Model Answers to Exercise 2 of February 28, 2017

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Problem 1  

Reflection of Passband Signal

(i) Since $x_{BB}$ is the baseband representation of $x_{PB}$,

$$x_{PB}(t) = 2 \text{Re}(x_{BB}(t) e^{i2\pi f_c t}), \quad t \in \mathbb{R}. \tag{1}$$

Consequently, we can express $x_{PB}$ in the more suggestive way

$$x_{PB}(t) = x_{PB}(-t) = 2 \text{Re}(x_{BB}(-t) e^{-i2\pi f_c t}) = 2 \text{Re}(x_{BB}^*(-t) e^{i2\pi f_c t}) = 2 \text{Re}(x_{BB}^*(t) e^{i2\pi f_c t}), \quad t \in \mathbb{R},$$

where in the third equality we used the fact that the real part of a complex number is equal to the real part of its complex conjugate. Since $x_{BB}$ is bandlimited to $W/2$ Hz, so is $x_{BB}^*$. We thus conclude from (1) using Proposition 7.6.9 that the baseband representation of $x_{PB}$ is $x_{BB}^*$.

(ii) From the first part we know that the baseband representation of $y_{PB}$ is $y_{BB}^*$. Consequently, by Theorem 7.6.10,

$$\langle x_{PB}, y_{PB} \rangle = 2 \text{Re}(\langle x_{BB}, y_{BB}^* \rangle) = 2 \text{Re} \left( \int_{-\infty}^{\infty} x_{BB}(t) y_{BB}(-t) dt \right) = 2 \text{Re} \left( (x_{BB} * y_{BB})(0) \right).$$
Problem 2

Symmetries of the FT

(i) The proof relies on the fact that conjugation and integration commute (Proposition 2.3.1).

If \( x \) is a real-valued signal with Fourier Transform \( \hat{x}(\cdot) \), then

\[
\hat{x}(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft} \, dt
\]

\[
= \int_{-\infty}^{\infty} x^*(t) (e^{i2\pi ft})^* \, dt
\]

\[
= \left( \int_{-\infty}^{\infty} x(t)e^{i2\pi ft} \, dt \right)^*
\]

\[
= \hat{x}^*(-f), \quad f \in \mathbb{R},
\]

where the first equality is the definition of the Fourier Transform; the second follows from our assumption that \( x \) is real and is thus equal to its complex conjugate; the third equality follows by swapping integration and conjugation; and the fourth equality follows again from the definition of the Fourier Transform.

(ii) If \( x \) is purely imaginary, then we can define the real signal \( y \) as

\[
y(t) = \frac{1}{i} x(t), \quad t \in \mathbb{R}.
\]

Since \( y \) is real, its FT \( \hat{y} \) is conjugate symmetric. And by the linearity of the FT, \( \hat{x} = iy \). Thus

\[
\hat{x}(f) = iy(f)
\]

\[
= (-i)^* \hat{y}(f)
\]

\[
= - (iy(-f))^*
\]

\[
= -\hat{x}^*(-f), \quad f \in \mathbb{R}.
\]

(iii) Let \( x \) be a complex integrable signal. Then \( x \) can be expressed as a sum

\[
x = x_R + x_I,
\]

where \( x_R \) is the integrable real signal

\[
x_R(t) = \text{Re}(x(t)), \quad t \in \mathbb{R},
\]

and where \( x_I \) is the integrable purely-imaginary signal

\[
x_I(t) = i \text{Im}(x(t)), \quad t \in \mathbb{R}.
\]

By the linearity of the FT,

\[
\hat{x} = \hat{x}_R + \hat{x}_I,
\]

thus expressing the FT of \( x \) as a sum of the conjugate symmetric function \( \hat{x}_R \) (because \( x_R \) is real) and the conjugate antisymmetric function \( \hat{x}_I \) (because \( x_I \) is purely imaginary).

To prove uniqueness, suppose that

\[
\hat{x}(f) = g_{\text{cs}}(f) + g_{\text{cas}}(f), \quad f \in \mathbb{R},
\]

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where $g_{cs}$ is conjugate symmetric
\begin{equation}
    g_{cs}(-f) = g_{cs}^*(f), \quad f \in \mathbb{R},
\end{equation}
and $g_{cas}$ is conjugate antisymmetric
\begin{equation}
    g_{cas}(-f) = -g_{cas}^*(f), \quad f \in \mathbb{R}.
\end{equation}

Then
\begin{equation}
    \hat{x}^*(-f) = g_{cs}^*(-f) + g_{cas}^*(-f)
    = g_{cs}(f) - g_{cas}(f), \quad f \in \mathbb{R},
\end{equation}
where the second equality follows from (3) and (4).

