Problem 1  

Manipulating Inner Products

We first show that the inner product $\langle u + v, 3u + v + iw \rangle$ is well-defined by showing that the signals $u + v$ and $3u + v + iw$ are energy-limited. The signal $u + v$ is energy-limited because both $u$ and $v$ are, by assumption, energy-limited, and because, by the Triangle Inequality (Proposition 3.4.1), $\|u + v\|_2 \leq \|u\|_2 + \|v\|_2$. Similarly, the signal $3u + v + iw$ is energy-limited because by the Triangle Inequality $\|3u + v + iw\|_2 \leq \|3u\|_2 + \|v\|_2 + \|iw\|_2 \leq 3\|u\|_2 + \|v\|_2 + \|w\|_2$.

We next use the properties of inner products to compute the desired inner product.

$$\langle u + v, 3u + v + iw \rangle = \langle u, 3u \rangle + \langle u, v \rangle + \langle u, iw \rangle + \langle v, 3u \rangle + \langle v, v \rangle + \langle v, iw \rangle \leq 3\|u\|_2^2 + \|v\|_2^2 + \|iw\|_2^2 \leq 3\|u\|_2^2 + \|v\|_2^2 + \|w\|_2^2.$$  

where (1) follows from Equations (3.6)–(3.10); (2) follows from (3.11) and (3.6)–(3.9); and finally (3) follows by (3.6). Notice that the inner products on the right-hand side of (3) are all well-defined because $u, v, w$ are all energy-limited signals.

Problem 2  

Orthogonality to All Signals

We first prove that for an energy-limited signal $u$ the condition that $u$ is indistinguishable from the all-zero signal implies that for every $v \in L_2$ the inner product $\langle u, v \rangle$ is zero. We thus assume that $u \in L_2$ is such that $u \equiv 0$ and proceed to prove that $\langle u, v \rangle = 0$ whenever $v \in L_2$. This follows directly from the Cauchy-Schwarz Inequality:

$$|\langle u, v \rangle| \leq \|u\|_2 \|v\|_2 = 0 \|v\|_2 = 0, \quad v \in L_2,$$

where the first inequality follows from the Cauchy-Schwarz Inequality, and the subsequent equality follows because $u$ is indistinguishable from the all-zero signal and is thus of zero energy. (To see that $u \equiv 0$ implies $\|u\|_2 = 0$ substitute $0$ for $v$ in (2.18).)
Figure 0.1: Energy-limited signal $x(\cdot)$ and a shifted version $t \mapsto x(t-t_0)$.

We next prove the other direction, namely, that if some $u \in L_2$ is orthogonal to every energy-limited signal, then it must be indistinguishable from the all-zero signal. Assume then that $u$ is energy-limited and orthogonal to every $v \in L_2$. Since $u$ is in $L_2$, and since it is orthogonal to every signal in $L_2$, it is a fortiori also orthogonal to itself. Thus, $\langle u, u \rangle = 0$, i.e., $\|u\|_2 = 0$. We thus conclude that $u \equiv 0$, because any signal of zero energy is indistinguishable from the all-zero signal. (To see that $\|u\|_2 = 0$ implies $u \equiv 0$, substitute 0 for $v$ in (2.18).)

**Problem 3**

**Finite-Energy Signals**

(i) Define the signal $y$ as

$$y(t) = x(t-t_0), \quad t \in \mathbb{R}.$$ 

Its energy is then

$$\|y\|_2^2 = \int_{-\infty}^{\infty} |y(t)|^2 \, dt$$

Figure 0.1 depicts an energy-limited signal $x(\cdot)$ and its time shift $t \mapsto x(t-t_0)$.

(ii) For the energy of the signal $\bar{x}$ we obtain

$$\|ar{x}\|_2^2 = \int_{-\infty}^{\infty} |ar{x}(t)|^2 \, dt$$

where the third equation follows by substituting $t' \triangleq -t$. An energy-limited signal $x(\cdot)$ and its mirror image $\bar{x}(\cdot)$ are plotted in Figure 0.2.
(iii) From (7) and (13) we conclude that
\[ \|y\|_2^2 = \|\tilde{x}\|_2^2 = \|x\|_2^2, \tag{15} \]
i.e., that shifting or reversing a signal in time does not change its energy.

**Problem 4**

*Inner Products of Mirror Images*

By Problem 3.3, Part b), it follows that if \( x \) and \( y \) are energy-limited signals, then so are also their mirror images \( \tilde{x} \) and \( \tilde{y} \). Therefore, the inner product \( \langle \tilde{x}, \tilde{y} \rangle \) is well-defined.

Indeed,
\[
\langle \tilde{x}, \tilde{y} \rangle = \int_{-\infty}^{\infty} x(-t) y^*(t) \, dt \\
= \int_{-\infty}^{\infty} x(t') y^*(t') \, dt' \\
= \langle x, y \rangle,
\]
where the second equality follows by the substitution \( t' = -t \).

**Problem 5**

*Truncated Polynomials*

(i) The energy in \( u \) is
\[
\|u\|_2^2 = \int_{-\infty}^{\infty} |u(t)|^2 \, dt \\
= \int_{0}^{1} (t + 2)^2 \, dt \\
= \int_{0}^{1} (t^2 + 4t + 4) \, dt \\
= \left[ \frac{t^3}{3} + 2t^2 + 4t \right]_{0}^{1} \\
= \frac{1}{3} + 2 + 4 \\
= \frac{19}{3}.
\]
Likewise, the energy in \( v \) is

\[
\|v\|_2^2 = \int_{-\infty}^{\infty} |v(t)|^2 \, dt \\
= \int_{0}^{1} (t^2 - 2t - 3)^2 \, dt \\
= \int_{0}^{1} (t^4 - 4t^3 - 2t^2 + 12t + 9) \, dt \\
= \left( \frac{1}{5}t^5 - \frac{2}{3}t^3 + 6t^2 + 9t \right) \bigg|_{0}^{1} \\
= \frac{1}{5} - 1 - \frac{2}{3} + 6 + 9 \\
= \frac{203}{15}.
\]

Thus, \( u \) and \( v \) are energy-limited signals.

(ii) Since \( u, v \in L_2 \), the inner product \( \langle u, v \rangle \) is well-defined. It is given by

\[
\langle u, v \rangle = \int_{-\infty}^{\infty} u(t)v(t) \, dt \\
= \int_{0}^{1} u(t)v(t) \, dt \\
= \int_{0}^{1} (t + 2)(t^2 - 2t - 3) \, dt \\
= \int_{0}^{1} (t^3 - 7t - 6) \, dt \\
= \left( \frac{1}{4}t^4 - \frac{7}{2}t^2 - 6t \right) \bigg|_{0}^{1} \\
= \frac{1}{4} - \frac{7}{2} - 6 \\
= \frac{37}{4}.
\]