Signal Processing with Factor Graphs: Beamforming and Hilbert Transform

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***Abstract*—Continuous-time linear state space models with discrete-time observations enable digital estimation of continuous-time signals with arbitrary temporal resolution by means of Kalman filtering/smoothing or Gaussian message passing in the corresponding factor graph. In this paper, we demonstrate the application of this approach to time-domain sensor array processing and to an emulation of the Hilbert transform.**

I. Introduction

Linear Gaussian state space models and Kalman filtering have long been standard tools in signal processing [1]. Nonetheless, such models are even more versatile than is commonly appreciated (cf. [2]–[5] for some pertinent examples). In this paper, we further illustrate this point by two examples that are almost obvious (with hindsight), but do not appear to be widely known and are perhaps even new. First, we point out that continuous-time models with discrete-time observations can be used for beamforming. Second, we propose an emulation of the Hilbert transform for such state space models, which can be used to define and to estimate the instantaneous amplitude of a signal.

We will use both discrete-time and continuous-time models. In the former, a real state vector X_k evolves according to

$$X_{k+1} = AX_k + BU_k \tag{1}$$

$$Y_k = CX_k + Z_k \tag{2}$$

where A is a real square matrix, $\{U_k\}$ and $\{Z_k\}$ are white Gaussian noise, and B and C are real vectors or matrices of appropriate dimensions. In continuous-time models, the state vector $X(t)$ evolves according to

$$dX(t) = AX(t)\,dt + BU(t)\,dt \tag{3}$$

where $U(t)$ is white Gaussian noise, and we observe

$$Y_k \triangleq CX(t_k) + Z_k \tag{4}$$

for discrete moments $\{t_k\}$. Such a continuous-time model may be viewed as a special case of a discrete-time model as in (1) and (2) (with different matrices A and B).

As is well known, MAP/MMSE/LMMSE estimation in such models can be carried out by Kalman filtering/smoothing [1] or, equivalently, by Gaussian message passing in the corresponding factor graph [8], [7]. In continuous-time models, all pertinent signals $X(t)$, $Y(t) \triangleq CX(t)$, and even $U(t)$ can be estimated with arbitrary temporal resolution [6], [7].

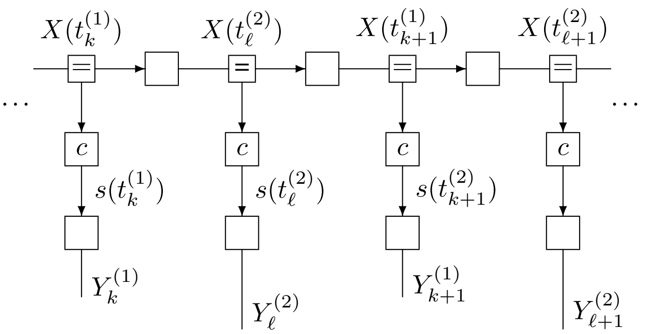

Fig. 1. Factor graph of beamforming as in Section II-A with a state space model of the source/target and with interleaved observations from two sensors.

II. Beamforming

Assume that some signal of interest $s(t)$ is measured by N sensors such that sensor n receives this signal with delay τ_n, $n = 1, \ldots, N$. Every sensor produces a discrete-time output signal; the output signal of sensor n will be denoted $\{Y_k^{(n)} : k = 1, 2, 3, \ldots\}$. From all these discrete-time signals (from all sensors), we wish to estimate $s(t)$.

We will consider two different special cases of this general problem. In both cases, we will assume that the delays $\tau_1, \ldots, \tau_N$ are known; estimating these delays is beyond the scope of this paper.

For general background on beamforming, we refer to [9], [10].

A. State Space Model of Signal with Wide-Band Sensors

In this case, we assume that we have a continuous-time state space model for $s(t)$ with $X(t)$ as in (3) and $s(t) = cX(t)$ for some row vector c. The output signal of sensor n at time $t_k^{(n)} + \tau_n$ is

$$Y_k^{(n)} = s(t_k^{(n)}) + Z_k^{(n)} \tag{5}$$

where $\{Z_k^{(n)}\}$ is white Gaussian noise.

