Communication and Detection Theory

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Practice Exam

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *Transforming an Energy-Limited Signal*

Let $\mathbf{v}$ be an energy-limited signal of self-similarity function $\mathsf{R}_{\mathbf{vv}}$, and let $\mathbf{p}$ be the signal

$$p(t) = \mathsf{A}v(\alpha t), \quad t \in \mathbb{R},$$

where A and α are real numbers (not necessarily positive) and $\alpha \neq 0$. Is $\mathbf{p}$ energy-limited? If so, relate its self-similarity function $\mathsf{R}_{\mathbf{pp}}$ to $\mathsf{R}_{\mathbf{vv}}$.

Problem 2 — *Bandlimited Signals and Bandwidth*

For every $\lambda \in (0,1)$ let $\check{\mathbf{g}}_\lambda$ be the Inverse Fourier Transform of $\mathbf{g}_\lambda$, where

$$g_\lambda(f) = \begin{cases} 0 & \text{if } f < -\lambda W \text{ or } f > (1-\lambda)W, \\ \frac{1}{\lambda W}(f + \lambda W) & \text{if } -\lambda W \leq f \leq 0, \\ -\frac{1}{(1-\lambda)W}\bigl(f - (1-\lambda)W\bigr) & \text{if } 0 < f \leq (1-\lambda)W, \end{cases} \qquad f \in \mathbb{R},$$

and where W is a fixed positive constant.

(i) Plot $\mathbf{g}_\lambda$ for a λ of your choice.

(ii) For which values of λ, if any, is $\check{\mathbf{g}}_\lambda$ bandlimited to W Hz?

(iii) For which values of λ, if any, is $\check{\mathbf{g}}_\lambda$ bandlimited to $W/4$ Hz?

(iv) What is the bandwidth of $\check{\mathbf{g}}_\lambda$? Express your answer in terms of λ and W.

(v) Which λ minimizes the bandwidth of $\check{\mathbf{g}}_\lambda$? What is the corresponding minimal bandwidth?

Consider $\mathbf{h}_\lambda$ and $\mathbf{u}_\lambda$ defined for $f_c > 0$ by

$$h_\lambda(f) = g_\lambda(f - f_c) + g_\lambda(f + f_c), \quad f \in \mathbb{R},$$

and

$$u_\lambda(f) = g_\lambda(|f| - f_c), \quad f \in \mathbb{R}.$$

Let $\check{\mathbf{h}}_\lambda$ and $\check{\mathbf{u}}_\lambda$ be the Inverse Fourier Transforms of $\mathbf{h}_\lambda$ and $\mathbf{u}_\lambda$.

(vi) For which positive values of f_c is $\check{\mathbf{h}}_\lambda$ a passband signal around f_c? For which positive values of f_c is $\check{\mathbf{u}}_\lambda$ a passband signal around f_c?

For the remainder assume that $\check{\mathbf{h}}_\lambda$ and $\check{\mathbf{u}}_\lambda$ are passband signals around f_c.

(vii) What is the bandwidth of $\check{\mathbf{h}}_\lambda$ around f_c? What is the bandwidth of $\check{\mathbf{u}}_\lambda$ around f_c?

(viii) For which values of λ, if any, is $\check{\mathbf{h}}_\lambda$ a *real* passband signal around f_c? For which values of λ, if any, is $\check{\mathbf{u}}_\lambda$ a *real* passband signal around f_c?

Problem 3 — *Smoothing a PAM Signal*

Let $\big(X(t)\big)$ be the result of mapping the IID random bits $D_1, \ldots, D_{\mathsf{K}}$ to the real numbers $X_1, \ldots, X_{\mathsf{N}}$ using $\mathbf{enc}\colon \{0,1\}^{\mathsf{K}} \to \mathbb{R}^{\mathsf{N}}$ and then mapping these symbols to the waveform

$$X(t) = \mathsf{A} \sum_{\ell=1}^{\mathsf{N}} X_\ell \, g(t - \ell \mathsf{T}_s), \quad t \in \mathbb{R},$$

where $\mathsf{A} > 0$, where $\mathbf{g}$ is an energy-limited pulse shape, and where $\mathsf{T}_s > 0$ is the baud period. Define the stochastic process $\big(Y(t)\big)$ as

$$Y(t) = \frac{1}{17} \int_t^{t+17} X(\tau) \, \mathrm{d}\tau, \quad t \in \mathbb{R}.$$

Can $\big(Y(t)\big)$ be viewed as a PAM signal? If so, of what pulse shape?

