Communication and Detection Theory

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http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 ***Reflection of Passband Signal***

(i) Since $\mathbf{x}_{\text{BB}}$ is the baseband representation of $\mathbf{x}_{\text{PB}}$,

$$x_{\text{PB}}(t) = 2\,\text{Re}\big(x_{\text{BB}}(t)\,e^{\text{i}2\pi f_c t}\big), \quad t \in \mathbb{R}.$$

Consequently, we can express $\overleftarrow{\mathbf{x}}_{\text{PB}}$ in the more suggestive way

$$\begin{aligned} \overleftarrow{x}_{\text{PB}}(t) &= x_{\text{PB}}(-t) \\ &= 2\,\text{Re}\big(x_{\text{BB}}(-t)\,e^{-\text{i}2\pi f_c t}\big) \\ &= 2\,\text{Re}\big(x^*_{\text{BB}}(-t)\,e^{\text{i}2\pi f_c t}\big) \\ &= 2\,\text{Re}\big(\overleftarrow{x}^*_{\text{BB}}(t)\,e^{\text{i}2\pi f_c t}\big), \quad t \in \mathbb{R}, \end{aligned} \tag{1}$$

where in the third equality we used the fact that the real part of a complex number is equal to the real part of its complex conjugate. Since $\mathbf{x}_{\text{BB}}$ is bandlimited to $W/2$ Hz, so is $\overleftarrow{\mathbf{x}}^*_{\text{BB}}$. We thus conclude from (1) using Proposition 7.6.9 that the baseband representation of $\overleftarrow{\mathbf{x}}_{\text{PB}}$ is $\overleftarrow{\mathbf{x}}^*_{\text{BB}}$.

(ii) From the first part we know that the baseband representation of $\mathbf{y}_{\text{PB}}$ is $\overleftarrow{\mathbf{y}}^*_{\text{BB}}$. Consequently, by Theorem 7.6.10,

$$\begin{aligned} \langle \mathbf{x}_{\text{PB}}, \overleftarrow{\mathbf{y}}_{\text{PB}} \rangle &= 2\,\text{Re}\big(\langle \mathbf{x}_{\text{BB}}, \overleftarrow{\mathbf{y}}^*_{\text{BB}} \rangle\big) \\ &= 2\,\text{Re}\bigg(\int_{-\infty}^{\infty} x_{\text{BB}}(t)\,y_{\text{BB}}(-t)\,\text{d}t\bigg) \\ &= 2\,\text{Re}\Big(\big(\mathbf{x}_{\text{BB}} \star \mathbf{y}_{\text{BB}}\big)(0)\Big). \end{aligned}$$

(i) The proof relies on the fact that conjugation and integration commute (Proposition 2.3.1).

If $\mathbf{x}$ is a real-valued signal with Fourier Transform $\hat{x}(\cdot)$, then

$$\begin{aligned}\hat{x}(f) &= \int_{-\infty}^{\infty} x(t)e^{-\mathrm{i}2\pi ft}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} x^*(t)\left(e^{\mathrm{i}2\pi ft}\right)^*\,\mathrm{d}t \\ &= \left(\int_{-\infty}^{\infty} x(t)e^{\mathrm{i}2\pi ft}\,\mathrm{d}t\right)^* \\ &= \hat{x}^*(-f), \quad f \in \mathbb{R},\end{aligned}$$

where the first equality is the definition of the Fourier Transform; the second follows from our assumption that $\mathbf{x}$ is real and is thus equal to its complex conjugate; the third equality follows by swapping integration and conjugation; and the fourth equality follows again from the definition of the Fourier Transform.

(ii) If $\mathbf{x}$ is purely imaginary, then we can define the real signal $\mathbf{y}$ as

$$y(t) = \frac{1}{\mathrm{i}}x(t), \quad t \in \mathbb{R}.$$

Since $\mathbf{y}$ is real, its FT $\hat{\mathbf{y}}$ is conjugate symmetric. And by the linearity of the FT, $\hat{\mathbf{x}} = \mathrm{i}\hat{\mathbf{y}}$. Thus

$$\begin{aligned}\hat{x}(f) &= \mathrm{i}\hat{y}(f) \\ &= (-\mathrm{i})^*\hat{y}(f) \\ &= -\big(\mathrm{i}\hat{y}(-f)\big)^* \\ &= -\hat{x}^*(-f), \qquad f \in \mathbb{R}.\end{aligned}$$