We then immediately have a factor graph as in Fig. 1. The unlabeled boxes in the top row of Fig. 1 represent the evolution of the state between discrete moments of time, i.e., conditional densities of the form $f\big(x(t_k)|x(t_\ell)\big)$, and the unlabeled boxes in the bottom row represent conditional densities $f(y_k^{(n)}|s(t_k^{(n)}))$ according to (5).

MAP/MMSE/LMMSE estimation of $s(t)$ then amounts to Gaussian message passing in this factor graph as described in

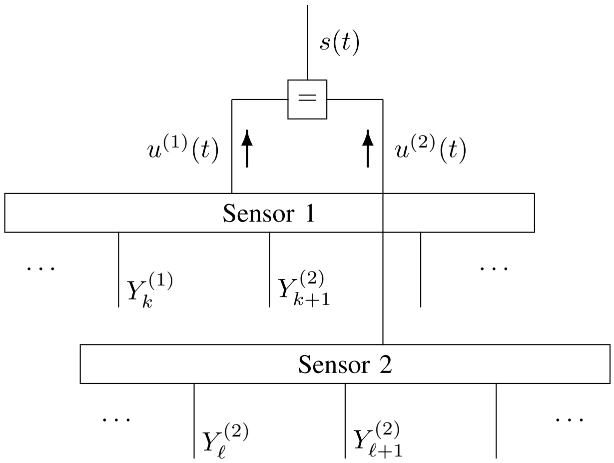

Fig. 2. Beamforming as in Section II-B with two sensors and with a state space model of each sensor. (This figure is not really a factor graph.)

[6], [7]. The complexity of this computation is linear in the number of sensors N.

By adding a suitable glue factor [2]–[5], the state space model can be augmented to model pulse-like signals that are localized at some unknown time. The detection of such pulses can then profit from the combined information of all sensors.

B. Wide-Band Signal with State Space Models of Sensors

In this case, we assume that we have a continuous-time state space model as in (3) for every sensor, each with input signal

$$u^{(n)}(t) = s(t). \tag{6}$$

We first estimate the input signal $u^{(n)}(t)$ of each sensor individually, disregarding (6); we compute this estimate under the assumption that $u^{(n)}(t)$ is white Gaussian noise (as discussed in [6], [7]), thus avoiding any *a priori* assumptions about the spectrum of $s(t)$. In a second step, we combine these estimates to the final estimate $\hat{s}(t)$, as illustrated in Fig. 2.

Arguing as in the proof of Theorem 1 of [7], we finally obtain

$$\hat{s}(t) = \sigma_S^2 \sum_{n=1}^{N} b^{\mathsf{T}} \overleftarrow{W}^{(n)} \left(\overrightarrow{m}_{X^{(n)}(t)} - \overleftarrow{m}_{X^{(n)}(t)} \right) \tag{7}$$

where

$$\overleftarrow{W}^{(n)} \triangleq \left(\overrightarrow{V}_{X^{(n)}(t)} + \overleftarrow{V}_{X^{(n)}(t)} \right)^{-1}, \tag{8}$$

where the quantities $\overrightarrow{m}_{X^{(n)}(t)}$ and $\overrightarrow{V}_{X^{(n)}(t)}$ denote the mean vector and the covariance matrix, respectively, of the forward message in the state space model, $\overleftarrow{m}_{X^{(n)}(t)}$ and $\overleftarrow{V}_{X^{(n)}(t)}$ denote the corresponding quantities for the backward message, and σ_S^2 is the *a priori* variance of $\int_0^1 s(t)\,dt$.

It should be noted that (7) is the correct MAP/MMSE/LMMSE estimate only for $N = 1$; for $N > 1$, some dependencies are neglected. Nonetheless, (7) is an obvious and useful estimate, and the complexity of its computation is linear in N.

