A STATE-SPACE APPROACH FOR THE ANALYSIS OF WAVE AND DIFFUSION FIELDS

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ABSTRACT

The analysis of wave and diffusion fields is a task central to a myriad of applications. Wave fields are encountered in acoustic, radar, and geophysics to name a few. Diffusion fields are found, for example, in physics, chemistry, and biology.

In this paper, we introduce a state-space approach allowing us to model both wave and diffusion fields within the same framework. Using state-space models we are able to model directly the partial differential equation describing the field. We use the sum-product algorithm to compute the likelihood of the observations in a computationally efficient manner.

We show how it is possible to estimate the parameters of the field, locate a point source, reconstruct the field at an arbitrary location.

***Index Terms*—** Wave equation, Diffusion equation, State-space model, Factor graph.

1. INTRODUCTION

In this work, we propose a state-space approach for the analysis of wave and diffusion fields. Using state-space models (SSMs) we are able to model directly the partial differential equation (PDE) describing the field. We consider the analysis of fields

$$v(t, r) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \tag{1}$$

described by linear PDEs with constant coefficients. Fields of this form include the one-dimensional wave equation and the diffusion equation. Many others exist [1].

In particular, we are interested in fitting a statistical model to observed data and to estimate unknown field parameters. We consider observations $\{Y_k^{(\ell)}\}$ of the field taken at uniform temporal instants $\{t_k\}_{k=1,\dots,K}$, at arbitrary spatial locations $\mathcal{S} = \{r_\ell\}_{\ell=1,\dots,L}$, and corrupted by i.i.d. additive Gaussian noise

$$Y_k^{(\ell)} = v(t_k, r_\ell) + Z_k^{(\ell)} \tag{2}$$

where $Z_k^{(\ell)} \sim \mathcal{N}(0, \sigma_\ell^2)$.

The analysis of wave fields has been thoroughly investigated within the array processing community. The many beamforming techniques developed in decades [2] have applications in diverse domains such as acoustic, radar, and geophysics. Indeed the estimation of direction of arrival and velocity of propagation is a task central to many applications.

Diffusion processes allow to model several physical and biological phenomena. Analysis of diffusion fields has received considerable attention in recent years [3, 4].

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2. PROPOSED APPROACH

First, we derive a SSM representation of the PDE describing the field. Second, we formulate another SSM that enables us to derive sufficient statistics used by the former SSM. Third, we describe how it is possible to compute likelihood of the observations $f(\boldsymbol{\theta})$ for a given parameter vector $\boldsymbol{\theta}$ by message passing on a factor graph. At last, we show how it is possible to reconstruct the field at arbitrary locations.

In the following, we limit our exposition to second-order PDEs. The extension to higher order is straightforward.

We denote scalars, vectors, matrices, and sets as $a, \boldsymbol{a}, \mathbf{A}, \mathcal{A}$, respectively. Random variables, either scalar or vector, are always capitalized.

2.1. A SSM for the field

We Fourier transform with respect to time a linear constant-coefficients PDE describing a field of the form in (1) and obtain the ordinary differential equation (ODE)

$$\partial_r^2 u(\omega, r) + \partial_r a_1 u(\omega, r) + a_2 u(\omega, r) = 0\,, \tag{3}$$

where $\partial_r = \partial/\partial r$ and $u(\omega, r)$ denotes the Fourier transform of $v(t, r)$.

The solutions of (3) are of the form

$$u(\omega, r) = c_1(\omega) e^{\lambda_1 r} + c_2(\omega) e^{\lambda_2 r}\,, \tag{4}$$

where $c_1(\omega), c_2(\omega)$ are arbitrary Hermitian-symmetric functions and λ_1, λ_2 are distinct roots of the characteristic equation $\lambda^2 + \lambda a_1 + a_2 = 0$.

Introducing the auxiliary variables u_1 and u_2, we rewrite (3) as a matrix differential equation

$$\partial_r \begin{bmatrix} u_1(\omega, r) \\ u_2(\omega, r) \end{bmatrix} = \mathbf{A}' \begin{bmatrix} u_1(\omega, r) \\ u_2(\omega, r) \end{bmatrix}\,, \tag{5}$$

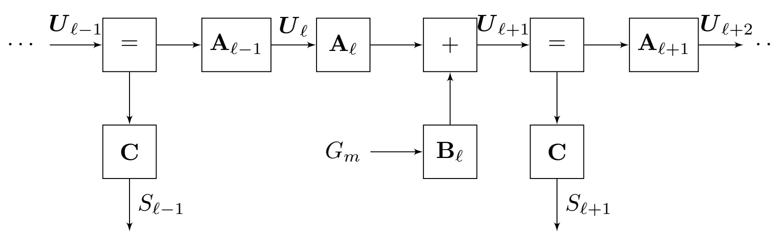

Fig. 1: Factor graph of (11) with $r_{\ell-1}, r_{\ell+1} \in \mathcal{S}$ and $r_\ell \in \mathcal{T}$.

