

Communication and Detection Theory

Signal and Information
Processing Laboratory

Institut für Signal- und
Informationsverarbeitung



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Prof. Dr. A. Lapidoth

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<http://www.isi.ee.ethz.ch/teaching/courses/cdt>

Problem 1

Hypothesis Testing

Let H take on the values 0 and 1 equiprobably. Conditional on $H = 0$, the observable Y is equal to $a + Z$, where Z is independent of H and has the Laplace distribution

$$f_Z(z) = \frac{1}{2} e^{-|z|}, \quad z \in \mathbb{R},$$

and $a > 0$ is a given constant. Conditional on $H = 1$, the observable Y is given by $-a + Z$.

- (i) Find and draw the densities $f_{Y|H=0}(\cdot)$ and $f_{Y|H=1}(\cdot)$.
- (ii) Find an optimal rule for guessing H based on Y .
- (iii) Compute the optimal probability of error.
- (iv) Compute the Bhattacharyya Bound.

Problem 2

Binary Hypothesis Testing

Let H take on the values 0 and 1 according to the prior (π_0, π_1) , and let the observation consist of the RV Y . Conditional on H , the densities of Y are given for every $y \in \mathbb{R}$ by

$$f_{Y|H=0}(y) = e^{-y} \mathbf{I}\{y \geq 0\}, \quad f_{Y|H=1}(y) = \beta e^{-\frac{y^2}{2}} \mathbf{I}\{y \geq 0\},$$

where $\beta > 0$ is some constant.

- (i) Determine β .
- (ii) Find a decision rule that minimizes the probability of error.
- (iii) For the rule that you have found, compute $\Pr(\text{error}|H = 0)$.

Hint: Different priors can lead to dramatically different decision rules.

Problem 3

Hypothesis Testing with a Random Parameter

Let $Y = X + AZ$, where X , A , and Z are independent random variables with X taking on the values ± 1 equiprobably, A taking on the values 2 and 3 equiprobably, and $Z \sim \mathcal{N}(0, \sigma^2)$.

- (i) Find an optimal rule for guessing X based on the pair (Y, A) .
- (ii) Repeat when you observe only Y .

Problem 4

Artifacts of Suboptimality

Let H take on the values 0 and 1 equiprobably. Conditional on $H = 0$, the observation Y is $\mathcal{N}(1, \sigma^2)$, and, conditional on $H = 1$, it is $\mathcal{N}(-1, \sigma^2)$. Alice guesses “ $H = 0$ ” if $Y > 2$ and guesses “ $H = 1$ ” otherwise.

- (i) Compute the probability that Alice errs as a function of σ^2 .
- (ii) Show that this probability is not monotonically nondecreasing in σ^2 .
- (iii) Does her guessing rule minimize the probability of error?
- (iv) Show that if you are obliged to use her rule, then adding noise to Y prior to feeding it to her detector may be beneficial.