(iii) Let $\mathbf{x}$ be a complex integrable signal. Then $\mathbf{x}$ can be expressed as a sum

$$\mathbf{x} = \mathbf{x}_{\mathrm{R}} + \mathbf{x}_{\mathrm{I}},$$

where $\mathbf{x}_{\mathrm{R}}$ is the integrable real signal

$$x_{\mathrm{R}}(t) = \mathrm{Re}\big(x(t)\big), \quad t \in \mathbb{R},$$

and where $\mathbf{x}_{\mathrm{I}}$ is the integrable purely-imaginary signal

$$x_{\mathrm{I}}(t) = \mathrm{i}\,\mathrm{Im}\big(x(t)\big), \quad t \in \mathbb{R}.$$

By the linearity of the FT,

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_{\mathrm{R}} + \hat{\mathbf{x}}_{\mathrm{I}},$$

thus expressing the FT of $\mathbf{x}$ as a sum of the conjugate symmetric function $\hat{\mathbf{x}}_{\mathrm{R}}$ (because $\mathbf{x}_{\mathrm{R}}$ is real) and the conjugate antisymmetric function $\hat{\mathbf{x}}_{\mathrm{I}}$ (because $\mathbf{x}_{\mathrm{I}}$ is purely imaginary).

To prove uniqueness, suppose that

$$\hat{x}(f) = g_{\mathrm{cs}}(f) + g_{\mathrm{cas}}(f), \quad f \in \mathbb{R}, \tag{2}$$

where $\mathbf{g}_{\text{cs}}$ is conjugate symmetric

$$g_{\text{cs}}(-f) = g_{\text{cs}}^*(f), \quad f \in \mathbb{R}, \tag{3}$$

and $\mathbf{g}_{\text{cas}}$ is conjugate antisymmtric

$$g_{\text{cas}}(-f) = -g_{\text{cas}}^*(f), \quad f \in \mathbb{R}. \tag{4}$$

Then

$$\begin{aligned} \hat{x}^*(-f) &= g_{\text{cs}}^*(-f) + g_{\text{cas}}^*(-f) \\ &= g_{\text{cs}}(f) - g_{\text{cas}}(f), \quad f \in \mathbb{R}, \end{aligned} \tag{5}$$

where the second equality follows from (3) and (4).

Adding (2) and (5) we obtain the unique solution

$$g_{\text{cs}}(f) = \frac{\hat{x}(f) + \hat{x}^*(-f)}{2}, \quad f \in \mathbb{R}, \tag{6}$$

and substituting (5) from (2) yields the unique solution

$$g_{\text{cas}}(f) = \frac{\hat{x}(f) - \hat{x}^*(-f)}{2}, \quad f \in \mathbb{R}. \tag{7}$$

Problem 3 *Phase Shift*

(i) We first express $z_{\text{PB}}(\cdot)$ in the more suggesting form

$$\begin{aligned} z_{\text{PB}}(t) &= x(t)\sin(2\pi f_{\text{c}} t + \phi) \\ &= 2\,\text{Re}\Big(-\frac{\mathrm{i}}{2}\,x(t)\,e^{\mathrm{i}(2\pi f_{\text{c}} t + \phi)}\Big) \\ &= 2\,\text{Re}\Big(-\frac{\mathrm{i}}{2}\,x(t)\,e^{\mathrm{i}\phi}\,e^{\mathrm{i}2\pi f_{\text{c}} t}\Big), \quad t \in \mathbb{R}, \end{aligned}$$

where in the second equality we used the fact that $\mathbf{x}$ is real. It now follows from Proposition 7.6.9 that the baseband representation $\mathbf{z}_{\text{BB}}$ of $\mathbf{z}_{\text{PB}}$ is

$$\mathbf{z}_{\text{BB}} = -\frac{\mathrm{i}}{2}\,e^{\mathrm{i}\phi}\mathbf{x}.$$

(ii) By the pevious part,

$$\hat{z}_{\text{BB}}(f) = -\frac{\mathrm{i}}{2}\,e^{\mathrm{i}\phi}\hat{x}(f), \quad f \in \mathbb{R}.$$