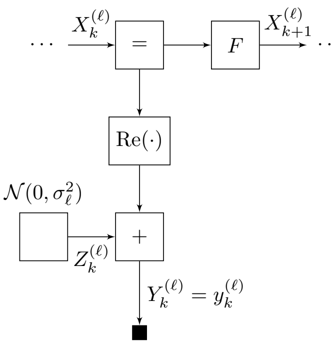

Fig. 2: Factor graph of (13).

with

$$\mathbf{A}' = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} . \tag{6}$$

This matrix is the companion matrix of the characteristic equation associated to (3). Observe that, in general, $\mathbf{A}'$ depends on ω.

In order to deal with the spatial sampling at the sensor locations $\mathcal{S}$, we solve the matrix differential equation in (5) for an arbitrary step $\Delta r_\ell = r_{\ell+1} - r_\ell$ as

$$\begin{bmatrix} u_1(\omega, r_{\ell+1}) \\ u_2(\omega, r_{\ell+1}) \end{bmatrix} = \mathbf{A}_\ell \begin{bmatrix} u_1(\omega, r_\ell) \\ u_2(\omega, r_\ell) \end{bmatrix} . \tag{7}$$

This latter expression has the form of the state equation of a discrete SSM with state-transition matrix

$$\mathbf{A}_\ell = \exp\left(\mathbf{A}'\Delta r_\ell\right) . \tag{8}$$

The matrix exponential can be computed analytically exploiting the factorization of $\mathbf{A}'$.

In light of the previous developments, we formulate the following discrete SSM

$$\boldsymbol{U}_{\ell+1} = \mathbf{A}_\ell \boldsymbol{U}_\ell \tag{9a}$$
$$S_\ell = \mathbf{C}\boldsymbol{U}_\ell \tag{9b}$$

for $\ell = 1, \ldots, L$ and with $\mathbf{C} = [1, 0]$. The state vector $\boldsymbol{U}_\ell$ can be interpreted as $[u(\omega, r_\ell), \partial_r u(\omega, r_\ell)]^\mathsf{T}$, i.e., as the Fourier transform of the field and its spatial derivative at position r_ℓ. The quantity S_ℓ can also be interpreted as $u(\omega, r_\ell)$ and will be discussed in Sec. 2.3.

The SSM in (9) couples the Fourier transforms of the field at different locations according to the law described by (3).

The SSM in (9) can be augmented to model multiple frequencies $\omega_n = 2\pi n/K$. It is sufficient to replace the state-transition matrix with a block diagonal matrix having as main diagonal blocks $\mathbf{A}_\ell(\omega_n)$. The state vector and the readout vector should be enlarged accordingly. The scalar quantity S_ℓ is replaced by a vector $\boldsymbol{S}_\ell$.

2.2. Point sources

The ODE in (3) can be extended to model the presence of point sources at locations $\mathcal{T} = \{r_m\}_{m=1,\ldots,M}$ by including an additional term

$$\partial_r^2 u(\omega, r) + \partial_r a_1 u(\omega, r) + a_2 u(\omega, r) = \sum_{m=1}^{M} \delta(r - r_m) g_m(\omega) , \tag{10}$$

where $g_m(t)$ is the signal emitted by the m-th source.

Similarly to the homogeneous case, we rewrite (10) in matrix form and discretize the spatial dimension at locations $r_\ell \in \mathcal{S} \cup \mathcal{T}$. We obtain a SSM with input

$$\boldsymbol{U}_{\ell+1} = \mathbf{A}_\ell \boldsymbol{U}_\ell + \mathbf{B}_\ell G_m \mathbf{1}_\mathcal{T}(r_\ell) \tag{11a}$$
$$S_\ell = \mathbf{C}\boldsymbol{U}_\ell \mathbf{1}_\mathcal{S}(r_\ell) , \tag{11b}$$

where G_m is interpreted as $g_m(\omega)$, $\mathbf{B}_\ell = \mathbf{A}_\ell[0, 1]^\mathsf{T}$, and $\mathbf{1}_\mathcal{A}(x)$ is the indicator function

$$\mathbf{1}_\mathcal{A}(x) = \begin{cases} 1 & x \in \mathcal{A} \\ 0 & x \notin \mathcal{A} \end{cases} . \tag{12}$$

We assume that there is no sensor located at the same position of a source, i.e. $\mathcal{S} \cap \mathcal{T} = \varnothing$.