Problem 4 — *Hypothesis Testing*

Let the binary random variable H take on the values 0 and 1 equiprobably. Let $\mathbf{W}$ be a standard Gaussian 3-vector that is independent of H. Consider the problem of guessing H based on the observation $\mathbf{Y}$, where conditional on $H = 0$,

$$\mathbf{Y} = \mathsf{A}\mathbf{W},$$

and conditional on $H = 1$,

$$\mathbf{Y} = \mathsf{B}\mathbf{W},$$

where A and B are the deterministic matrices

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} 0 & 0 & \beta \\ 0 & \alpha & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and where α and β are positive real numbers and $\beta \neq 1$.

(i) If you must guess H based on exactly two out of the three components of

$$\mathbf{Y} = (Y^{(1)}, Y^{(2)}, Y^{(3)})^{\mathsf{T}},$$

which two components would you choose in order to minimize the probability of error? Why?

(ii) Determine the conditional densities $f_{\mathbf{Y}|H=0}$ and $f_{\mathbf{Y}|H=1}$.

(iii) Find a one-dimensional sufficient statistic for guessing H based on $\mathbf{Y}$.

(iv) Describe an optimal decision rule for guessing H based on $\mathbf{Y}$.

(v) Compute the Bhattacharyya Bound on the optimal probability of error $p^{\star}(\text{error})$.

(vi) Compute $\lim_{\beta\to\infty} p^{\star}(\text{error})$.

Problem 5 — *A Guessing Rule*

Let the received waveform $\bigl(Y(t)\bigr)$ be

$$Y(t) = X\,s(t) + N(t), \quad t \in \mathbb{R},$$

where $\mathbf{s}$ is a real, deterministic, integrable signal that is bandlimited to W Hz, X is a RV taking on the values ± 1 equiprobably, and $\bigl(N(t)\bigr)$ is noise.

(i) Consider a decoder that guesses "$X = +1$" if $(\mathbf{Y} \star \mathbf{h})(0)$ is positive and guesses "$X = -1$" otherwise. Here $\mathbf{h}$ is some real, deterministic, integrable signal that is bandlimited to W_c Hz. What is the probability of error of this decoder under the assumption that $\bigl(N(t)\bigr)$ is white Gaussian noise of PSD $\mathsf{N}_0/2$ with respect to $\max\{\mathsf{W}, \mathsf{W}_\mathrm{c}\}$? Express your answer using the $\mathcal{Q}$-function, $\mathbf{s}$, $\mathbf{h}$, and N_0.

(ii) Find an $\mathbf{h}$ (of whichever bandwidth you like) that minimizes the probability of error.

(iii) Evaluate the probability of error when $s(t) = \mathsf{A}\,\mathrm{sinc}^2(\mathsf{W}t)$ for all $t \in \mathbb{R}$ and the frequency response of $\mathbf{h}$ closely resembles that of an ideal unit-gain LPF of cutoff frequency W_c. Which choice of W_c minimizes the probability of error?

(iv) Suppose now that X takes on the values $\pm 1, \pm 3$ equiprobably, and consider the guessing rule

$$\text{Guess} = \begin{cases} +3 & \text{if } (\mathbf{Y} \star \mathbf{h})(0) > \alpha, \\ +1 & \text{if } \alpha \geq (\mathbf{Y} \star \mathbf{h})(0) > 0, \\ -1 & \text{if } 0 \geq (\mathbf{Y} \star \mathbf{h})(0) > -\alpha, \\ -3 & \text{if } -\alpha \geq (\mathbf{Y} \star \mathbf{h})(0), \end{cases}$$

where α is some real number. Assume that $(\mathbf{s} \star \mathbf{h})(0) > 0$. Which choice of α minimizes the probability of error?