Communication and Detection Theory

Signal and Information Processing Laboratory
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Model Answers to Exercise 1 of February 21, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *Manipulating Inner Products*

We first show that the inner product $\langle \mathbf{u}+\mathbf{v}, 3\mathbf{u}+\mathbf{v}+\mathrm{i}\mathbf{w}\rangle$ is well-defined by showing that the signals $\mathbf{u}+\mathbf{v}$ and $3\mathbf{u}+\mathbf{v}+\mathrm{i}\mathbf{w}$ are energy-limited. The signal $\mathbf{u}+\mathbf{v}$ is energy-limited because both $\mathbf{u}$ and $\mathbf{v}$ are, by assumption, energy-limited, and because, by the Triangle Inequality (Proposition 3.4.1), $\|\mathbf{u}+\mathbf{v}\|_2 \leq \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2$. Similarly, the signal $3\mathbf{u}+\mathbf{v}+\mathrm{i}\mathbf{w}$ is energy-limited because by the Triangle Inequality

$$\begin{aligned}\|3\mathbf{u}+\mathbf{v}+\mathrm{i}\mathbf{w}\|_2 &\leq \|3\mathbf{u}+\mathbf{v}\|_2 + \|\mathrm{i}\mathbf{w}\|_2 \\ &\leq \|3\mathbf{u}\|_2 + \|\mathbf{v}\|_2 + \|\mathrm{i}\mathbf{w}\|_2 \\ &= 3\,\|\mathbf{u}\|_2 + \|\mathbf{v}\|_2 + \|\mathbf{w}\|_2 .\end{aligned}$$

We next use the properties of inner products to compute the desired inner product.

$$\begin{aligned}&\langle \mathbf{u}+\mathbf{v}, 3\mathbf{u}+\mathbf{v}+\mathrm{i}\mathbf{w}\rangle \\ &= \langle \mathbf{u}, 3\mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathrm{i}\mathbf{w}\rangle + \langle \mathbf{v}, 3\mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathrm{i}\mathbf{w}\rangle && (1)\\ &= 3\,\|\mathbf{u}\|_2^2 + \langle \mathbf{u}, \mathbf{v}\rangle - \mathrm{i}\,\langle \mathbf{u}, \mathbf{w}\rangle + 3\,\langle \mathbf{v}, \mathbf{u}\rangle + \|\mathbf{v}\|_2^2 - \mathrm{i}\,\langle \mathbf{v}, \mathbf{w}\rangle && (2)\\ &= 3\,\|\mathbf{u}\|_2^2 + \|\mathbf{v}\|_2^2 + \langle \mathbf{u}, \mathbf{v}\rangle + 3\,\langle \mathbf{u}, \mathbf{v}\rangle^* - \mathrm{i}\,\langle \mathbf{u}, \mathbf{w}\rangle - \mathrm{i}\,\langle \mathbf{v}, \mathbf{w}\rangle, && (3)\end{aligned}$$

where (1) follows from Equations (3.6)–(3.10); (2) follows from (3.11) and (3.6)–(3.9); and finally (3) follows by (3.6). Notice that the inner products on the right-hand side of (3) are all well-defined because $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are all energy-limited signals.

Problem 2 — *Orthogonality to All Signals*

We first prove that for a signal $\mathbf{u} \in \mathcal{L}_2$ the condition that $\mathbf{u}$ is indistinguishable from the all-zero signal implies that for every $\mathbf{v} \in \mathcal{L}_2$ the inner product $\langle \mathbf{u}, \mathbf{v}\rangle$ is zero. We thus assume that $\mathbf{u} \in \mathcal{L}_2$ is such that $\mathbf{u} \equiv \mathbf{0}$ and proceed to prove that the $\langle \mathbf{u}, \mathbf{v}\rangle = 0$ whenever $\mathbf{v} \in \mathcal{L}_2$. This follows directly from the Cauchy-Schwarz Inequality:

$$\begin{aligned}|\langle \mathbf{u}, \mathbf{v}\rangle| &\leq \|\mathbf{u}\|_2\,\|\mathbf{v}\|_2 \\ &= 0\,\|\mathbf{v}\|_2 \\ &= 0, \quad \mathbf{v} \in \mathcal{L}_2,\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz Inequality, and the subsequent equality follows because $\mathbf{u}$ is indistinguishable from the all-zero signal and is thus of zero energy. (To see that $\mathbf{u} \equiv \mathbf{0}$ implies $\|\mathbf{u}\|_2 = 0$ substitue $\mathbf{0}$ for $\mathbf{v}$ in (2.18).)