2.3. A SSM for the Fourier transform

We consider another SSM to provide estimates of $u(\omega, r_\ell)$, used as the measurement $S_\ell = X_K^{(\ell)}$ in (9) or (11). For $k = 1, \ldots, K$

$$X_{k+1}^{(\ell)} = F X_k^{(\ell)} \tag{13a}$$
$$Y_k^{(\ell)} = \mathrm{Re}(X_k^{(\ell)}) + Z_k^{(\ell)} , \tag{13b}$$

with $F = \exp(i\omega\Delta t)$ and Δt being the uniform temporal sampling step.

The effect of the SSM (13) is akin to the effect of a discrete Fourier transform. More details are available in [5]. It can be augmented to account for multiple frequencies ω_n using a diagonal state-transition matrix $\mathbf{F}$.

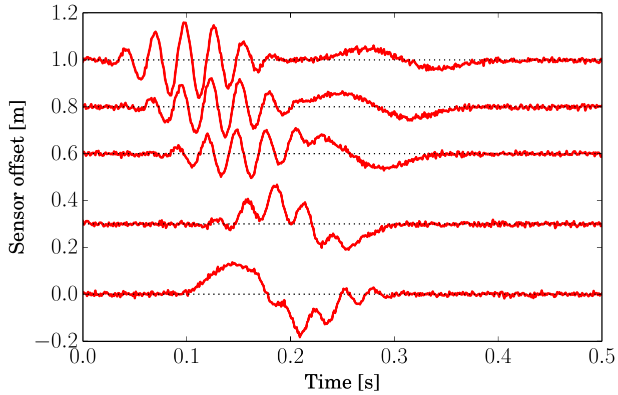

Fig. 3: Measured wave field at different sensors for the damped wave equation example. The signal amplitude is not to scale.

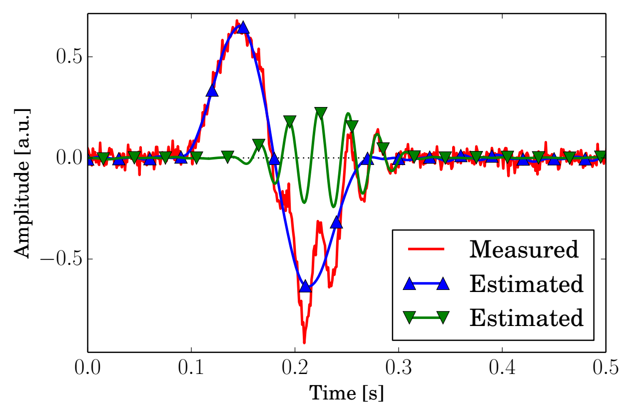

Fig. 4: Measured signal at $r = 0$ (in red) and the reconstructed waves (in blue and green).

2.4. Likelihood computation

The SSMs in (9), (11), and (13) precisely describe the relationships between all the quantities involved, from the measurements $\{Y_k^{(\ell)}\}$ to the Fourier transform of the measurements $\{S^{(\ell)}\}$, and to the field $\{\boldsymbol{U}_\ell\}$. We model such relationships using a factor graph [6] and use the sum-product algorithm to compute the likelihood of the observations.

Fig. 1 depicts a part of the factor graph of (11). Two sensors and a source are modeled. Fig. 2 depicts a part of the factor graph of (13) for a single measurement.

We consider message passing on the graph using scaled Gaussian messages. The message on the edge X, in the direction of the arrow, has the form

$$\overrightarrow{\mu}_X(x) = \overrightarrow{\gamma}_X e^{-x^\mathsf{T}\overrightarrow{\mathbf{W}}_X x/2 + x^\mathsf{T}\overrightarrow{\mathbf{W}}_X \overrightarrow{\boldsymbol{m}}_X}, \qquad (14)$$

and is parametrized by scale factor $\overrightarrow{\gamma}_X$, mean vector $\overrightarrow{\boldsymbol{m}}_X$, and inverse covariance matrix $\overrightarrow{\mathbf{W}}_X$. For the sake of exposition, so far we considered complex vectors and matrices. For the actual message passing implementation we consider their real equivalent and use the rules for Gaussian messages given in [6].

