Communication and Detection Theory

Signal and Information Processing Laboratory
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Spring Semester 2017

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Exercise 12 of May 16, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 ***Hypothesis Testing with Two Observations***

Let H take on the values 0 and 1 equiprobably. Let $\mathbf{Y}_1$ be a random vector taking values in $\mathbb{R}^2$, and let Y_2 be a random variable. Conditional on $H = 0$,

$$\mathbf{Y}_1 = \boldsymbol{\mu} + \mathbf{Z}_1, \quad Y_2 = \alpha + Z_2,$$

and, conditional on $H = 1$,

$$\mathbf{Y}_1 = -\boldsymbol{\mu} + \mathbf{Z}_1, \quad Y_2 = -\alpha + Z_2.$$

Here H, $\mathbf{Z}_1$, and Z_2 are independent with the components of $\mathbf{Z}_1$ being IID $\mathcal{N}(0, 1)$, with Z_2 having the mean-one exponential distribution, and with $\boldsymbol{\mu} \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ being deterministic.

(i) Find an optimal rule for guessing H based on $\mathbf{Y}_1$. Find a one-dimensional sufficient statistic.

(ii) Find an optimal rule for guessing H based on Y_2.

(iii) Find a two-dimensional sufficient statistic (T_1, T_2) for guessing H based on $(\mathbf{Y}_1, Y_2)$.

(iv) Find an optimal rule for guessing H based on the pair (T_1, T_2).

Problem 2 ***Optimality Does Not Imply Sufficiency***

Let H take value in the set $\{0, 1\}$, and let $d = 2$. Suppose that

$$Y_j = (1 - 2H) + \Theta Z_j, \quad j = 1, \ldots, d,$$

where $H, \Theta, Z_1, \ldots, Z_d$ are independent with Θ taking on the distinct positive values σ_0 and σ_1 with probability ρ_0 and ρ_1 respectively, and with $Z_1, \ldots, Z_d$ being IID $\mathcal{N}(0, 1)$. Let $T = \sum_j Y_j$.

(i) Show that (T, Θ) forms a sufficient statistic for guessing H based on $Y_1, \ldots, Y_d$ when Θ *is* observed.

(ii) Show that T does *not* form a sufficient statistic for guessing H based on $Y_1, \ldots, Y_d$ when Θ *is not* observed.

(iii) Show that notwithstanding Part (ii), if H has a uniform prior, then the decision rule that guesses "$H = 0$" whenever $T \geq 0$ is optimal both when Θ is observed and when it is not observed.

Problem 3 *Covariance Matrices*

Which of the following matrices cannot be a covariance matrix of some real random vector?

$$\mathsf{A} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 5 & 1 \\ 2 & 2 \end{pmatrix}, \qquad \mathsf{C} = \begin{pmatrix} 2 & 10 \\ 10 & 1 \end{pmatrix}, \qquad \mathsf{D} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Problem 4 *Multivariate Gaussians*

Show that if Z is a univariate Gaussian, then the random vector $(Z, Z)^{\mathsf{T}}$ is a Gaussian vector. What is its canonical representation?

Problem 5 *Manipulating Gaussians*

Let $W_1, W_2, \ldots, W_5$ be IID $\mathcal{N}(0,1)$. Define $Y = 3W_1 + 4W_2 - 2W_3 + W_4 - W_5$ and $Z = W_1 - 4W_2 - 2W_3 + 3W_4 - W_5$. What is the joint distribution of (Y, Z)?

Problem 6 *Independence, Uncorrelatedness and Gaussianity*

Let the random variables X and H be independent with $X \sim \mathcal{N}(0,1)$ and with H taking on the values ± 1 equiprobably. Let $Y = HX$ denote their product.

(i) Find the density of Y.

(ii) Are X and Y correlated?

(iii) Compute $\Pr\big[|X| \geq 1\big]$ and $\Pr\big[|Y| \geq 1\big]$.

(iv) Compute the probability that both $|X|$ and $|Y|$ exceed 1.

(v) Are X and Y independent?

(vi) Is the vector $(X, Y)^{\mathsf{T}}$ a Gaussian vector?