Problem 1  
**Separation between Signals**

Given $u_1, u_2 \in \mathcal{L}_2$, let $\mathcal{V}$ be the set of all complex signals $v$ that are equidistant to $u_1$ and $u_2$:

$$\mathcal{V} = \{v \in \mathcal{L}_2 : \|v - u_1\|_2 = \|v - u_2\|_2\}.$$

(i) Show that

$$\mathcal{V} = \{v \in \mathcal{L}_2 : \text{Re}(\langle v, u_2 - u_1 \rangle) = \frac{\|u_2\|^2 - \|u_1\|^2}{2}\}.$$

(ii) Is $\mathcal{V}$ a linear subspace of $\mathcal{L}_2$?

(iii) Show that $(u_1 + u_2)/2 \in \mathcal{V}$.

Problem 2  
**Orthogonal Subspace**

Given signals $v_1, \ldots, v_n \in \mathcal{L}_2$, define the set

$$\mathcal{U} = \{u \in \mathcal{L}_2 : \langle u, v_1 \rangle = \langle u, v_2 \rangle = \cdots = \langle u, v_n \rangle = 0\}.$$

Show that $\mathcal{U}$ is a linear subspace of $\mathcal{L}_2$.

Problem 3  
**Constructing an Orthonormal Basis**

Let $T_s$ be a positive constant. Consider the signals $s_1: t \mapsto I\{0 \leq t \leq T_s/2\} - I\{T_s/2 < t \leq T_s\}$; $s_2: t \mapsto I\{0 \leq t \leq T_s\}$; $s_3: t \mapsto I\{0 \leq t \leq T_s/4\} + I\{3T_s/4 \leq t \leq T_s\}$; and $s_4: t \mapsto I\{0 \leq t \leq T_s/4\} - I\{3T_s/4 \leq t \leq T_s\}$.

(i) Plot $s_1$, $s_2$, $s_3$, and $s_4$.

(ii) Find an orthonormal basis for $\text{span}(s_1, s_2, s_3, s_4)$.

(iii) Express each of the signals $s_1$, $s_2$, $s_3$, and $s_4$ as a linear combination of the basis vectors found in Part (ii).
Problem 4  

**Expansion of a Function**

Expand the function $t \mapsto \text{sinc}^2(t/2)$ as an orthonormal expansion in the functions

$$
\ldots, t \mapsto \text{sinc}(t + 2), \ t \mapsto \text{sinc}(t + 1), \ t \mapsto \text{sinc}(t), \ t \mapsto \text{sinc}(t - 1), \ t \mapsto \text{sinc}(t - 2), \ldots
$$

Problem 5  

**Inner Product with a Bandlimited Signal**

Show that if $x$ is an energy-limited signal that is bandlimited to $W$ Hz, and if $y \in \mathcal{L}_2$, then

$$(x, y) = T_s \sum_{\ell = -\infty}^{\infty} x(\ell T_s) y_{\text{LPF}}^*(\ell T_s),$$

where $y_{\text{LPF}}$ is the result of passing $y$ through an ideal unit-gain lowpass filter of bandwidth $W$ Hz, and where $T_s = 1/(2W)$.

Problem 6  

**Inner Product between Passband Signals**

Let $x_{\text{PB}}$ and $y_{\text{PB}}$ be energy-limited passband signals that are bandlimited to $W$ Hz around the carrier frequency $f_c$. Let $x_{\text{BB}}$ and $y_{\text{BB}}$ be their corresponding baseband representations. Let $T = 1/W$. Show that

$$
\langle x_{\text{PB}}, y_{\text{PB}} \rangle = 2T \Re \left( \sum_{\ell = -\infty}^{\infty} x_{\text{BB}}(\ell T) y_{\text{BB}}^*(\ell T) \right).
$$