III. Emulating the Hilbert Transform

A. Background: the Hilbert Transform

The basic idea of the Hilbert transform [11] may be stated as follows. For a given real signal/function $r(t)$, a corresponding imaginary signal $is(t)$ (where $s(t)$ is real and $i \triangleq \sqrt{-1}$) is created according to the following principles:

1) For $r(t) = \cos(\omega t + \varphi)$, we define $s(t) \triangleq \sin(\omega t + \varphi)$ and thus $r(t) + is(t) = e^{i(\omega t + \varphi)}$.
2) The mapping $r(t) \mapsto s(t)$ is linear.

These two principles lead to the following definition of the Hilbert transform of a signal/function $r(t)$ with a well-defined Fourier transform. Let $f^{\mathcal{F}}$ denote the Fourier transform of f, i.e.,

$$f^{\mathcal{F}}(\omega) \triangleq \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\,dt. \tag{9}$$

Then

$$s^{\mathcal{F}}(\omega) \triangleq \begin{cases} -ir^{\mathcal{F}}(\omega) & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ ir^{\mathcal{F}}(\omega) & \text{for } \omega < 0 \end{cases} \tag{10}$$

and thus

$$(r + is)^{\mathcal{F}}(\omega) = \begin{cases} 2r^{\mathcal{F}}(\omega) & \text{for } \omega > 0 \\ r^{\mathcal{F}}(\omega) & \text{for } \omega = 0 \\ 0 & \text{for } \omega < 0. \end{cases} \tag{11}$$

More generally, the Hilbert transform of $r(t)$ is defined by the convolution of $r(t)$ with $\frac{1}{\pi t}$, which agrees with (10) if $r(t)$ has a Fourier transform. It should be noted, however, that a filter with impulse response $\frac{1}{\pi t}$ is not stable, and a discrete step in $r(t)$ creates a pulse of infinite magnitude in $s(t)$.

The Hilbert transform is often used to define the instantaneous amplitude (or positive envelope signal) $\sqrt{r^2(t) + s^2(t)}$ of a signal $r(t)$.

These remarks do not do justice to the rich theory of the Hilbert transform [11], but they suffice to motivate the development below.

B. A State-Space Hilbert Transform

We are now going to emulate the idea of the Hilbert transform (according to the principles stated above) in a state space setting as follows.

Consider the linear state space model

$$X_k = AX_{k-1} + bU_k \tag{12}$$

$$Y_k = cX_k \tag{13}$$

where A is a real square matrix, where b is a real column vector, and where c is a real row vector. We will assume that all eigenvalues of A are nonzero (which excludes finite-impulse-response filters).

First, we consider the special case of (12) where

$$A = \rho \begin{pmatrix} \cos\Omega & -\sin\Omega \\ \sin\Omega & \cos\Omega \end{pmatrix} \tag{14}$$

with real Ω and $\rho > 0$. The state-space impulse response of such a system has the form $X_k = (0,0)^\mathsf{T}$ for $k < 0$ and

$$X_k = a\rho^k \begin{pmatrix} \text{Re}\left(e^{i(\Omega k+\varphi)}\right) \\ \text{Im}\left(e^{i(\Omega k+\varphi)}\right) \end{pmatrix} \tag{15}$$

for $k \geq 0$, where the amplitude a and the phase φ are determined by $X_0 = b$.

For $c = (1,0)$, we then have the impulse response

$$Y_k = \begin{cases} a\rho^k \cos(\Omega k + \varphi) & \text{for } k \geq 0 \\ 0 & \text{for } k < 0, \end{cases} \tag{16}$$

and we define its state-space Hilbert transform as

$$Y_k^{\mathcal{H}} \triangleq (0,1)X_k \tag{17}$$

$$= \begin{cases} a\rho^k \sin(\Omega k + \varphi) & \text{for } k \geq 0 \\ 0 & \text{for } k < 0. \end{cases} \tag{18}$$

It should be noted that (18) is *not* the Hilbert transform of (16), but it may be viewed as embodying the same idea for the state space model at hand. Note that $\sqrt{Y_k^2 + (Y_k^{\mathcal{H}})^2}$ is arguably a better instantaneous amplitude (or envelope signal) than its counterpart from the Hilbert transform.