Consequently, by Proposition 7.6.8,

$$\hat{z}_{\text{PB}}(f) = -\frac{\mathrm{i}}{2}\,e^{\mathrm{i}\phi}\hat{x}(f - f_{\text{c}}) + \frac{\mathrm{i}}{2}\,e^{-\mathrm{i}\phi}\hat{x}^*(-f - f_{\text{c}}), \quad f \in \mathbb{R}.$$

Since $\mathbf{x}$ is real, its FT is conjugate symmetric, so we can also write this as

$$\hat{z}_{\text{PB}}(f) = -\frac{\mathrm{i}}{2}\,e^{\mathrm{i}\phi}\hat{x}(f - f_{\text{c}}) + \frac{\mathrm{i}}{2}\,e^{-\mathrm{i}\phi}\hat{x}(f + f_{\text{c}}), \quad f \in \mathbb{R}.$$

(i) Let $\mathbf{x}_{\mathrm{PB}}$ be a real integrable passband signal that is bandlimited to W Hz around the carrier frequency f_{c}. The FT $\hat{\mathbf{x}}_{\mathrm{BB}}$ of its baseband repsentation $\mathbf{x}_{\mathrm{BB}}$ is related to its FT $\hat{\mathbf{x}}_{\mathrm{PB}}$ via the relation

$$\hat{x}_{\mathrm{BB}}(f) = \hat{x}_{\mathrm{PB}}(f + f_{\mathrm{c}})\,\mathrm{I}\Big\{|f| \le \frac{W}{2}\Big\}, \quad f \in \mathbb{R} \tag{8}$$

(Proposition 7.6.5). Consequently, the condition

$$\hat{x}_{\mathrm{PB}}(f_{\mathrm{c}} - \delta) = \hat{x}^{*}_{\mathrm{PB}}(f_{\mathrm{c}} + \delta), \quad |\delta| \le \frac{W}{2}$$

is equivalent to the condition

$$\hat{x}_{\mathrm{BB}}(-\delta) = \hat{x}^{*}_{\mathrm{BB}}(\delta), \quad |\delta| \le \frac{W}{2}. \tag{9}$$

It remains to argue that this latter condition is equivalent to the condition that $\mathbf{x}_{\mathrm{BB}}$ is real.

One direction is obvious: if $\mathbf{x}_{\mathrm{BB}}$ is real, then its FT is conjugate symmetric (Exercise 6.1).

The other direction is almost as obvious: The signal $\mathbf{x}_{\mathrm{BB}}$ is an integrable signal that is bandlimited to $W/2$ Hz (Theorem 7.6.5). Consequently, it is equal to the IFT of its FT $\hat{\mathbf{x}}_{\mathrm{BB}}$ (Proposition 6.4.10). Thus, if (9) holds then

$$\begin{aligned}
x_{\mathrm{BB}}(t) &= \int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{\mathrm{BB}}(f)\, e^{\mathrm{i}2\pi f t}\,\mathrm{d}f \\
&= \left(\int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}^{*}_{\mathrm{BB}}(f)\, e^{-\mathrm{i}2\pi f t}\,\mathrm{d}f\right)^{*} \\
&= \left(\int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{\mathrm{BB}}(-f)\, e^{-\mathrm{i}2\pi f t}\,\mathrm{d}f\right)^{*} \\
&= \left(\int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{\mathrm{BB}}(\tilde{f})\, e^{\mathrm{i}2\pi \tilde{f} t}\,\mathrm{d}\tilde{f}\right)^{*} \\
&= x^{*}_{\mathrm{BB}}(t), \quad t \in \mathbb{R}.
\end{aligned}$$

(ii) By (8), the condition

$$\hat{x}_{\mathrm{PB}}(f_{\mathrm{c}} - \delta) = -\hat{x}^{*}_{\mathrm{PB}}(f_{\mathrm{c}} + \delta), \quad |\delta| \le \frac{W}{2}$$

is equivalent to the condition

$$\hat{x}_{\mathrm{BB}}(-\delta) = -\hat{x}^{*}_{\mathrm{BB}}(\delta), \quad |\delta| \le \frac{W}{2}. \tag{10}$$

It remains to argue that this latter condition is equivalent to the condition that $\mathbf{x}_{\mathrm{BB}}$ is purely imaginary.

