Control-Based Analog-to-Digital Conversion Without Sampling and Quantization

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***Abstract*—A new approach to analog-to-digital conversion is proposed. The analog part of such a converter is a continuous-time linear system/filter that is subject to digital control; the digital part infers the signal based only on the digital control. An exact transfer-function analysis is possible.**

I. Introduction

In this paper, we further develop the new approach to analog-to-digital conversion that was proposed in [1] and add an extra twist to it. The approach of [1] is illustrated in Figure 1. The analog part of such an analog-to-digital converter (ADC) is a continuous-time linear system (filter) with an output signal $y(t)$ (or with several such output signals) which is sampled and quantized. From these samples, the digital part of the ADC estimates the analog input signal $u(t)$.

A main feature of the approach of [1] is to submit the analog part of the ADC to digital control (as in Figure 1), which makes it possibles to use a linear system/filter that is unstable (when not controlled). However, no explicit example of such an ADC was given in [1], and no attempt was made to analyze the behaviour of such a converter.

In this paper, we will show that the signal $y(t)$ in Figure 1 can be omitted; the signal $u(t)$ will then be estimated only from the digital control bits. The thresholds for these control bits need not be implemented with precision, and the (im-) precision of these thresholds does not affect the accuracy of the estimation of $u(t)$ (provided that the control is successful). We also present a transfer function analysis of the overall ADC, which shows, in particular, that the ADC is free of aliasing.

The proposed ADCs resemble sigma-delta ADCs [2] (and indeed, the analysis of Section II can be adapted to sigma-delta ADCs). In contrast to sigma-delta ADCs, we have no stability issues with high-order analog filters (because we are free to design a suitable control for any filter). The conceptual relation between our approach and the idea of beta-expansion converters [3] is discussed in [1]. For general background on analog-to-digital conversion, we refer to [4].

II. Input Signal Estimation: Transfer Functions

For ease of exposition, we will assume both $|u(t)| < 1$ and $|y(t)| < 1$. Let $\breve{y}(t)$ be the (hypothetical) output signal of the uncontrolled analog filter and assume that the digital control is such that

$$y(t) = \breve{y}(t) - q(t) \tag{1}$$

for some (presumably very complicated) control signal $q(t)$, which is (in principle) known to the digital estimator of $u(t)$.

Loosely speaking, the main idea is this: if there is sufficient gain in the analog filter, we expect $|\breve{y}(t)|$ to be much larger than $|y(t)|$ so that $\breve{y}(t) \approx q(t)$ and the "small" signal $y(t)$ can be replaced by 0 for estimating $u(t)$.

For the following analysis, we assume that the uncontrolled analog filter is time-invariant and stable with impulse response $g(t)$, i.e.,

$$\breve{y}(t) = (u * g)(t) \tag{2}$$

where "$*$" denotes convolution. We propose to estimate $u(t)$ by

$$\hat{u}(t) \triangleq (q * h)(t) \tag{3}$$

$$= (u * g * h)(t) - (y * h)(t) \tag{4}$$

for some suitable filter with impulse response $h(t)$. (The actual computation of $\hat{u}(t)$ need not follow (3), but can be carried out as outlined in Section IV.) Note that (3) is a continuous-time estimate which does not entail any aliasing.

Let $G(\omega)$ and $H(\omega)$ be the Fourier transforms of $g(t)$ and $h(t)$, respectively. A natural (with hindsight) choice of the estimation filter is

$$H(\omega) = \frac{\overline{G(\omega)}}{|G(\omega)|^2 + \eta^2} \tag{5}$$

where $\overline{x}$ denotes the complex conjugate of $x \in \mathbb{C}$ and where $\eta \in \mathbb{R}$ is a design parameter. The decomposition (4) can then be interpreted as follows. The first term in (4) is the signal path: the signal $u(t)$ (with $|u(t)| < 1$) is passed through a filter with transfer function

$$G(\omega)H(\omega) = \frac{|G(\omega)|^2}{|G(\omega)|^2 + \eta^2} \tag{6}$$

The second term in (4) is essentially the conversion error: the (unknown) signal $y(t)$ (with $|y(t)| < 1$) is passed through a filter with transfer function $H(\omega)$.

