exists an $n'(\delta,\epsilon) > 0$ such that for all $(R_1,R_2) \in \mathcal{R}'(\epsilon)$ and $n > n'(\delta,\epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1-\hat{\mathbf{S}}_1\|^2\right] - \frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1-\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] < \xi_1'\delta + \xi_2'\epsilon$$

where ξ_1' and ξ_2' depend only on σ^2, γ_{13}, P_1, P_2, and N. □

C) Upper Bound on Expected Distortion: We now derive an upper bound on the achievable distortion for the proposed vector-quantizer scheme. By Corollary F.1, it suffices to analyze the genie-aided scheme. Using that $\hat{\mathbf{S}}_1^{\mathrm{G}} = \gamma_{11}\mathbf{U}_1^* + \gamma_{12}\mathbf{U}_2^* + \gamma_{13}\mathbf{Y}$, we have

$$\begin{aligned}\frac{1}{n}\mathsf{E}\Big[\|\mathbf{S}_1-\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\Big] &= \frac{1}{n}\Big(\mathsf{E}\left[\|\mathbf{S}_1\|^2\right] - 2\gamma_{11}\mathsf{E}\left[\langle\mathbf{S}_1,\mathbf{U}_1^*\rangle\right] - 2\gamma_{12}\mathsf{E}\left[\langle\mathbf{S}_1,\mathbf{U}_2^*\rangle\right] - 2\gamma_{13}\mathsf{E}\left[\langle\mathbf{S}_1,\mathbf{Y}\rangle\right] + \gamma_{11}^2\mathsf{E}\left[\|\mathbf{U}_1^*\|^2\right] \\ &\quad + 2\gamma_{11}\gamma_{12}\mathsf{E}\left[\langle\mathbf{U}_1^*,\mathbf{U}_2^*\rangle\right] + 2\gamma_{11}\gamma_{13}\mathsf{E}\left[\langle\mathbf{U}_1^*,\mathbf{Y}\rangle\right] + \gamma_{12}^2\mathsf{E}\left[\|\mathbf{U}_2^*\|^2\right] + 2\gamma_{12}\gamma_{13}\mathsf{E}\left[\langle\mathbf{U}_2^*,\mathbf{Y}\rangle\right] + \gamma_{13}^2\mathsf{E}\left[\|\mathbf{Y}\|^2\right]\Big). \end{aligned} \tag{187}$$

Some of the expectation terms are bounded straightforwardly. In particular, we have $\mathsf{E}\left[\|\mathbf{S}_1\|^2\right] = n\sigma^2$, $\mathsf{E}\left[\|\mathbf{U}_1^*\|^2\right] = n\sigma^2(1-2^{-2R_1})$, and $\mathsf{E}\left[\|\mathbf{U}_2^*\|^2\right] = n\sigma^2(1-2^{-2R_2})$. For three further terms we take over the bounds from the analysis of the vector-quantizer scheme. That is, by Lemma D.14, we have that for every $\delta > 0$ and $0 < \epsilon < 0.3$ and every positive integer n

$$\begin{aligned}\frac{1}{n}\mathsf{E}\left[\langle\mathbf{S}_1,\mathbf{U}_1^*\rangle\right] &\geq \sigma^2(1-2^{-2R_1}) - \zeta_1(\delta,\epsilon) \\ &= \mathsf{c}_{11} - \zeta_1(\delta,\epsilon) \end{aligned} \tag{188}$$

$$\begin{aligned}\frac{1}{n}\mathsf{E}\left[\|\hat{\mathbf{S}}_1\|^2 - \|\hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] &\leq 4\sigma^2\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] + 16\gamma_{13}\sigma^2\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] + 36\gamma_{13}(P_1+P_2)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\ &\quad + 36\gamma_{13}\big((P_1+P_2)(\epsilon + \Pr\left[\mathcal{E}_{\mathbf{S}}\right]) + (P_1+P_2+N)(1+\epsilon)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] + N(\epsilon+\Pr\left[\mathcal{E}_{\mathbf{Z}}\right])\big) \\ &\leq 4\big(\sigma^2(1+4\gamma_{13}) + 9\gamma_{13}(P_1+P_2+N)(1+\epsilon)\big)\Pr\left[\mathcal{E}_{\hat{\mathbf{U}}}\right] \\ &\quad + 36\gamma_{13}\big((P_1+P_2)\Pr\left[\mathcal{E}_{\mathbf{S}}\right] + N\Pr\left[\mathcal{E}_{\mathbf{Z}}\right] + (P_1+P_2+N)\epsilon\big). \end{aligned}$$

