Acoustics II: electrical-mechanical-acoustical analogies

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Motivation

- microphones, loudspeakers consist of mechanical, acoustical and electrical subsystems
- due to their interaction an analysis has to consider all subsystems in integral way
- the fundamental differential equations have identical form in all systems
- introduction of analogies and conversion of mechanical and acoustical systems into electrical ones
- excellent tools available for analysis of electrical networks
analogies
mechanical systems
describing quantities
mechanical elements
Mechanical sources
Mechanical resonance - spring pendulum
acoustical systems
describing quantities
Acoustical elements
analogies
potential and flow quantities
Impedances
Measurement of acoustical impedances
distributed elements
coupling
interfaces
Dual conversion
thin absorbers
measurement of the impedance at the ear
Model of the middle ear
Diagnosis
Experiments
back

mechanical systems
Describing quantities

At a specific point of a mechanical system, the two quantities of interest are:

- velocity \( u = \frac{dx}{dt} \)
- force \( F \)

From \( u \), further quantities are derived to describe the movement:

- displacement \( x = \int u \, dt \)
- acceleration \( a = \frac{du}{dt} \)
mechanical elements
Mechanical mass

ideal mass:

- incompressible → each point of the mass has identical velocity
- physical description: $F_{res} = m \cdot a$ (Newton)

ports 1 and 2:

$u_1 = u_2 = u$

$F_1 - F_2 = m \frac{du}{dt}$
Spring

ideal spring:

- stiffness $s$ independent of excursion $x$
- physical description: $F = s \cdot x$ (Hook’s law)
- no force difference along the spring

\[ F = s \int (u_1 - u_2) \, dt \]
Friction

ideal friction:

- proportionality between friction force and velocity
- physical description: \( F = R \cdot u \)
- no force difference along the friction element

\[
F_1 = F_2 = F \\
F = R(u_1 - u_2)
\]
ideal link:

- connecting element for mechanical building blocks
- no mass and incompressible
**Lever**

**Ideal lever:**
- frictionless and massless
- mechanical transformer

\[ F_1 l_1 - F_2 l_2 = 0 \]
\[ u_1 l_2 + u_2 l_1 = 0 \]
Mechanical sources:

- force source
  - example: conductor in a magnetic field $\rightarrow$ force
    $\sim B \times I$ but independent of velocity

- velocity source
  - example: motor with a flywheel $\rightarrow$ velocity
    independent of load (force)
resonance system: spring pendulum

- given: external force: \( F(t) = \hat{F} \sin(\omega t) \)
- unknown: movement of mass
  - excursion \( x(t) \)
  - velocity \( u(t) = \frac{dx}{dt} \)
  - acceleration \( a(t) = \frac{d^2x}{dt^2} \)
resonance system: spring pendulum: solution?

equilibrium of forces:

\[ F_{\text{acceleration}} + F_{\text{friction}} + F_{\text{spring}} = F \]

differential equation for \( x \) in complex writing for harmonic excitation:

\[ m \frac{d^2x}{dt^2} + R \frac{dx}{dt} + sx = \hat{F} e^{j\omega t} \]
resonance system: spring pendulum:
solution?

general solution of the differential equation:

\[ x(t) = \frac{\hat{F}e^{i\omega t}}{(j\omega)^2 m + j\omega R + s} = \frac{\hat{F}e^{i\omega t}}{j\omega \left(j\omega m + R + \frac{s}{j\omega}\right)} \]
resonance system: spring pendulum: solution?

with

\[ Z_m = j\omega m + R + \frac{s}{j\omega} \]

the quantities become:

\[ x(t) = \frac{\hat{F}e^{j\omega t}}{j\omega Z_m} \]

\[ u(t) = \frac{\hat{F}e^{j\omega t}}{Z_m} \]

\[ a(t) = \frac{j\omega \hat{F}e^{j\omega t}}{Z_m} \]
resonance system: spring pendulum:
solution?

amplitude responses:

\[ 20 \log(x, u, a) \]

- 12 dB/ Okt
- 6 dB/ Okt

below resonance above
resonance system: spring pendulum

consequences for microphones:

- majority of mics use a membrane (mechanical spring-mass resonance system)
- condition for flat amplitude response:
  - operation below resonance if electrical output proportional to membrane excursion
  - operation at resonance if electrical output proportional to membrane velocity
acoustical systems
describing quantities

fundamental acoustical quantities:

- sound pressure \( p \)
- sound particle velocity \( v \) or volume flow \( Q = \int_S v \, dS \)

movie: rohr-open5-4
Acoustical elements
Acoustical mass

Acoustical mass:

