Communication and Detection Theory

Signal and Information Processing Laboratory
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http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *Hypothesis Testing*

Let H take on the values 0 and 1 equiprobably. Conditional on $H = 0$, the observable Y is equal to $a + Z$, where Z is independent of H and has the Laplace distribution

$$f_Z(z) = \frac{1}{2}\, e^{-|z|}, \ \ z \in \mathbb{R},$$

and $a > 0$ is a given constant. Conditional on $H = 1$, the observable Y is given by $-a + Z$.

(i) Find and draw the densities $f_{Y|H=0}(\cdot)$ and $f_{Y|H=1}(\cdot)$.

(ii) Find an optimal rule for guessing H based on Y.

(iii) Compute the optimal probability of error.

(iv) Compute the Bhattacharyya Bound.

Problem 2 — *Binary Hypothesis Testing*

Let H take on the values 0 and 1 according to the prior (π_0, π_1), and let the observation consist of the RV Y. Conditional on H, the densities of Y are given for every $y \in \mathbb{R}$ by

$$f_{Y|H=0}(y) = e^{-y}\, \mathrm{I}\{y \geq 0\}, \qquad f_{Y|H=1}(y) = \beta\, e^{-\frac{y^2}{2}}\, \mathrm{I}\{y \geq 0\},$$

where $\beta > 0$ is some constant.

(i) Determine β.

(ii) Find a decision rule that minimizes the probability of error.

(iii) For the rule that you have found, compute $\Pr(\text{error}|H = 0)$.

Hint: Different priors can lead to dramatically different decision rules.

Problem 3 — *Hypothesis Testing with a Random Parameter*

Let $Y = X + AZ$, where X, A, and Z are independent random variables with X taking on the values ± 1 equiprobably, A taking on the values 2 and 3 equiprobably, and $Z \sim \mathcal{N}\big(0, \sigma^2\big)$.

(i) Find an optimal rule for guessing X based on the pair (Y, A).

(ii) Repeat when you observe only Y.

Problem 4 ***Artifacts of Suboptimality***

Let H take on the values 0 and 1 equiprobably. Conditional on $H = 0$, the observation Y is $\mathcal{N}(1, \sigma^2)$, and, conditional on $H = 1$, it is $\mathcal{N}(-1, \sigma^2)$. Alice guesses "$H = 0$" if $Y > 2$ and guesses "$H = 1$" otherwise.

(i) Compute the probability that Alice errs as a function of σ^2.

(ii) Show that this probability is not monotonically nondecreasing in σ^2.

(iii) Does her guessing rule minimize the probability of error?

(iv) Show that if you are obliged to use her rule, then adding noise to Y prior to feeding it to her detector may be beneficial.