For the setting without sources as in (9), it can be shown that the log-likelihood is

$$\ln f(\boldsymbol{\theta}) = \frac{1}{2}\overrightarrow{\boldsymbol{m}}_{U_L}^\mathsf{T}\overrightarrow{\mathbf{W}}_{U_L}\overrightarrow{\boldsymbol{m}}_{U_L} + \ln \overrightarrow{\gamma}_{U_L}. \qquad (15)$$

This expression allows to perform maximum likelihood (ML) estimation of the field parameters $\boldsymbol{\theta} = (a_1, a_2)^\mathsf{T}$.

We observe that the messages pertaining the SSM in (13) do not depend on $\boldsymbol{\theta}$, i.e. are sufficient statistics, and can be computed only once. In contrast, the message passing for (9) need to be repeated for different values of $\boldsymbol{\theta}$.

Likelihood computation in presence of multiple sources may require an iterative approach such as what used by the authors in [7].

2.5. Field reconstruction

In absence of sources, knowledge of $u(\omega, r)$ and $\partial_r u(\omega, r)$ at one specific location enables us to find $c_1(\omega)$ and $c_2(\omega)$ in (4) by solving

$$\begin{bmatrix} u(\omega, r) \\ \partial_r u(\omega, r) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 r} & e^{\lambda_2 r} \\ \lambda_1 e^{\lambda_1 r} & \lambda_2 e^{\lambda_2 r} \end{bmatrix} \begin{bmatrix} c_1(\omega) \\ c_2(\omega) \end{bmatrix}. \qquad (16)$$

Once c_1, c_2 are known, it is then possible to reconstruct the field at any location r.

Given the ML estimate of the field parameters $\boldsymbol{\theta}$, the mean vector $\overrightarrow{\boldsymbol{m}}_{U_L}$ contains the ML estimates of $u(\omega, r_L)$ and $\partial_r u(\omega, r_L)$. These estimates can be used for the field reconstruction as in (16).

When sources are present, an estimate of $g_m(\omega)$ can also be obtained by message passing from $\overleftarrow{\boldsymbol{m}}_{G_m}$. Field reconstruction in this case needs to account for the particular solutions of (10).

3. EXAMPLES

We consider two numerical examples. First, a wave field modeled by a damped wave equation without point sources. Second, a diffusion field with a single point source.

3.1. Damped wave equation

Consider the one-dimensional damped wave equation

$$\partial_t^2 v(t, r) - c^2 \partial_r^2 v(t, r) + \gamma \partial_t v(t, r) = 0,$$

where c is the medium velocity and γ is a damping factor. We take the Fourier transform and obtain the Helmholtz equation

$$\partial_r^2 u(\omega, r) + \kappa^2 u(\omega, r) = 0,$$

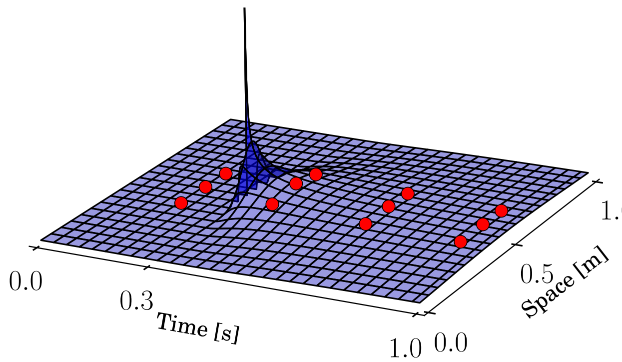

Fig. 5: The diffusion field and the measurements (red dots).

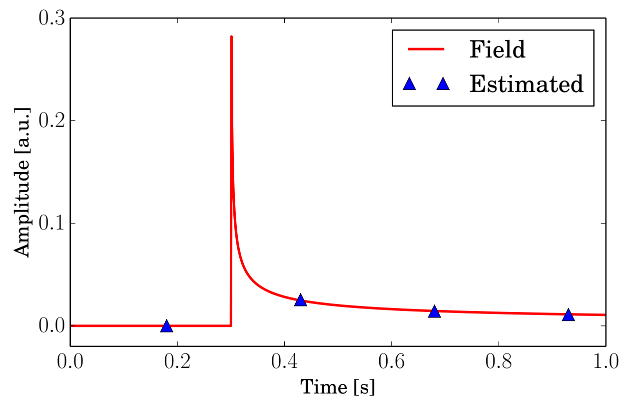

Fig. 6: Estimated diffusion field (blue triangles) at the estimated source location. Actual field (red line).

where $\kappa^2 = (\omega^2 - i\gamma\omega)/c^2$. The complex wavenumber accounts for wave dispersion and attenuation.