The magnitude of the ratio of these transfer functions is

$$\frac{|G(\omega)H(\omega)|}{|H(\omega)|} = |G(\omega)|, \tag{7}$$

which may be used as a proxy for the frequency-dependent signal-to-noise ratio of the ADC. The parameter η in (5) determines the bandwidth of the ADC, which is roughly given by $0 \leq \omega < \omega_{\text{crit}}$ with ω_{crit} determined by $|G(\omega_{\text{crit}})| = \eta$.

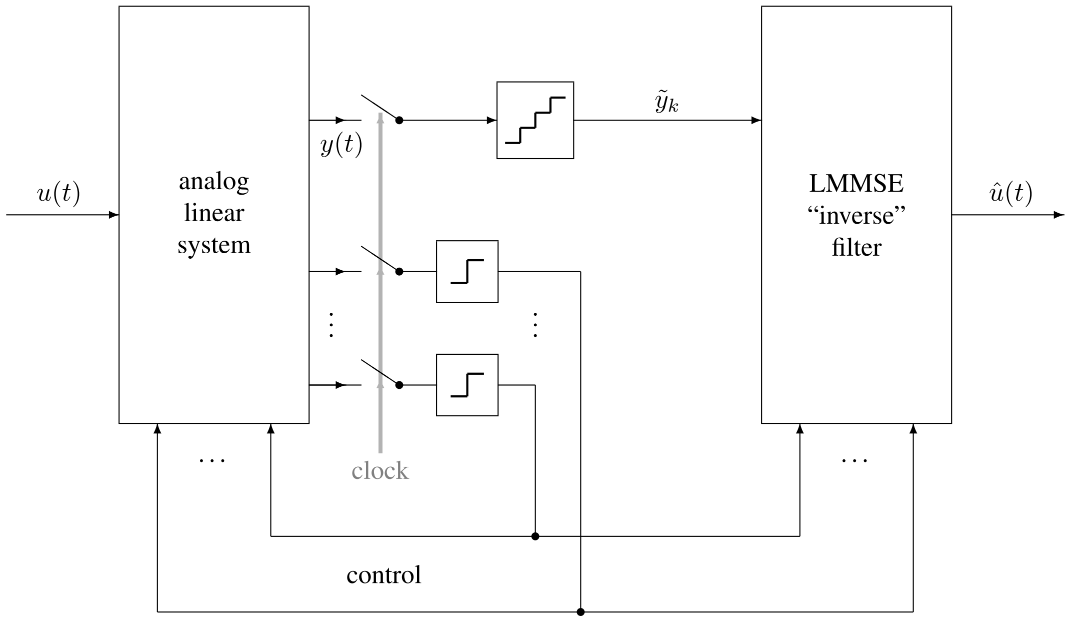

Fig. 1. Digital-to-analog conversion as in [1]. In the present paper, the signal path via $y(t)$ and $\tilde{y}_k$ is dropped.

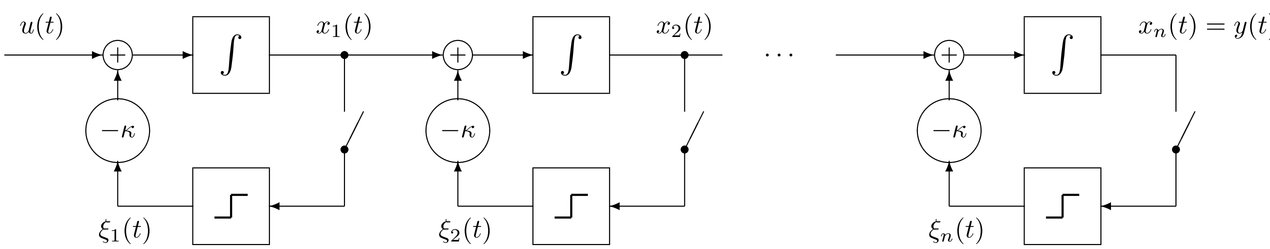

Fig. 2. Analog part for the example of Section III.