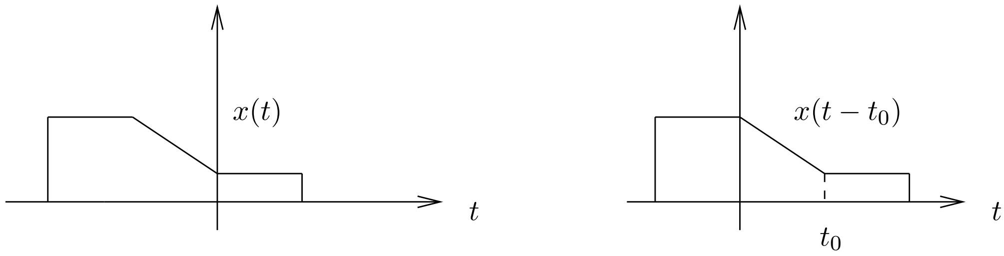

Figure 0.1: Energy-limited signal $x(\cdot)$ and a shifted version $t \mapsto x(t-t_0)$.

We next prove the other direction, namely, that if some $\mathbf{u} \in \mathcal{L}_2$ is orthogonal to every energy-limited signal, then it must be indistinguishable from the all-zero signal. Assume then that $\mathbf{u}$ is energy limited and orthogonal to every $\mathbf{v} \in \mathcal{L}_2$. Since $\mathbf{u}$ is in $\mathcal{L}_2$, and since it is orthogonal to every signal in $\mathcal{L}_2$, it is *a-fortiori* also orthogonal to itself. Thus, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$, i.e., $\|\mathbf{u}\|_2 = 0$. We thus conclude that $\mathbf{u} \equiv \mathbf{0}$, because any signal of zero energy is indistinguishable from the all-zero signal. (To see that $\|\mathbf{u}\|_2 = 0$ implies $\mathbf{u} \equiv \mathbf{0}$, substitue $\mathbf{0}$ for $\mathbf{v}$ in (2.18).)

Problem 3 — *Finite-Energy Signals*

(i) Define the signal $\mathbf{y}$ as

$$y(t) = x(t-t_0), \quad t \in \mathbb{R}.$$

Its energy is then

$$\begin{aligned}
\|\mathbf{y}\|_2^2 &= \int_{-\infty}^{\infty} |y(t)|^2 \, \mathrm{d}t && (4)\\
&= \int_{-\infty}^{\infty} |x(t-t_0)|^2 \, \mathrm{d}t && (5)\\
&= \int_{-\infty}^{\infty} |x(t')|^2 \, \mathrm{d}t' && (6)\\
&= \|\mathbf{x}\|_2^2 && (7)\\
&< \infty, && (8)
\end{aligned}$$

where the third equation follows by substituting $t' \triangleq t - t_0$.

In Figure 0.1 an energy-limited signal $x(\cdot)$ and the corresponding shifted version $t \mapsto x(t-t_0)$ are plotted.

(ii) For the energy of the signal $\overleftarrow{\mathbf{x}}$ we obtain

$$\begin{aligned}
\|\overleftarrow{\mathbf{x}}\|_2^2 &= \int_{-\infty}^{\infty} |\overleftarrow{x}(t)|^2 \, \mathrm{d}t && (9)\\
&= \int_{-\infty}^{\infty} |x(-t)|^2 \, \mathrm{d}t && (10)\\
&= -\int_{\infty}^{-\infty} |x(t')|^2 \, \mathrm{d}t' && (11)\\
&= \int_{-\infty}^{\infty} |x(t')|^2 \, \mathrm{d}t' && (12)\\
&= \|\mathbf{x}\|_2^2 && (13)\\
&< \infty && (14)
\end{aligned}$$