where $\zeta_1(\delta,\epsilon)$ is such that $\lim_{\delta,\epsilon\to 0}\zeta_1(\delta,\epsilon) = 0$. By Lemma D.15, we have that for every $\delta > 0$ and $0 < \epsilon < 0.3$ there exists an $n_2'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n_2'(\delta,\epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle\right] \le \sigma^2\rho(1-2^{-2R_1})(1-2^{-2R_2}) + \zeta_2(\delta,\epsilon) = \mathsf{k}_{12} + \zeta_2(\delta,\epsilon) \tag{189}$$

where $\zeta_2(\delta,\epsilon)$ is such that $\lim_{\delta,\epsilon\to 0}\zeta_2(\delta,\epsilon) = 0$, and by Lemma D.16, we have that for every $\delta > 0$ and $0 < \epsilon < 0.3$ there exists an $n'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n'(\delta,\epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] \ge \sigma^2\rho(1-2^{-2R_2}) - \zeta_3(\delta,\epsilon) = \mathsf{c}_{12} - \zeta_3(\delta,\epsilon) \tag{190}$$

where $\zeta_3(\delta,\epsilon)$ is such that $\lim_{\delta,\epsilon\to 0}\zeta_3(\delta,\epsilon) = 0$. Next, recalling that $\mathbf{Y} = \alpha_1\mathbf{S}_1 + \beta_1\mathbf{U}_1^* + \alpha_2\mathbf{S}_2 + \beta_2\mathbf{U}_2^* + \mathbf{Z}$, gives

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{Y}\rangle\right] \\
&= \frac{1}{n}\Big(\alpha_1\mathsf{E}\left[\|\mathbf{S}_1\|^2\right] + \beta_1\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_1^*\rangle\right] + \alpha_2\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{S}_2\rangle\right] \\
&\quad + \beta_2\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] + \underbrace{\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{Z}\rangle\right]}_{=0}\Big) \\
&\overset{a)}{\ge} \alpha_1\sigma^2 + \beta_1(\mathsf{c}_{11} - \zeta_1(\delta,\epsilon)) + \alpha_2\rho\sigma^2 + \beta_2(\mathsf{c}_{12} - \zeta_3(\delta,\epsilon)) \\
&= \mathsf{c}_{13} - \zeta_4(\delta,\epsilon)
\end{aligned} \tag{191}$$

where in $a)$ we have used (188), (189) and (190), and where $\zeta_4(\delta,\epsilon)$ is such that $\lim_{\delta,\epsilon\to 0}\zeta_4(\delta,\epsilon) = 0$. For the remaining terms in (187), it can be shown, similarly as for (188) and (190), that for every $\delta > 0$ and $0 < \epsilon < 0.3$ there exists an $n''(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n''(\delta,\epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_1^*\rangle\right] \le \mathsf{c}_{11} + \zeta_5(\delta,\epsilon) \tag{192}$$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_2, \mathbf{U}_1^*\rangle\right] \le \mathsf{c}_{21} + \zeta_6(\delta,\epsilon) \tag{193}$$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] \le \mathsf{c}_{12} + \zeta_7(\delta,\epsilon) \tag{194}$$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{S}_2, \mathbf{U}_2^*\rangle\right] \le \mathsf{c}_{22} + \zeta_8(\delta,\epsilon) \tag{195}$$

where $\zeta_j(\delta,\epsilon)$, $j \in \{5,\ldots 8\}$, are such that $\lim_{\delta,\epsilon\to 0}\zeta_j(\delta,\epsilon) = 0$. Using (189) and (192)–(195), we now get that for every $\delta > 0$ and $0 < \epsilon < 0.3$ there exists an $\tilde{n}_1(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > \tilde{n}_1(\delta,\epsilon)$

