Exercise 9 of April 25, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

The questions about the operational PSD are optional.

Problem 1

π/4-QPSK

In QPSK or 4-QAM the data bits are mapped to complex symbols \( (C_\ell) \) which take value in the set \( \{\pm 1 \pm i\} \) and which are then transmitted using the signal \( (X(t)) \) defined in (18.24). Consider now π/4-QPSK where, prior to transmission, the complex symbols \( (C_\ell) \) are rotated to form the complex symbols

\[
\tilde{C}_\ell = \alpha^\ell C_\ell, \quad \ell \in \mathbb{Z},
\]

where \( \alpha = e^{i\pi/4} \). The transmitted signal is then

\[
2A \text{Re} \left( \sum_{\ell=-\infty}^{\infty} \tilde{C}_\ell g(t - \ell T_s) e^{2\pi f_c t} \right), \quad t \in \mathbb{R}.
\]

Compute the power and the operational PSD of the π/4-QPSK signal when \( (C_\ell) \) is a zero-mean WSS CSP of autocovariance function \( K_{CC} \). Compare the power and operational PSD of π/4-QPSK with those of QPSK. How do they compare when the symbols \( (C_\ell) \) are IID?

*Hint: See Exercise 17.22.*

Problem 2

The Power in the In-Phase and Quadrature Components

Consider the setup of Theorem 18.3.1 with the additional assumptions that the real part (and hence, by Exercise 17.17, also the imaginary part) of \( (C_\ell, \ell \in \mathbb{Z}) \) is WSS and that the pulse shape \( g \) is real. Compute the power in each of the signals

\[
t \mapsto 2A \sum_{\ell=-\infty}^{\infty} \text{Re}(C_\ell) g(t - \ell T_s) \cos(2\pi f_c t),
\]
\[
t \mapsto -2A \sum_{\ell=-\infty}^{\infty} \text{Im}(C_\ell) g(t - \ell T_s) \sin(2\pi f_c t),
\]

and show that these powers add up to the power in \( X(\cdot) \). Give an intuitive explanation for this result. Do you expect a similar result for the operational PSD?

*Hint: To compute their power, express the signals as QAM signals and use Theorem 18.3.1.*
Problem 3  
**Sums of Independent Gaussians**

Let $X_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(0, \sigma_2^2)$ be independent. Convolve their densities to show that $X_1 + X_2$ is Gaussian.

Problem 4  
**Computing Probabilities**

Let $X \sim \mathcal{N}(1, 3)$ and $Y \sim \mathcal{N}(-2, 4)$ be independent. Express the probabilities $\Pr[X \leq 2]$ and $\Pr[2X + 3Y > -2]$ using the $Q$-function with nonnegative arguments.

Problem 5  
**Bounds on the $Q$-Function**

Prove (19.19). We suggest changing the integration variable in (19.9) to $\zeta \triangleq \xi - \alpha$ and then proving (19.19) using the inequality

$$1 - \frac{\zeta^2}{2} \leq \exp \left( -\frac{\zeta^2}{2} \right) \leq 1, \quad \xi \in \mathbb{R}.$$