Adding (2) and (5) we obtain the unique solution
\begin{equation}
    g_{cs}(f) = \frac{\hat{x}(f) + \hat{x}^*(-f)}{2}, \quad f \in \mathbb{R},
\end{equation}
and substituting (5) from (2) yields the unique solution
\begin{equation}
    g_{cas}(f) = \frac{\hat{x}(f) - \hat{x}^*(-f)}{2}, \quad f \in \mathbb{R}.
\end{equation}

**Problem 3**

*Phase Shift*

(i) We first express $z_{PB}(\cdot)$ in the more suggesting form
\begin{equation*}
    z_{PB}(t) = x(t) \sin(2\pi f_c t + \phi)
    = 2 \text{Re} \left( -\frac{i}{2} x(t) e^{i(2\pi f_c t + \phi)} \right)
    = 2 \text{Re} \left( -\frac{i}{2} x(t) e^{i\phi} e^{i2\pi f_c t} \right), \quad t \in \mathbb{R},
\end{equation*}
where in the second equality we used the fact that $x$ is real. It now follows from Proposition 7.6.9 that the baseband representation $z_{BB}$ of $z_{PB}$ is
\begin{equation*}
    z_{BB} = -\frac{i}{2} e^{i\phi} x.
\end{equation*}

(ii) By the previous part,
\begin{equation*}
    \hat{z}_{BB}(f) = -\frac{i}{2} e^{i\phi} \hat{x}(f), \quad f \in \mathbb{R}.
\end{equation*}

Consequently, by Proposition 7.6.8,
\begin{equation*}
    \hat{z}_{PB}(f) = -\frac{i}{2} e^{i\phi} \hat{x}(f - f_c) + \frac{i}{2} e^{-i\phi} \hat{x}^*(-f - f_c), \quad f \in \mathbb{R}.
\end{equation*}

Since $x$ is real, its FT is conjugate symmetric, so we can also write this as
\begin{equation*}
    \hat{z}_{PB}(f) = -\frac{i}{2} e^{i\phi} \hat{x}(f - f_c) + \frac{i}{2} e^{-i\phi} \hat{x}(f + f_c), \quad f \in \mathbb{R}.
\end{equation*}
Problem 4
Purely Real and Purely Imaginary
Baseband Representations

(i) Let $x_{PB}$ be a real integrable passband signal that is bandlimited to $W$ Hz around the carrier frequency $f_c$. The FT $\hat{x}_{BB}$ of its baseband representation $x_{BB}$ is related to its FT $\hat{x}_{PB}$ via the relation

$$\hat{x}_{BB}(f) = \hat{x}_{PB}(f + f_c) I\{|f| \leq \frac{W}{2}\}, \quad f \in \mathbb{R}$$  \hspace{1cm} (8)

(Proposition 7.6.5). Consequently, the condition

$$\hat{x}_{PB}(f_c - \delta) = \hat{x}_{PB}^*(f_c + \delta), \quad |\delta| \leq \frac{W}{2}$$

is equivalent to the condition

$$\hat{x}_{BB}(-\delta) = \hat{x}_{BB}^*(\delta), \quad |\delta| \leq \frac{W}{2}.$$  \hspace{1cm} (9)

It remains to argue that this latter condition is equivalent to the condition that $x_{BB}$ is real. One direction is obvious: if $x_{BB}$ is real, then its FT is conjugate symmetric (Exercise 6.1).

The other direction is almost as obvious: The signal $x_{BB}$ is an integrable signal that is bandlimited to $W/2$ Hz (Theorem 7.6.5). Consequently, it is equal to the IFT of its FT $\hat{x}_{BB}$ (Proposition 6.4.10). Thus, if (9) holds then

$$x_{BB}(t) = \int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{BB}(f) e^{i2\pi ft} df = \left( \int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{BB}^*(f) e^{-i2\pi ft} df \right)^* = \left( \int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{BB}(-f) e^{-i2\pi ft} df \right)^* = \left( \int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{BB}(\tilde{f}) e^{i2\pi \tilde{f} t} d\tilde{f} \right)^* = x_{BB}^*(t), \quad t \in \mathbb{R}.$$  \hspace{1cm} (ii)

(ii) By (8), the condition

$$\hat{x}_{PB}(f_c - \delta) = -\hat{x}_{PB}^*(f_c + \delta), \quad |\delta| \leq \frac{W}{2}$$

is equivalent to the condition

$$\hat{x}_{BB}(-\delta) = -\hat{x}_{BB}^*(\delta), \quad |\delta| \leq \frac{W}{2}.$$  \hspace{1cm} (10)

It remains to argue that this latter condition is equivalent to the condition that $x_{BB}$ is purely imaginary.

