Communication and Detection Theory

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http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *A Multi-Antenna Receiver*

Let H take on the values 0 and 1 equiprobably. We wish to guess H based on the random variables Y_1 and Y_2. Conditional on $H = 0$,

$$Y_1 = \mathsf{A} + Z_1, \qquad Y_2 = \mathsf{A} + Z_2,$$

and conditional on $H = 1$,

$$Y_1 = -\mathsf{A} + Z_1, \qquad Y_2 = -\mathsf{A} + Z_2.$$

Here A is a positive constant, and $Z_1 \sim \mathcal{N}(0, \sigma_1^2)$, $Z_2 \sim \mathcal{N}(0, \sigma_2^2)$, and H are independent.

(i) Find an optimal rule for guessing H based on (Y_1, Y_2).

(ii) Draw the decision regions in the (Y_1, Y_2)-plane for the special case where $\sigma_1 = 2\sigma_2$.

(iii) Returning to the general case, find a one-dimensional sufficient statistic.

(iv) Find the optimal probability of error in terms of σ_1^2, σ_2^2, and A.

(v) Consider a suboptimal receiver that declares "$H = 0$" if $Y_1 + Y_2 > 0$, and otherwise declares "$H = 1$." Evaluate the probability of error for this decoder as a function of σ_1^2, σ_2^2, and A.

Problem 2 — *Ternary Gaussian Detection*

Consider the following special case of the problem discussed in Section 21.6. Here M is uniformly distributed over the set $\{1, 2, 3\}$, and the mean vectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ are given by

$$\mathbf{s}_1 = \mathbf{0}, \quad \mathbf{s}_2 = \mathbf{s}, \quad \mathbf{s}_3 = -\mathbf{s},$$

where $\mathbf{s}$ is some deterministic nonzero vector in $\mathbb{R}^{\mathsf{J}}$. Find the conditional probability of error of the MAP rule conditional on each hypothesis.

Problem 3 — *4-PSK Detection*

Consider the setup of Section 21.4 with $\mathsf{M} = 4$ and

$$(a_1, b_1) = (0, \mathsf{A}), \;\; (a_2, b_2) = (-\mathsf{A}, 0), \;\; (a_3, b_3) = (0, -\mathsf{A}), \;\; (a_4, b_4) = (\mathsf{A}, 0).$$

(i) Sketch the decision regions of the MAP decision rule.

(ii) Using the $\mathcal{Q}$-function, express the conditional probabilities of error of this rule conditional on each hypothesis.

(iii) Compute an upper bound on $p_{\text{MAP}}(\text{error}|M = 1)$ using Proposition 21.5.3. Indicate on the figure which events are summed two or three times. Can you improve the bound by summing only over a subset of the alternative hypotheses?

Hint: In Part (ii) first find the probability of correct detection.

Problem 4 — *A 7-ary QAM problem*

Consider the problem addressed in Section 21.4 in the special case where $\mathsf{M} = 7$ and

$$a_m = \mathsf{A}\cos\Bigl(\frac{2\pi m}{6}\Bigr), \quad b_m = \mathsf{A}\sin\Bigl(\frac{2\pi m}{6}\Bigr), \quad m = 1, \ldots, 6,$$

$$a_7 = 0, \quad b_7 = 0.$$

(i) Illustrate the decision regions of the MAP (nearest-neighbor) guessing rule.

(ii) Let $\mathbf{Z} = (Z^{(1)}, Z^{(2)})^{\mathsf{T}}$ be a random vector whose components are IID $\mathcal{N}(0, \sigma^2)$. Show that for every message $m \in \{1, \ldots, 7\}$ the conditional probability of error $p_{\text{MAP}}(\text{error}|M = m)$ can be upper-bounded by the probability that the Euclidean norm of $\mathbf{Z}$ exceeds $\mathsf{A}/2$. Calculate this probability.

(iii) What is the upper bound on $p_{\text{MAP}}(\text{error}|M = m)$ that Proposition 21.5.3 yields in this case? Can you improve it by including fewer terms?

(iv) Compare the different bounds.