Following the steps described in Sec. 2.1, we obtain a discrete SSM as in (9) with state-transition matrix

$$\mathbf{A}_\ell = \exp\left(\begin{bmatrix} 0 & 1 \\ -\kappa^2 & 0 \end{bmatrix} \Delta r_\ell\right).$$

We generate a synthetic wave field with $c = 8, \gamma = 3$, and with two waves traveling in opposite directions. The wave field is measured at five locations and the noisy measurements are shown in Fig. 3. It is possible to see the effect of attenuation on the wave amplitude as the waves travel further.

The wave field is modeled with the proposed approach as in (9). The wave field parameters $\boldsymbol{\theta} = (c, \gamma)$ are unknown to the algorithm and are estimated with a ML approach.

Fig. 4 compares the signal measured at location $r = 0$ and the reconstructed field. The waves are reconstructed using equation (16). The waves traveling in opposite directions are separated and additive Gaussian noise is suppressed.

We emphasize that both the waves traveling in opposite directions are modeled jointly. This is conceptually different from beamforming techniques where only one direction of propagation is considered.

3.2. Diffusion process

Consider the one-dimensional heat equation with with thermal diffusivity $\alpha > 0$

$$\partial_t v(t, r) - \alpha \partial_r^2 v(t, r) = 0\,. \tag{17}$$

The state-transition matrix for this field is

$$\mathbf{A}_\ell = \exp\left(\begin{bmatrix} 0 & 1 \\ i\frac{\omega}{\alpha} & 0 \end{bmatrix} \Delta r_\ell\right). \tag{18}$$

Fig. 5 depicts a diffusion field generated by a single impulsive point source source. The field is measured at three locations in the proximity of the source. The measurements are depicted with red dots.

We model the field with the SSM with input of (11). The position of the source is unknown to the algorithm. The constant α is known.

We find the ML estimate of the source position with a search over possible locations $r_m \in [0, 1]$. As the source location is found, an ML estimate of $g(\omega)$ is also obtained by message passing.

Fig. 6 depicts the diffusion field at the source position. The actual diffusion field (red line) and the estimated value of the field (blue triangles) are compared. The estimated field is the inverse Fourier transform of the estimated $g(\omega)$.

4. CONCLUSIONS

We proposed an approach for the analysis of wave and diffusion fields using SSMs and factor graphs. Within the proposed framework, it is possible to model fields described by different PDEs.

The state-space formalism enables us to model directly the PDE describing the field and to account for localized sources. The likelihood of the observations is computed efficiently exploiting the factor graph structure using the sum-product algorithm. Field parameters and source locations can be estimated with a ML approach. Estimates of the field, the spatial derivative of the field, and source signal are found by message passing.

We constructed numerical examples for a wave field and a diffusion field. We showed how it is possible to separate waves traveling in different directions and jointly estimate wave field parameters. We showed how to estimate the source position in a diffusion field and to reconstruct the diffusion field at the source location.

5. REFERENCES

[1] S. Farlow, *Partial Differential Equations for Scientists and Engineers*, ser. Dover books on advanced mathematics. Dover Publications, 1993.

[2] H. L. Van Trees, *Optimum Array Processing*. John Wiley & Sons, Inc., 2002.

[3] J. Murray-Bruce and P. Dragotti, "Spatio-temporal sampling and reconstruction of diffusion fields induced by point sources," in *Acoustics, Speech and Signal Processing (ICASSP), 2014 IEEE International Conference on*, May 2014, pp. 31–35.

[4] Y. M. Lu and M. Vetterli, "Distributed spatio-temporal sampling of diffusion fields from sparse instantaneous sources," in *Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP), 2009 3rd IEEE International Workshop on*, 2009, pp. 205–208.

[5] C. Reller, H.-A. Loeliger, and S. Maranò, "Multi-sensor estimation and detection of phase-locked sinusoids," in *Proc. IEEE Int. Conf. Acoustics, Speech, and Signal Processing*, Prague, Czech Republic, May 2011, pp. 3872–3875.

[6] H.-A. Loeliger, J. Dauwels, J. Hu, S. Korl, L. Ping, and F. R. Kschischang, "The factor graph approach to model-based signal processing," *Proc. IEEE*, vol. 95, no. 6, pp. 1295–1322, June 2007.

[7] S. Maranò, C. Reller, H.-A. Loeliger, and D. Fäh, "Seismic waves estimation and wavefield decomposition: Application to ambient vibrations," *Geophys. J. Int.*, vol. 191, no. 1, pp. 175–188, Oct. 2012.