One direction is obvious: if $x_{BB}$ is purely imaginary, then its FT is conjugate antisymmetric (Exercise 6.1).

The other direction is almost as obvious: The signal $x_{BB}$ is an integrable signal that is bandlimited to $W/2$ Hz (Theorem 7.6.5). Consequently, it is equal to the IFT of its FT $\hat{x}_{BB}$
(Proposition 6.4.10). Thus, if (10) holds then
\[
x_{BB}(t) = \int_{-W/2}^{W/2} \hat{x}_{BB}(f) e^{i2\pi ft} df
\]
\[
= \left( \int_{-W/2}^{W/2} \hat{x}_{BB}(f) e^{-i2\pi ft} df \right)^*
\]
\[
= -\left( \int_{-W/2}^{W/2} \hat{x}_{BB}(-f) e^{-i2\pi ft} df \right)^*
\]
\[
= -\left( \int_{-W/2}^{W/2} \hat{x}_{BB}(f) e^{i2\pi ft} df \right)^*
\]
\[
= -x_{BB}^*(t), \quad t \in \mathbb{R}.
\]

**Problem 5**

**Symmetry around the Carrier Frequency**

Let \( x_{PB} \) be a real integrable passband signal that is bandlimited to \( W \) Hz around the carrier frequency \( f_c \).

(i) If \( x_{PB} \) is equal to the signal \( t \mapsto w(t) \cos(2\pi f_c t) \) for some real integrable signal \( w \) that is bandlimited to \( W/2 \) Hz, then, by Proposition 7.6.9, \( x_{BB} \) is equal to \( w/2 \), because
\[
x_{PB}(t) = w(t) \cos(2\pi f_c t)
\]
\[
= 2 \text{Re} \left( \frac{w(t)}{2} e^{i2\pi f_c t} \right),
\]
where the second equality follows because \( w \) is real. In this case \( x_{PB} \) has a real baseband representation, namely \( w/2 \), and by Exercise 7.1
\[
\hat{x}_{PB}(f + \delta) = \hat{x}_{PB}^*(f - \delta), \quad |\delta| \leq \frac{W}{2}. \tag{11}
\]
Conversely, if (11) holds, then by Exercise 7.1 the baseband representation of \( x_{PB} \) is real, so
\[
x_{PB}(t) = 2 \text{Re} \left( x_{BB}(t) e^{i2\pi f_c t} \right)
\]
\[
= 2x_{BB}(t) \cos(2\pi f_c t),
\]
and \( x_{PB} \) has the desired representation with \( w = 2x_{BB} \).

(ii) If \( x_{PB} \) is equal to the signal \( t \mapsto w(t) \sin(2\pi f_c t) \) for some real integrable signal \( w \) that is bandlimited to \( W/2 \) Hz, then, by Proposition 7.6.9, \( x_{BB} \) is equal to \( w/(2i) \), because
\[
x_{PB}(t) = w(t) \sin(2\pi f_c t)
\]
\[
= 2 \text{Re} \left( \frac{w(t)}{2i} e^{i2\pi f_c t} \right),
\]
where the second equality follows because \( w \) is real. In this case \( x_{PB} \) has a purely imaginary baseband representation, namely \( w/(2i) \), and by Exercise 7.1
\[
\hat{x}_{PB}(f + \delta) = -\hat{x}_{PB}^*(f - \delta), \quad |\delta| \leq \frac{W}{2}. \tag{12}
\]
Conversely, if (12) holds, then by Exercise 7.1 the baseband representation of \( x_{PB} \) is purely imaginary, so

\[
x_{PB}(t) = 2 \text{Re} \left( x_{BB}(t) e^{i2\pi f_c t} \right) \\
= 2ix_{BB}(t) \sin(2\pi f_c t),
\]

and \( x_{PB} \) has the desired representation with \( w = 2ix_{BB} \).