Problem 1  

A Multi-Antenna Receiver

Let $H$ take on the values 0 and 1 equiprobably. We wish to guess $H$ based on the random variables $Y_1$ and $Y_2$. Conditional on $H = 0$,

$$Y_1 = A + Z_1, \quad Y_2 = A + Z_2,$$

and conditional on $H = 1$,

$$Y_1 = -A + Z_1, \quad Y_2 = -A + Z_2.$$

Here $A$ is a positive constant, and $Z_1 \sim \mathcal{N}(0, \sigma_1^2)$, $Z_2 \sim \mathcal{N}(0, \sigma_2^2)$, and $H$ are independent.

(i) Find an optimal rule for guessing $H$ based on $(Y_1, Y_2)$.

(ii) Draw the decision regions in the $(Y_1, Y_2)$-plane for the special case where $\sigma_1 = 2\sigma_2$.

(iii) Returning to the general case, find a one-dimensional sufficient statistic.

(iv) Find the optimal probability of error in terms of $\sigma_1^2$, $\sigma_2^2$, and $A$.

(v) Consider a suboptimal receiver that declares “$H = 0$” if $Y_1 + Y_2 > 0$, and otherwise declares “$H = 1$.” Evaluate the probability of error for this decoder as a function of $\sigma_1^2$, $\sigma_2^2$, and $A$.

Problem 2  

Ternary Gaussian Detection

Consider the following special case of the problem discussed in Section 21.6. Here $M$ is uniformly distributed over the set $\{1, 2, 3\}$, and the mean vectors $s_1, s_2, s_3$ are given by

$$(a_1, b_1) = (0, A), \quad (a_2, b_2) = (-A, 0), \quad (a_3, b_3) = (0, -A), \quad (a_4, b_4) = (A, 0).$$

Problem 3  

4-PSK Detection

Consider the setup of Section 21.4 with $M = 4$ and

$$(a_1, b_1) = (0, A), \quad (a_2, b_2) = (-A, 0), \quad (a_3, b_3) = (0, -A), \quad (a_4, b_4) = (A, 0).$$
(i) Sketch the decision regions of the MAP decision rule.

(ii) Using the $Q$-function, express the conditional probabilities of error of this rule conditional on each hypothesis.

(iii) Compute an upper bound on $p_{\text{MAP}}(\text{error}|M = 1)$ using Proposition 21.5.3. Indicate on the figure which events are summed two or three times. Can you improve the bound by summing only over a subset of the alternative hypotheses?

*Hint: In Part (ii) first find the probability of correct detection.*

**Problem 4**

*A 7-ary QAM problem*

Consider the problem addressed in Section 21.4 in the special case where $M = 7$ and

$$a_m = A \cos\left(\frac{2\pi m}{6}\right), \quad b_m = A \sin\left(\frac{2\pi m}{6}\right), \quad m = 1, \ldots, 6,$$

$$a_7 = 0, \quad b_7 = 0.$$

(i) Illustrate the decision regions of the MAP (nearest-neighbor) guessing rule.

(ii) Let $Z = (Z^{(1)}, Z^{(2)})^T$ be a random vector whose components are IID $\mathcal{N}(0, \sigma^2)$. Show that for every message $m \in \{1, \ldots, 7\}$ the conditional probability of error $p_{\text{MAP}}(\text{error}|M = m)$ can be upper-bounded by the probability that the Euclidean norm of $Z$ exceeds $A/2$. Calculate this probability.

(iii) What is the upper bound on $p_{\text{MAP}}(\text{error}|M = m)$ that Proposition 21.5.3 yields in this case? Can you improve it by including fewer terms?

(iv) Compare the different bounds.