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{Y}\rangle\right] \\
&= \frac{1}{n}\Big(\alpha_1\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{S}_1\rangle\right] + \beta_1\mathsf{E}\left[\|\mathbf{U}_1^*\|^2\right] + \alpha_2\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{S}_2\rangle\right] \\
&\quad + \beta_2\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle\right] + \underbrace{\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{Z}\rangle\right]}_{=0}\Big) \\
&\le \alpha_1(\mathsf{k}_{11} + \zeta_5(\delta,\epsilon)) + \beta_1\mathsf{k}_{11} + \alpha_2(\mathsf{c}_{21} + \zeta_6(\delta,\epsilon)) \\
&\quad + \beta_2(\mathsf{k}_{12} + \zeta_2(\delta,\epsilon)) \\
&= \mathsf{k}_{13} + \tilde{\zeta}_1(\delta,\epsilon)
\end{aligned} \tag{196}$$

where $\tilde{\zeta}_1(\delta,\epsilon)$ is such that $\lim_{\delta,\epsilon\to 0}\tilde{\zeta}_1(\delta,\epsilon) = 0$. Similarly, it can be shown that for every $\delta > 0$ and $0 < \epsilon < 0.3$ there exists an $\tilde{n}_2(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > \tilde{n}_2(\delta,\epsilon)$

$$\frac{1}{n}\mathsf{E}\left[\langle \mathbf{U}_2^*, \mathbf{Y}\rangle\right] \le \mathsf{k}_{23} + \tilde{\zeta}_2(\delta,\epsilon) \tag{197}$$

where $\tilde{\zeta}_2(\delta,\epsilon)$ is such that $\lim_{\delta,\epsilon\to 0}\tilde{\zeta}_2(\delta,\epsilon) = 0$, and finally, we have that for every $\delta > 0$ and $0 < \epsilon < 0.3$ there exists an $\tilde{n}_3(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > \tilde{n}_3(\delta,\epsilon)$

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\|\mathbf{Y}\|^2\right] \\
&= \frac{1}{n}\Big(\alpha_1^2\mathsf{E}\left[\|\mathbf{S}_1\|^2\right] + 2\alpha_1\beta_1\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_1^*\rangle\right] \\
&\quad + 2\alpha_1\alpha_2\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{S}_2\rangle\right] + 2\alpha_1\beta_2\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{U}_2^*\rangle\right] \\
&\quad + 2\alpha_1\underbrace{\mathsf{E}\left[\langle \mathbf{S}_1, \mathbf{Z}\rangle\right]}_{=0} + \beta_1^2\mathsf{E}\left[\|\mathbf{U}_1^*\|^2\right] \\
&\quad + 2\beta_1\alpha_2\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{S}_2\rangle\right] + 2\beta_1\beta_2\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{U}_2^*\rangle\right] \\
&\quad + 2\beta_1\underbrace{\mathsf{E}\left[\langle \mathbf{U}_1^*, \mathbf{Z}\rangle\right]}_{=0} + \alpha_2^2\mathsf{E}\left[\|\mathbf{S}_2\|^2\right] \\
&\quad + 2\alpha_2\beta_2\mathsf{E}\left[\langle \mathbf{S}_2, \mathbf{U}_2^*\rangle\right] + 2\alpha_2\underbrace{\mathsf{E}\left[\langle \mathbf{S}_2, \mathbf{Z}\rangle\right]}_{=0} \\
&\quad + \beta_2^2\mathsf{E}\left[\|\mathbf{U}_2^*\|^2\right] + 2\beta_2\underbrace{\mathsf{E}\left[\langle \mathbf{U}_2^*, \mathbf{Z}\rangle\right]}_{=0} + \mathsf{E}\left[\|\mathbf{Z}\|^2\right]\Big) \\
&\le \alpha_1^2\sigma^2 + 2\alpha_1\beta_1(\mathsf{c}_{11} + \zeta_5(\delta,\epsilon)) + 2\alpha_1\alpha_2\rho\sigma^2 \\
&\quad + 2\alpha_1\beta_2(\mathsf{c}_{12} + \zeta_7(\delta,\epsilon)) + \beta_1^2\mathsf{k}_{11} \\
&\quad + 2\beta_1\alpha_2(\mathsf{c}_{21} + \zeta_6(\delta,\epsilon)) + 2\beta_1\beta_2(\mathsf{k}_{12} + \zeta_2(\delta,\epsilon)) \\
&\quad + \alpha_2^2\sigma^2 + 2\alpha_2\beta_2(\mathsf{c}_{22} + \zeta_8(\delta,\epsilon)) + \beta_2^2\mathsf{k}_{22} + N \\
&= \mathsf{k}_{33} + \tilde{\zeta}_3(\delta,\epsilon))
\end{aligned} \tag{198}$$