- accelerated but not compressed air
- realization: tube of length \( l \), diameter \( d \) where \( \lambda \gg l \) and \( \lambda \gg d \)

\[
\Delta F = \rho A l \frac{dv}{dt} \quad \text{Newton}
\]

where 
\( \rho \): density of air
Acoustical compliance:

- compressed but not accelerated air
- realization: cavity $V$ with opening area $A$ (largest dimension $l \ll \lambda$)
Acoustical compliance

- experiment: virtual piston presses on opening $A$ with force $F$
- piston sinks in by $\Delta l = \frac{F}{s}$ ( $s$: stiffness of the compliance )

with the assumption of adiabatic behavior (no temperature exchange):

$$s = c^2 \rho \frac{A^2}{V}$$

where

$c$: speed of sound
Acoustical compliance

- position of the opening is irrelevant

![Diagram showing the relationship between impulse and flow.](attachment:image.png)
Acoustical resistance

Acoustical resistance:

- element introduces loss (conversion of sound energy into heat)
- realization: porous material, small tube

Resistance of a small tube:

\[ \Delta p = \nu \frac{8l \eta}{r^2} \]

where

- \( \Delta p \): sound pressure difference on both sides
- \( \nu \): sound particle velocity
- \( l \): length of the tube
- \( r \): radius of the tube \((r \ll l)\)
- \( \eta \): dynamic viscosity of air: \(1.82 \times 10^{-5}\) Nsm\(^{-2}\)
Acoustical resistance

- Acoustical resistance of a tube is always accompanied by an acoustical mass
- Mass behavior can be neglected for small diameters and low frequencies
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**Analog quantities**

<table>
<thead>
<tr>
<th>potential quantity</th>
<th>flow quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>electrical system:</td>
<td>voltage</td>
</tr>
<tr>
<td>mechanical system:</td>
<td>velocity $u$</td>
</tr>
<tr>
<td>acoustical system:</td>
<td>sound pressure $p$</td>
</tr>
</tbody>
</table>

→ dual analogy would be possible as well
Analog quantities

Amplitude scaling:

- conversion of acoustical and mechanical quantities into electrical ones requires amplitude and unit conversion factors
- arbitrary amplitude conversion factors are allowed
- here: $= 1$
Analog quantities

**mechanical → electrical:**

\[ U = G_1 u \quad \text{with} \quad G_1 = \frac{1}{\text{m}} \text{Vs} \]

\[ I = \frac{1}{G_2} F \quad \text{with} \quad G_2 = \frac{1}{\text{A}} \text{N} \]
Analog quantities

acoustical $\rightarrow$ electrical:

$$U = G_3 p \quad \text{with} \quad G_3 = 1 \frac{V m^2}{N}$$

$$I = \frac{1}{G_4} Q \quad \text{with} \quad G_4 = 1 \frac{m^3}{As}$$
Impedances: general

Conversion of mechanical and acoustical elements into analog electrical ones is performed using impedance definition:

\[
\text{impedance} = \frac{\text{potentialQuantity}}{\text{flowQuantity}}
\]
Impedances: mechanical mass

Impedance $Z$ of the mechanical mass:

$$Z = \frac{u}{F} = \frac{u}{m \frac{du}{dt}}$$

with $u = u_0 e^{j\omega t}$ follows $\frac{du}{dt} = u_0 j\omega e^{j\omega t}$ and finally

$$Z = \frac{1}{j\omega m}$$

- mechanical mass $\triangleq$ electrical capacitor
- inertia of the mass is understood relative to reference system at rest $\rightarrow$ capacitors always at ground potential
Impedances: mechanical spring

impedance $Z$ of the mechanical spring:

$$Z = \frac{u}{F} = \frac{u}{s \int u \, dt}$$

with $u = u_0 e^{j\omega t}$ follows $\int u \, dt = u_0 \frac{1}{j\omega} e^{j\omega t}$ and finally

$$Z = j\omega \frac{1}{s}$$

- mechanical spring $\triangleq$ electrical inductance
Impedances: mechanical friction

impedance $Z$ of a mechanical friction element:

$$Z = \frac{u}{F} = \frac{u}{R_u} = \frac{1}{R}$$

- mechanical friction element $\cong$ electrical resistance
Impedances: acoustical mass

impedance $Z$ of the acoustical mass:

$$Z = \frac{p}{Q} = \frac{\Delta F}{A\nu} = \frac{\rho A l \frac{dv}{dt}}{AA\nu}$$