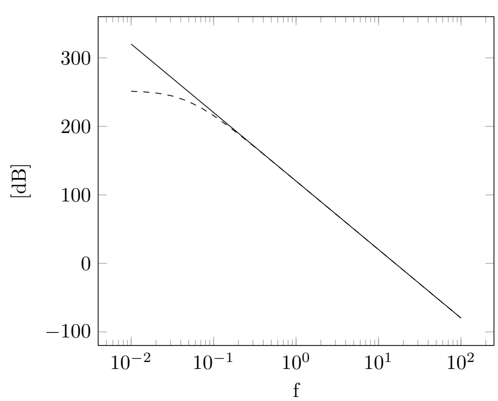

Fig. 3. Amplitude response $|G(2\pi f)|$ of the analog filter in dB ($= 20\log_{10}(|G(2\pi f)|)$) for the numerical example in Section III. Solid: $\rho = 0$; dashed: $\rho = 0.3$.

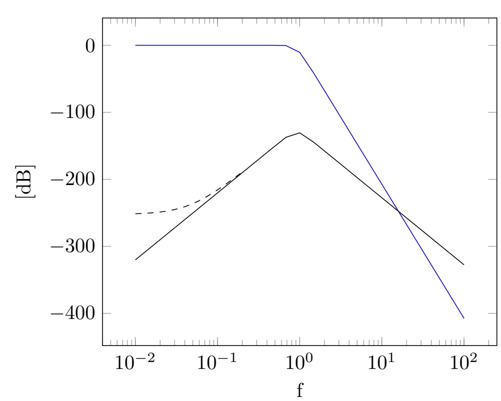

Fig. 4. Transfer functions for the ADC with analog filter as in Figure 3. Top: amplitude response $|G(2\pi f)H(2\pi f)|$ of ADC. Bottom: amplitude response $|H(2\pi f)|$ of error signal $y(t)$ for $\rho = 0$ (solid) and for $\rho = 0.3$ (dashed).

III. An Example

An example of the analog part of such an ADC is shown in Figure 2. The state variables $x_1, \ldots, x_n \in \mathbb{R}$ in Figure 2 obey the differential equation

$$\frac{d}{dt}x_\ell(t) = -\rho x_\ell(t) + \beta x_{\ell-1} - \kappa\xi_\ell(t), \tag{8}$$

$\ell = 1, \ldots n$, with $\rho \geq 0$, with $\beta > 0$, with $x_0(t) \triangleq u(t)$, and where $\xi_\ell(t) \in \{+1, -1\}$ is a control bit. For $\rho > 0$, the integrators are leaky and the (uncontrolled) integrator chain is stable; for $\rho = 0$, the (uncontrolled) integrator chain is unstable.

The switches in Figure 2 represent sample-and-hold circuits that are controlled by a digital clock (as in Figure 1) with period T. The threshold elements in Figure 2 produce $\xi_\ell(t) \in \{+1, -1\}$ depending on the sign of $x_\ell(kT)$ at sampling time kT immediately preceding t.

We will assume and require both $|u(t)| < 1$ and

$$|x_\ell(t)| < 1 \tag{9}$$

for $\ell = 1, \ldots n$, which constrains the admissible values of β, κ, ρ, and T.

Let $y(t) \triangleq x_n(t)$ be the (unused) output signal of the analog filter. The transfer function $G(\omega)$ (as defined in Section II) is then

$$G(\omega) = \left(\frac{\beta}{i\omega + \rho}\right)^n, \tag{10}$$

which is plotted in Figure 3 for $n = 5$, $\beta = 100$, and $\rho \in \{0, 0.3\}$. The resulting transfer functions (6) and (5) (with $\eta^2 = 300^5$) are plotted in Figure 4. The quantity (7) at $\omega = \omega_{\text{crit}}$ is $\eta \approx 124\,\text{dB}$. Note that the signal-to-noise ratio at low frequencies increases for $\rho \to 0$. Indeed, the obvious choice for a practical implementation is $\rho = 0$.