where $\tilde{\zeta}_3(\delta,\epsilon)$ is such that $\lim_{\delta,\epsilon\to 0}\tilde{\zeta}_3(\delta,\epsilon) = 0$. Thus, combining (188)–(191) and (196)–(198) with (187) gives that for every $\delta > 0$ and $0 < \epsilon < 0.3$ there exists an $n'(\delta,\epsilon) \in \mathbb{N}$ such that for all $n > n'(\delta,\epsilon)$

$$\begin{aligned}
&\frac{1}{n}\mathsf{E}\left[\|\mathbf{S}_1 - \hat{\mathbf{S}}_1^{\mathrm{G}}\|^2\right] \\
&\le \sigma^2 - 2\gamma_{11}\mathsf{c}_{11} - 2\gamma_{12}\mathsf{c}_{12} - 2\gamma_{13}\mathsf{c}_{13} + \gamma_{11}^2\mathsf{k}_{11} \\
&\quad + 2\gamma_{11}\gamma_{12}\mathsf{k}_{12} + 2\gamma_{11}\gamma_{13}\mathsf{k}_{13} + \gamma_{12}^2\mathsf{k}_{22} \\
&\quad + \gamma_{12}\gamma_{13}\mathsf{k}_{23} + \gamma_{13}^2\mathsf{k}_{33} + \zeta'(\delta,\epsilon) \\
&= \sigma^2 - 2\gamma_{11}\mathsf{c}_{11} - 2\gamma_{12}\mathsf{c}_{12} - 2\gamma_{13}\mathsf{c}_{13} \\
&\quad + \begin{pmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13}\end{pmatrix}\mathsf{K}\begin{pmatrix}\gamma_{11}\\ \gamma_{12}\\ \gamma_{13}\end{pmatrix} + \zeta'(\delta,\epsilon) \\
&\overset{a)}{=} \sigma^2 - 2\gamma_{11}\mathsf{c}_{11} - 2\gamma_{12}\mathsf{c}_{12} - 2\gamma_{13}\mathsf{c}_{13} \\
&\quad + \begin{pmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13}\end{pmatrix}\mathsf{K}\mathsf{K}^{-1}\begin{pmatrix}\mathsf{c}_{11}\\ \mathsf{c}_{12}\\ \mathsf{c}_{13}\end{pmatrix} + \zeta'(\delta,\epsilon) \\
&= \sigma^2 - 2\gamma_{11}\mathsf{c}_{11} - 2\gamma_{12}\mathsf{c}_{12} - 2\gamma_{13}\mathsf{c}_{13} \\
&\quad + \begin{pmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13}\end{pmatrix}\begin{pmatrix}\mathsf{c}_{11}\\ \mathsf{c}_{12}\\ \mathsf{c}_{13}\end{pmatrix} + \zeta'(\delta,\epsilon) \\
&= \sigma^2 - \gamma_{11}\mathsf{c}_{11} - \gamma_{12}\mathsf{c}_{12} - \gamma_{13}\mathsf{c}_{13} + \zeta'(\delta,\epsilon)
\end{aligned} \tag{199}$$

where we have used the shorthand notation K for $\mathsf{K}(R_1, R_2)$, and where in $a)$ we have used the definition of the coefficients γ_{ij} in (44), and where $\zeta'(\delta, \epsilon)$ satisfies $\lim_{\delta,\epsilon \rightarrow 0} \zeta'(\delta, \epsilon) = 0$. Now, letting in (199) first $n \rightarrow \infty$ and then $\delta, \epsilon \rightarrow 0$, and combining the result with Corollary F.1 gives

$$\overline{\lim_{n \rightarrow \infty}} \mathsf{E}\|\mathbf{S}_1 - \hat{\mathbf{S}}_1\|^2 \leq \sigma^2 - \gamma_{11}\mathsf{c}_{11} - \gamma_{12}\mathsf{c}_{12} - \gamma_{13}\mathsf{c}_{13}$$

whenever (R_1, R_2) satisfy (169)–(171).

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