with $v = v_0 e^{j\omega t}$ follows $\frac{dv}{dt} = v_0 j\omega e^{j\omega t}$ and finally

$$Z = j\omega \frac{\rho l}{A}$$

- acoustical mass $\triangleq$ electrical inductance
Impedance: acoustical compliance

impedance $Z$ of the acoustical compliance:

$$Z = \frac{p}{Q} = \frac{\Delta F}{A} = \frac{c^2 \rho \frac{A^2}{V} \Delta l}{AAv} = \frac{c^2 \rho \frac{A^2}{V} \int v dt}{AAv}$$

with $v = v_0 e^{j\omega t}$ follows $\int v dt = v_0 \frac{1}{j\omega} e^{j\omega t}$ and finally

$$Z = \frac{c^2 \rho}{j\omega V}$$

▶ acoustical compliance $\triangleq$ electrical capacitor
Impedance: acoustical resistance

impedance $Z$ of the acoustical resistance:

$$Z = \frac{p}{Q} = \frac{8l\eta}{\pi r^4}$$

where

$l$: length of the tube
$r$: radius of the tube ($r \ll l$)

$\eta$: dynamic viscosity in air $= 1.82 \times 10^{-5} \text{ Nsm}^{-2}$

$\triangleright$ acoustical resistance $\triangleq$ electrical resistance
## Impedances: overview

<table>
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<tr>
<th></th>
<th>Electrical</th>
<th>Mechanical</th>
<th>Acoustical</th>
</tr>
</thead>
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<tr>
<td>$Z = R$</td>
<td>resistance</td>
<td>resistance</td>
<td>resistance</td>
</tr>
<tr>
<td>$Z = \frac{1}{j\omega C}$</td>
<td>capacitor</td>
<td>mass</td>
<td>compliance</td>
</tr>
<tr>
<td>$Z = j\omega L$</td>
<td>inductance</td>
<td>spring</td>
<td>mass</td>
</tr>
</tbody>
</table>

Impedances:
- Measurement of acoustical impedances
- Electrical mechanical acoustical

### Equations

- $Z = R$
- $Z = \frac{1}{j\omega C}$
- $Z = j\omega L$
### Impedances: overview

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<tr>
<td>$Z = R$</td>
<td>$R$</td>
<td>$R = \frac{1}{R_m} G_1 G_2$</td>
</tr>
<tr>
<td>$Z = \frac{1}{j\omega C}$</td>
<td>$C$</td>
<td>$C = m \frac{1}{G_1 G_2}$</td>
</tr>
<tr>
<td>$Z = j\omega L$</td>
<td>$L$</td>
<td>$L = \frac{1}{s} G_1 G_2$</td>
</tr>
</tbody>
</table>
Examples of acoustical and mechanical networks
Example: acoustical resonance circuit

combination of acoustical mass and compliance:

\[ P = P_0 + \Delta P \]
Example: acoustical resonance circuit

combination of acoustical mass and compliance:

\[ P = Po + p \]

\[ Z_{ein} \]

\[ P = Po + p \]

\[ Po + p \]

\[ Po \]

\[ C \]

\[ L \]

→ parallel resonance circuit
Example: acoustical resonance circuit

combination of acoustical mass and compliance:

\[ P = P_0 + p \]

\[ Z_{\text{ein}} \]
Example: acoustical resonance circuit

combination of acoustical mass and compliance:

$P = Po + p$

$P = Po$

$Z_{ein}$

$Po+p$

$L$

$C$

→ series resonance circuit
Example: acoustical muffler

- no damping for low frequency air flow
- high damping for high frequency sound
- → acoustical low pass filter
Example: acoustical transmission line

description of tube-like structures (cross sect. area = $A$) but length > $\lambda$:

- subdivision of the tube into small sections of length $l$ ($l \ll \lambda$)
- representation of each section by mass and compliance properties
- equivalent network: $L$, $C$, $L$ T-section with
  - $L = \frac{\rho l}{2A}$
  - $C = \frac{Al}{\rho c^2}$
Example: acoustical transmission line

numerical example:

- hard terminated tube of length 24 cm
- cross sectional area \( A = 10^{-4} \) m\(^2\)
- discretization: \( l = 0.04 \) m
Example: acoustical transmission line

Spice simulation of the voltage at the hard termination:

Theoretically expected resonance frequencies:

\[
\frac{1\lambda}{4} = 0.24\text{m} \rightarrow 354 \text{ Hz}, \quad \frac{3\lambda}{4} = 0.24\text{m} \rightarrow 1063 \text{ Hz}
\]

\[
\frac{5\lambda}{4} = 0.24\text{m} \rightarrow 1771 \text{ Hz}, \quad \frac{7\lambda}{4} = 0.24\text{m} \rightarrow 2479 \text{ Hz}
\]
Example: mechanical resonance circuit

combinations of mechanical mass and spring:

[Diagram of mechanical resonance circuit]
Example: mechanical resonance circuit

combinations of mechanical mass and spring:

![Mechanical resonance circuit diagram]
Example: mechanical resonance circuit

- sinusoidal force excitation
- analysis of the mechanical spring pendulum with help of an equivalent electrical circuit.
Example: mechanical spring pendulum

\[ x = \frac{F_F}{s} \]
\[ u = \frac{F_R}{R} \]
\[ a = \frac{F_M}{m} \]
Example: mechanical vibration damper

- strategy in case of sinusoidal vibration → implementation of an additional resonance system
Example: mechanical vibration damper

- additional series resonance circuit
- at resonance: impedance $\rightarrow 0$
- short-circuit for the excitation force
Measurement of acoustical impedances
Measurement of acoustical impedances

- investigation of unknown acoustical impedances
  - e.g. quality control of wind instruments
  - e.g. audiometric tests → input impedance of the middle ear

- measurement methods
  - combination of sound pressure and sound particle sensor, e.g. $\mu$Flown
  - two pressure sensors and known acoustical resistance
Measurement of acoustical impedances

method with two pressure sensors and known acoustical resistance

evaluation strategies:

▶ loudspeaker signal controlled for constant difference
\[ p_1 - p_2 \rightarrow p_2 \sim Z \]

▶ calculation from measured spectra
distributed acoustical elements
distributed acoustical elements

- simulation of acoustical behavior of long tubes:
  - series of short sections, each represented by lumped elements
  - or introduction of a distributed acoustical element → four pole
distributed acoustical elements

▷ assumption: 2 plane waves traveling in opposite directions

\[
\tilde{p}(x) = Ae^{-jkx} + Be^{jkx}
\]
\[
\tilde{v}(x) = \frac{A}{\rho c}e^{-jkx} - \frac{B}{\rho c}e^{jkx}
\]
distributing acoustical elements

definition:

\[ \ddot{p}(x = 0) \equiv \ddot{p}_1 \text{ and } \ddot{p}(x = d) \equiv \ddot{p}_2 \]

\[ \ddot{v}(x = 0) \equiv \ddot{v}_1 \text{ and } \ddot{v}(x = d) \equiv \ddot{v}_2 \]
distributed acoustical elements

insert above definition:

\[ \ddot{p}(x = 0) = \ddot{p}_1 \text{ and } \ddot{p}(x = d) = \ddot{p}_2 \]
\[ \ddot{v}(x = 0) = \ddot{v}_1 \text{ and } \ddot{v}(x = d) = \ddot{v}_2 \]

in:

\[
\ddot{p}(x) = Ae^{-jkx} + Be^{jkx}
\]

\[
\ddot{v}(x) = \frac{A}{\rho c}e^{-jkx} - \frac{B}{\rho c}e^{jkx}
\]

and resolve for \( \ddot{p}_2 \) and \( \ddot{v}_2 \)
distributed acoustical elements

yields the four pole equations:

\[
\ddot{p}_2 = \frac{\ddot{p}_1}{2} e^{-jkd} + \frac{\ddot{p}_1}{2} e^{jkd} + \frac{\ddot{v}_1 \rho c}{2} e^{-jkd} - \frac{\ddot{v}_1 \rho c}{2} e^{jkd}
\]

\[
= \ddot{p}_1 \cosh(jkd) - \ddot{v}_1 \rho c \sinh(jkd)
\]

\[
\ddot{v}_2 = \frac{1}{\rho c} \frac{\ddot{p}_1}{2} e^{-jkd} - \frac{1}{\rho c} \frac{\ddot{p}_1}{2} e^{jkd} + \frac{\ddot{v}_1}{2} e^{-jkd} + \frac{\ddot{v}_1}{2} e^{jkd}
\]

\[
= -\ddot{p}_1 \frac{1}{\rho c} \sinh(jkd) + \ddot{v}_1 \cosh(jkd)
\]
distributed acoustical elements

representation of the above four-pole relations by T-type circuit:

with:

\[
Z_1 = j \rho c \frac{1 - \cos(kd)}{\sin(kd)}
\]

\[
Z_2 = \frac{-j \rho c}{\sin(kd)}
\]
distributed acoustical elements

impedance at the entrance of tube of length \( d \) with hard termination:

\[
\frac{\ddot{p}_1}{\ddot{v}_1} = Z_1 + Z_2
\]

\[
\frac{\ddot{p}_1}{\ddot{v}_1} = \frac{j \rho c - j \rho c \cos(kd) - j \rho c}{\sin(kd)} = -j \rho c \cot(kd)
\]
coupling of mechanical, acoustical and electrical systems
interfaces
"linking" of different subsystems:

- transformation of different physical quantities by help of interfaces:
  - type 1: conversion of potential quantity into potential quantity and flow quantity into flow quantity
  - type 2: conversion of potential quantity into flow quantity and vice versa
Interfaces: transformer

interface type 1: transformer

relations for the transformer:

\[ U_2 = nU_1 \]  \hspace{1cm} (1)

\[ I_2 = \frac{1}{n}I_1 \]  \hspace{1cm} (2)
interface: gytor

interface type 2: gytor

\[ U_2 = I_1 m \]  \hspace{1cm} (3)

\[ I_2 = U_1 \frac{1}{m} \]  \hspace{1cm} (4)
Dual conversion
Dual conversion

motivation: elimination of gyrators

- selection of an arbitrary conversion factor $r$
- dual conversion of a suitable region of the network (cut in half all gyrators)
- gyrators $\rightarrow$ transformers
- series arrangement $\rightarrow$ parallel arrangement
- parallel arrangement $\rightarrow$ series arrangement

$\begin{align*}
\text{L} &\rightarrow C, \quad C \rightarrow \text{L}, \quad R \rightarrow 1/R
\end{align*}$
Dual conversion

- Analogies
- Mechanical systems
  - Describing quantities
  - Mechanical elements
  - Mechanical sources
  - Mechanical resonance - spring pendulum
- Acoustical systems
  - Describing quantities
  - Acoustical elements
- Analogies
  - Potential and flow quantities
  - Impedances
  - Measurement of acoustical impedances
  - Distributed elements
- Coupling
  - Interfaces
    - Dual conversion
- Thin absorbers
- Measurement of the impedance at the ear
  - Model of the middle ear
  - Diagnosis
  - Experiments
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Dual conversion

Analogies

Mechanical systems
- describing quantities
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Acoustical systems
- describing quantities
- Acoustical elements

Analogies
- potential and flow quantities
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Coupling
- interfaces
  - Dual conversion

Thin absorbers

Measurement of the impedance at the ear
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application:
thin absorber
in front of a hard wall
Thin absorber in front of a hard wall

discussion of the behavior of a thin porous layer in front of a hard wall by an equivalent network for perpendicular incidence

→ equivalent electrical network?
Thin absorber in front of a hard wall

\[ Z_0 \]
\[ Z_s \]
\[ R_s \]
\[ m \]

\[ i_L \]
\[ i_R \]
\[ u \]

\[ m r^2 \]

\[ \cot(k \times 10\text{cm}) \]

Diagram showing the electrical analogy of a mechanical system involving impedances and coupling effects.
Thin absorber in front of a hard wall

- absorption: dissipated power according to $i_R \cdot u$
- max. absorption for
  - $Z_S \to 0$
  - $R_S = Z_0$
  - $m \to \infty$ and $f \to \infty$
application: measurement of the impedance at the ear

ETH-Diss. Alfred Stirnemann
Impedance measurement for the diagnosis of middle ear diseases

idea:

- estimation of middle-ear transfer characteristics from measured impedance at the entrance of the ear canal
- detection of possible diseases in the middle-ear
- e.g. Otosclerosis
Measurement of the impedance at the ear - principal structure
Model of the middle ear - equivalent network
Model of the middle ear - equivalent network

complete equivalent network

[Diagram of an electrical network representing the middle ear]
Model of the middle ear - equivalent network

dual conversion
Model of the middle ear - equivalent network

Network after dual conversion and neglect of irrelevant elements

$v_{TR}$: velocity of ear drum
$v_{ST}$: velocity of inner ear window
Diagnosis

- measurement of the input impedance
- estimation of the element values
- comparison with limits → indication for possible disease
Experiments

experiments - model fit:
Experiments

experiments - comparison of impedances:
Experiments

experiments - velocity of inner ear window:

![Graph showing the velocity transfer function of the inner ear window.](image)
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