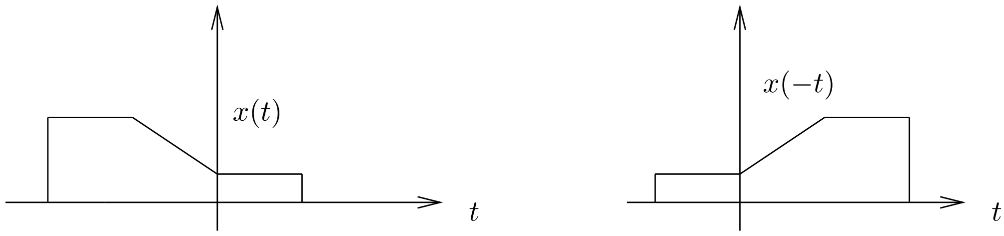

Figure 0.2: Energy-limited signal $x(\cdot)$ and the flipped version $t \mapsto x(-t)$.

where the third equation follows by substituting $t' \triangleq -t$. An energy-limited signal $x(\cdot)$ and its mirror image $\overleftarrow{x}(\cdot)$ are plotted in Figure 0.2.

(iii) From (7) and (13) we conclude that

$$\|\mathbf{y}\|_2^2 = \|\overleftarrow{\mathbf{x}}\|_2^2 = \|\mathbf{x}\|_2^2\,, \tag{15}$$

i.e., that shifting or reversing a signal in time does not change its energy.

Problem 4 — *Inner Products of Mirror Images*

By Problem 3.3, Part b), it follows that if $\mathbf{x}$ and $\mathbf{y}$ are energy-limited signals, then so are also their mirror images $\overleftarrow{\mathbf{x}}$ and $\overleftarrow{\mathbf{y}}$. Therefore, the inner product $\langle \overleftarrow{\mathbf{x}}, \overleftarrow{\mathbf{y}} \rangle$ is well-defined.
Indeed,

$$\begin{aligned}\langle \overleftarrow{\mathbf{x}}, \overleftarrow{\mathbf{y}} \rangle &= \int_{-\infty}^{\infty} x(-t)y^*(-t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x(t')y^*(t')\,\mathrm{d}t' \\ &= \langle \mathbf{x}, \mathbf{y} \rangle\,,\end{aligned}$$

where the second equality follows by the substitution $t' = -t$.

Problem 5 — *Truncated Polynomials*

(i) The energy in $\mathbf{u}$ is

$$\begin{aligned}\|\mathbf{u}\|_2^2 &= \int_{-\infty}^{\infty} |u(t)|^2\,\mathrm{d}t \\ &= \int_0^1 (t+2)^2\,\mathrm{d}t \\ &= \int_0^1 \left(t^2+4t+4\right)\mathrm{d}t \\ &= \left(\frac{1}{3}t^3+2t^2+4t\right)\Big|_0^1 \\ &= \frac{1}{3}+2+4 \\ &= \frac{19}{3}.\end{aligned}$$

Likewise, the energy in $\mathbf{v}$ is

$$\begin{aligned}
\|\mathbf{v}\|_2^2 &= \int_{-\infty}^{\infty} |v(t)|^2 \, \mathrm{d}t \\
&= \int_0^1 \left(t^2 - 2t - 3\right)^2 \, \mathrm{d}t \\
&= \int_0^1 \left(t^4 - 4t^3 - 2t^2 + 12t + 9\right) \, \mathrm{d}t \\
&= \left(\frac{1}{5}t^5 - t^4 - \frac{2}{3}t^3 + 6t^2 + 9t\right) \Big|_0^1 \\
&= \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9 \\
&= \frac{203}{15}.
\end{aligned}$$

Thus, $\mathbf{u}$ and $\mathbf{v}$ are energy-limited signals.

(ii) Since $\mathbf{u}, \mathbf{v} \in \mathcal{L}_2$, the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ is well-defined. It is given by

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= \int_{-\infty}^{\infty} u(t)v(t) \, \mathrm{d}t \\
&= \int_0^1 u(t)v(t) \, \mathrm{d}t \\
&= \int_0^1 (t+2)(t^2 - 2t - 3) \, \mathrm{d}t \\
&= \int_0^1 \left(t^3 - 7t - 6\right) \, \mathrm{d}t \\
&= \left(\frac{1}{4}t^4 - \frac{7}{2}t^2 - 6t\right) \Big|_0^1 \\
&= \frac{1}{4} - \frac{7}{2} - 6 \\
&= -\frac{37}{4}.
\end{aligned}$$