Problem 1  

**Transforming an Energy-Limited Signal**

Let \( v \) be an energy-limited signal of self-similarity function \( R_{vv} \), and let \( p \) be the signal

\[
p(t) = A v(\alpha t), \quad t \in \mathbb{R},
\]

where \( A \) and \( \alpha \) are real numbers (not necessarily positive) and \( \alpha \neq 0 \). Is \( p \) energy-limited? If so, relate its self-similarity function \( R_{pp} \) to \( R_{vv} \).

Problem 2  

**Bandlimited Signals and Bandwidth**

For every \( \lambda \in (0, 1) \) let \( \hat{g}_\lambda \) be the Inverse Fourier Transform of \( g_\lambda \), where

\[
g_\lambda(f) = \begin{cases} 
0 & \text{if } f < -\lambda W \text{ or } f > (1 - \lambda)W, \\
\frac{1}{\lambda W}(f + \lambda W) & \text{if } -\lambda W \leq f \leq 0, \\
\frac{1}{(1-\lambda)W}(f - (1 - \lambda)W) & \text{if } 0 < f \leq (1 - \lambda)W,
\end{cases} \quad f \in \mathbb{R},
\]

and where \( W \) is a fixed positive constant.

(i) Plot \( g_\lambda \) for a \( \lambda \) of your choice.

(ii) For which values of \( \lambda \), if any, is \( \hat{g}_\lambda \) bandlimited to \( W \) Hz?

(iii) For which values of \( \lambda \), if any, is \( \hat{g}_\lambda \) bandlimited to \( W/4 \) Hz?

(iv) What is the bandwidth of \( \hat{g}_\lambda \)? Express your answer in terms of \( \lambda \) and \( W \).

(v) Which \( \lambda \) minimizes the bandwidth of \( \hat{g}_\lambda \)? What is the corresponding minimal bandwidth?

Consider \( h_\lambda \) and \( u_\lambda \) defined for \( f_c > 0 \) by

\[
h_\lambda(f) = g_\lambda(f - f_c) + g_\lambda(f + f_c), \quad f \in \mathbb{R},
\]

and

\[
u_\lambda(f) = g_\lambda(|f| - f_c), \quad f \in \mathbb{R}.
\]

Let \( \hat{h}_\lambda \) and \( \hat{u}_\lambda \) be the Inverse Fourier Transforms of \( h_\lambda \) and \( u_\lambda \).
(vi) For which positive values of $f_c$ is $\tilde{h}_\lambda$ a passband signal around $f_c$? For which positive values of $f_c$ is $\tilde{u}_\lambda$ a passband signal around $f_c$?

For the remainder assume that $\tilde{h}_\lambda$ and $\tilde{u}_\lambda$ are passband signals around $f_c$.

(vii) What is the bandwidth of $\tilde{h}_\lambda$ around $f_c$? What is the bandwidth of $\tilde{u}_\lambda$ around $f_c$?

(viii) For which values of $\lambda$, if any, is $\tilde{h}_\lambda$ a real passband signal around $f_c$? For which values of $\lambda$, if any, is $\tilde{u}_\lambda$ a real passband signal around $f_c$?

Problem 3

Smoothing a PAM Signal

Let $(X(t))$ be the result of mapping the IID random bits $D_1, \ldots, D_K$ to the real numbers $X_1, \ldots, X_N$ using $\text{enc}: \{0, 1\}^K \rightarrow \mathbb{R}^N$ and then mapping these symbols to the waveform

$$X(t) = A \sum_{\ell=1}^{N} X_\ell g(t - \ell T_s), \quad t \in \mathbb{R},$$

where $A > 0$, where $g$ is an energy-limited pulse shape, and where $T_s > 0$ is the baud period. Define the stochastic process $(Y(t))$ as

$$Y(t) = \frac{1}{17} \int_{t}^{t+17} X(\tau) \, d\tau, \quad t \in \mathbb{R}.$$

Can $(Y(t))$ be viewed as a PAM signal? If so, of what pulse shape?

Problem 4

Hypothesis Testing

Let the binary random variable $H$ take on the values 0 and 1 equiprobably. Let $W$ be a standard Gaussian 3-vector that is independent of $H$. Consider the problem of guessing $H$ based on the observation $Y$, where conditional on $H = 0$,

$$Y = AW,$$

and conditional on $H = 1$,

$$Y = BW,$$

where $A$ and $B$ are the deterministic matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & \beta \\ 0 & \alpha & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and where $\alpha$ and $\beta$ are positive real numbers and $\beta \neq 1$.

(i) If you must guess $H$ based on exactly two out of the three components of

$$Y = (Y^{(1)}, Y^{(2)}, Y^{(3)})^T,$$

which two components would you choose in order to minimize the probability of error? Why?
(ii) Determine the conditional densities \( f_{Y|H=0} \) and \( f_{Y|H=1} \).

(iii) Find a one-dimensional sufficient statistic for guessing \( H \) based on \( Y \).

(iv) Describe an optimal decision rule for guessing \( H \) based on \( Y \).

(v) Compute the Bhattacharyya Bound on the optimal probability of error \( p^*(error) \).

(vi) Compute \( \lim_{\beta \to \infty} p^*(error) \).

**Problem 5**

**A Guessing Rule**

Let the received waveform \( (Y(t)) \) be

\[
Y(t) = X s(t) + N(t), \quad t \in \mathbb{R},
\]

where \( s \) is a real, deterministic, integrable signal that is bandlimited to \( W \) Hz, \( X \) is a RV taking on the values \( \pm 1 \) equiprobably, and \((N(t))\) is noise.

(i) Consider a decoder that guesses “\( X = +1 \)” if \((Y \ast h)(0)\) is positive and guesses “\( X = -1 \)” otherwise. Here \( h \) is some real, deterministic, integrable signal that is bandlimited to \( W_c \) Hz. What is the probability of error of this decoder under the assumption that \((N(t))\) is white Gaussian noise of PSD \( N_0/2 \) with respect to \( \max\{W, W_c\} \)? Express your answer using the \( Q \)-function, \( s \), \( h \), and \( N_0 \).

(ii) Find an \( h \) (of whichever bandwidth you like) that minimizes the probability of error.

(iii) Evaluate the probability of error when \( s(t) = A \text{sinc}^2(Wt) \) for all \( t \in \mathbb{R} \) and the frequency response of \( h \) closely resembles that of an ideal unit-gain LPF of cutoff frequency \( W_c \). Which choice of \( W_c \) minimizes the probability of error?

(iv) Suppose now that \( X \) takes on the values \( \pm 1, \pm 3 \) equiprobably, and consider the guessing rule

\[
\text{Guess} = \begin{cases} 
+3 & \text{if } (Y \ast h)(0) > \alpha, \\
+1 & \text{if } \alpha \geq (Y \ast h)(0) > 0, \\
-1 & \text{if } 0 \geq (Y \ast h)(0) > -\alpha, \\
-3 & \text{if } -\alpha \geq (Y \ast h)(0), 
\end{cases}
\]

where \( \alpha \) is some real number. Assume that \((s \ast h)(0) > 0\). Which choice of \( \alpha \) minimizes the probability of error?

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