Everything else then follows naturally. For $c = (0,1)$, the impulse response is

$$Y_k = \begin{cases} a\rho^k \sin(\Omega k + \varphi) & \text{for } k \geq 0 \\ 0 & \text{for } k < 0, \end{cases} \tag{19}$$

and we define

$$Y_k^{\mathcal{H}} \triangleq (-1,0)X_k \tag{20}$$

$$= \begin{cases} -a\rho^k \cos(\Omega k + \varphi) & \text{for } k \geq 0 \\ 0 & \text{for } k < 0. \end{cases} \tag{21}$$

For general $c = (c_1, c_2)$, we thus have

$$Y_k^{\mathcal{H}} \triangleq (-c_2, c_1)X_k \tag{22}$$

by linearity.

At this point, we have defined the state-space Hilbert transform (22) only for the impulse response of a system with A as in (14). However, for this same matrix A, (22) holds for any fixed input signal $\{U_k\}$ by linearity, and the further generalization to a stochastic input signal is immediate.

Beyond (14), we first consider the case where A is a real scalar, for which we define

$$Y_k^{\mathcal{H}} = 0. \tag{23}$$

Considering the impulse response and arguing as above, we note again that this definition does not agree with the Hilbert transform, but it can still be argued that it embodies the same idea for the particular model at hand. Again, $\sqrt{Y_k^2 + (Y_k^{\mathcal{H}})^2} = |Y_k|$ is arguably a better instantaneous amplitude than its counterpart from the Hilbert transform.

We next consider models of the form (12) and (13) where the real matrix A has the form

$$A = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_M \end{pmatrix} \tag{24}$$

where each block J_ℓ is either a 2×2 matrix as in (14) or else a real number. The corresponding generalization of (22) and (23) is

$$Y_k^{\mathcal{H}} \triangleq c \begin{pmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_M \end{pmatrix} X_k \tag{25}$$

with

$$R_\ell \triangleq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{26}$$

if J_ℓ is a 2×2 block and $R_\ell \triangleq 0$ if J_ℓ is a scalar. More generally, we propose the definition (25) for every matrix A in real Jordan form.

Finally, we consider models of the form (12) and (13) with arbitrary real matrix A (but still assuming that all eigenvalues of A are nonzero). Let T be a regular transformation matrix such that TAT^{-1} is in real Jordan form. With $\tilde{X}_k \triangleq TX_k$, we then have

$$\tilde{X}_{k+1} = TAT^{-1}\tilde{X}_k + TbU_k \tag{27}$$

$$Y_k = cT^{-1}\tilde{X}_k. \tag{28}$$

Applying (25) to this diagonalized model yields

$$Y_k^{\mathcal{H}} = cT^{-1} \begin{pmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_M \end{pmatrix} TX_k. \tag{29}$$

C. Remarks

The proposed state-space Hilbert transform (29) does not agree with the (standard) Hilbert transform [11], but it emulates its basic idea (according to the stated principles) for the state space model at hand. In particular, it can be used to define the instantaneous amplitude or envelope signal $\sqrt{Y_k^2 + (Y_k^{\mathcal{H}})^2}$, which may be more useful than its counterpart from the Hilbert transform.

We also note that the proposed state-space Hilbert transform is easily adapted to continuous-time models as in (3) and (4).

It should further be noted that the observation noise Z_k is an essential part of the signal models (1)–(4). In particular, the signal $\{Y_k\}$ and its state-space Hilbert transform $\{Y_k^{\mathcal{H}}\}$ are usually not directly observable, but need to be estimated from noisy observations by means of Kalman smoothing (forward-backward Gaussian message passing). In the continuous-time case, the instantaneous amplitude can thus be estimated with arbitrary temporal resolution.

Since (29) is purely local in time, the system model (12), (13) can easily be generalized to a time-varying model. In particular, the definition also applies to models with additional initial or final conditions, as well as to models for pulse-like signals that use glue factors [2]–[4].

IV. Conclusion

We have demonstrated the use of linear Gaussian state space models for beamforming and for a model-based emulation of the Hilbert transform. The former uses the continuous-time capability of such models as well as, in one version, input-signal estimation as in [7]. The latter uses the state space representation both for its definition and for estimating the corresponding signals from noisy observations.

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