IV. LMMSE Perspective

The filter (5) may be recognized as the Wiener filter [5], [6] for estimating a continuous-time zero-mean white Gaussian noise signal $U(t)$ from

$$\tilde{Y}(t) \triangleq (U * g)(t) + Z(t), \tag{11}$$

where $Z(t)$ is also zero-mean white Gaussian noise. (We switch to capital letters $U(t)$, $\tilde{Y}(t)$, $Z(t)$, and $X(t)$ because these quantities are now stochastic processes.) With this interpretation, we have

$$\eta^2 = \sigma_Z^2/\sigma_U^2, \tag{12}$$

where σ_U^2 and σ_Z^2 are the power spectral densities of U and Z, respectively.

According to (3) and (4), this filter is then applied to the signal $q(t) = \breve{y}(t) - y(t)$ instead of $\tilde{Y}(t)$.

This Wiener-filter interpretation makes it easy to translate the estimation filter into a state space form (i.e., into a Kalman-filter setting) as in [1], [7]. (See [6] for a translation in the opposite direction.) In the state space setting, the (known) control signals in Figure 1 can be plugged directly into the state space model, which amounts to replacing the observation (11) by its controlled version $\tilde{Y}(t) - q(t)$, and replacing $\tilde{Y}(t)$ by $q(t)$ as the observed signal (as above) then amounts to estimating $U(t)$ from the "observation" 0.

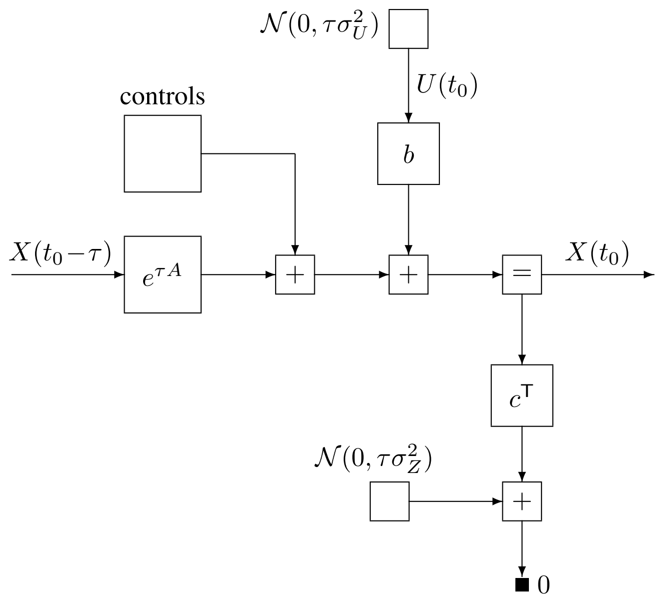

Fig. 5. A factor graph segment of the state space model for estimating $U(t)$. "$\mathcal{N}(m, \sigma^2)$" denotes a Gaussian density with mean m and variance σ^2. The representation is exact only in the limit $\tau \to 0$.

In summary, estimating $U(t)$ as in [1], [7] (with the known control signals plugged into the space space model) from the virtual observation $y(t) = 0$ coincides with the estimate $\hat{u}(t)$ from (3) and (5). The relevant factor graph [8] for this estimation is shown in Figure 5, which refers to the continuous-time state space model

$$dX(t) = AX(t)\,dt + bU(t), \tag{13}$$

and

$$y(t) = c^\mathsf{T} X(t) \tag{14}$$

with $X(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. The box labeled "controls" in Figure 5 represents the (known) control signals. The estimation can then be carried out by Gaussian message passing in this factor graph [7], [8]. A more detailed discussion of these computations will be given elsewhere.

V. Conclusion

We have proposed an approach to analog-to-digital conversion that is based entirely on digital control of an analog filter/system; there is no "signal path" with sampling and quantization. The digital control itself is (necessarily) based on sampling and thresholding of analog quantities, but the details (and the accuracy) of these operations are irrelevant for the accuracy of the proposed converter.

An exact continuous-time transfer-function analysis of such converters has been given. Such converters are not subject to aliasing, and the analog signal can (in principle) be estimated with arbitrary temporal resolution.

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