Communication and Detection Theory

Signal and Information Processing Laboratory
Institut für Signal- und Informationsverarbeitung

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Prof. Dr. A. Lapidoth

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http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *Separation between Signals*

Given $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{L}_2$, let $\mathcal{V}$ be the set of all complex signals $\mathbf{v}$ that are equidistant to $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\mathcal{V} = \left\{ \mathbf{v} \in \mathcal{L}_2 : \|\mathbf{v} - \mathbf{u}_1\|_2 = \|\mathbf{v} - \mathbf{u}_2\|_2 \right\}.$$

(i) Show that

$$\mathcal{V} = \left\{ \mathbf{v} \in \mathcal{L}_2 : \mathrm{Re}\Big(\langle \mathbf{v}, \mathbf{u}_2 - \mathbf{u}_1 \rangle \Big) = \frac{\|\mathbf{u}_2\|_2^2 - \|\mathbf{u}_1\|_2^2}{2} \right\}.$$

(ii) Is $\mathcal{V}$ a linear subspace of $\mathcal{L}_2$?

(iii) Show that $(\mathbf{u}_1 + \mathbf{u}_2)/2 \in \mathcal{V}$.

Problem 2 — *Orthogonal Subspace*

Given signals $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{L}_2$, define the set

$$\mathcal{U} = \left\{ \mathbf{u} \in \mathcal{L}_2 : \langle \mathbf{u}, \mathbf{v}_1 \rangle = \langle \mathbf{u}, \mathbf{v}_2 \rangle = \cdots = \langle \mathbf{u}, \mathbf{v}_n \rangle = 0 \right\}.$$

Show that $\mathcal{U}$ is a linear subspace of $\mathcal{L}_2$.

Problem 3 — *Constructing an Orthonormal Basis*

Let $\mathsf{T_s}$ be a positive constant. Consider the signals $\mathbf{s}_1 \colon t \mapsto \mathrm{I}\{0 \le t \le \mathsf{T_s}/2\} - \mathrm{I}\{\mathsf{T_s}/2 < t \le \mathsf{T_s}\}$; $\mathbf{s}_2 \colon t \mapsto \mathrm{I}\{0 \le t \le \mathsf{T_s}\}$; $\mathbf{s}_3 \colon t \mapsto \mathrm{I}\{0 \le t \le \mathsf{T_s}/4\} + \mathrm{I}\{3\mathsf{T_s}/4 \le t \le \mathsf{T_s}\}$; and $\mathbf{s}_4 \colon t \mapsto \mathrm{I}\{0 \le t \le \mathsf{T_s}/4\} - \mathrm{I}\{3\mathsf{T_s}/4 \le t \le \mathsf{T_s}\}$.

(i) Plot $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, and $\mathbf{s}_4$.

(ii) Find an orthonormal basis for $\mathrm{span}\,(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4)$.

(iii) Express each of the signals $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_3$, and $\mathbf{s}_4$ as a linear combination of the basis vectors found in Part (ii).

Problem 4 *Expansion of a Function*

Expand the function $t \mapsto \operatorname{sinc}^2(t/2)$ as an orthonormal expansion in the functions

$$\ldots, t \mapsto \operatorname{sinc}(t+2), t \mapsto \operatorname{sinc}(t+1), t \mapsto \operatorname{sinc}(t), t \mapsto \operatorname{sinc}(t-1), t \mapsto \operatorname{sinc}(t-2), \ldots$$

Problem 5 *Inner Product with a Bandlimited Signal*

Show that if $\mathbf{x}$ is an energy-limited signal that is bandlimited to W Hz, and if $\mathbf{y} \in \mathcal{L}_2$, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathsf{T}_{\mathrm{s}} \sum_{\ell=-\infty}^{\infty} x(\ell \mathsf{T}_{\mathrm{s}})\, y_{\mathrm{LPF}}^{*}(\ell \mathsf{T}_{\mathrm{s}}),$$

where $\mathbf{y}_{\mathrm{LPF}}$ is the result of passing $\mathbf{y}$ through an ideal unit-gain lowpass filter of bandwidth W Hz, and where $\mathsf{T}_{\mathrm{s}} = 1/(2W)$.

Problem 6 *Inner Product between Passband Signals*

Let $\mathbf{x}_{\mathrm{PB}}$ and $\mathbf{y}_{\mathrm{PB}}$ be energy-limited passband signals that are bandlimited to W Hz around the carrier frequency f_{c}. Let $\mathbf{x}_{\mathrm{BB}}$ and $\mathbf{y}_{\mathrm{BB}}$ be their corresponding baseband representations. Let $\mathsf{T} = 1/W$. Show that

$$\langle \mathbf{x}_{\mathrm{PB}}, \mathbf{y}_{\mathrm{PB}} \rangle = 2\mathsf{T} \operatorname{Re}\left(\sum_{\ell=-\infty}^{\infty} x_{\mathrm{BB}}(\ell \mathsf{T})\, y_{\mathrm{BB}}^{*}(\ell \mathsf{T}) \right).$$