One direction is obvious: if $\mathbf{x}_{\mathrm{BB}}$ is purely imaginary, then its FT is conjugate antisymmetric (Exercise 6.1).

The other direction is almost as obvious: The signal $\mathbf{x}_{\mathrm{BB}}$ is an integrable signal that is bandlimited to $W/2$ Hz (Theorem 7.6.5). Consequently, it is equal to the IFT of its FT $\hat{\mathbf{x}}_{\mathrm{BB}}$

(Proposition 6.4.10). Thus, if (10) holds then

$$\begin{aligned} x_{\mathrm{BB}}(t) &= \int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{\mathrm{BB}}(f)\, e^{i2\pi f t} \,\mathrm{d}f \\ &= \left(\int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}^*_{\mathrm{BB}}(f)\, e^{-i2\pi f t} \,\mathrm{d}f\right)^* \\ &= -\left(\int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{\mathrm{BB}}(-f)\, e^{-i2\pi f t} \,\mathrm{d}f\right)^* \\ &= -\left(\int_{-\frac{W}{2}}^{\frac{W}{2}} \hat{x}_{\mathrm{BB}}(\tilde{f})\, e^{i2\pi \tilde{f} t} \,\mathrm{d}\tilde{f}\right)^* \\ &= -x^*_{\mathrm{BB}}(t), \quad t \in \mathbb{R}. \end{aligned}$$

Problem 5 ***Symmetry around the Carrier Frequency***

Let $\mathbf{x}_{\mathrm{PB}}$ be a real integrable passband signal that is bandlimited to W Hz around the carrier frequency f_{c}.

(i) If $\mathbf{x}_{\mathrm{PB}}$ is equal to the signal $t \mapsto w(t)\cos(2\pi f_{\mathrm{c}} t)$ for some real integrable signal $\mathbf{w}$ that is bandlimited to $W/2$ Hz, then, by Proposition 7.6.9, $\mathbf{x}_{\mathrm{BB}}$ is equal to $\mathbf{w}/2$, because

$$\begin{aligned} x_{\mathrm{PB}}(t) &= w(t)\cos(2\pi f_{\mathrm{c}} t) \\ &= 2\,\mathrm{Re}\Big(\frac{w(t)}{2} e^{i2\pi f_{\mathrm{c}} t}\Big), \end{aligned}$$

where the second equality follows because $\mathbf{w}$ is real. In this case $\mathbf{x}_{\mathrm{PB}}$ has a real baseband representation, namely $\mathbf{w}/2$, and by Exercise 7.1

$$\hat{x}_{\mathrm{PB}}(f_{\mathrm{c}} + \delta) = \hat{x}^*_{\mathrm{PB}}(f_{\mathrm{c}} - \delta), \quad |\delta| \leq \frac{W}{2}. \tag{11}$$

Conversely, if (11) holds, then by Exercise 7.1 the baseband representation of $\mathbf{x}_{\mathrm{PB}}$ is real, so

$$\begin{aligned} x_{\mathrm{PB}}(t) &= 2\,\mathrm{Re}\Big(x_{\mathrm{BB}}(t) e^{i2\pi f_{\mathrm{c}} t}\Big) \\ &= 2x_{\mathrm{BB}}(t)\cos(2\pi f_{\mathrm{c}} t), \end{aligned}$$

and $\mathbf{x}_{\mathrm{PB}}$ has the desired representation with $\mathbf{w} = 2\mathbf{x}_{\mathrm{BB}}$.

(ii) If $\mathbf{x}_{\mathrm{PB}}$ is equal to the signal $t \mapsto w(t)\sin(2\pi f_{\mathrm{c}} t)$ for some real integrable signal $\mathbf{w}$ that is bandlimited to $W/2$ Hz, then, by Proposition 7.6.9, $\mathbf{x}_{\mathrm{BB}}$ is equal to $\mathbf{w}/(2i)$, because

$$\begin{aligned} x_{\mathrm{PB}}(t) &= w(t)\sin(2\pi f_{\mathrm{c}} t) \\ &= 2\,\mathrm{Re}\Big(\frac{w(t)}{2i} e^{i2\pi f_{\mathrm{c}} t}\Big), \end{aligned}$$

where the second equality follows because $\mathbf{w}$ is real. In this case $\mathbf{x}_{\mathrm{PB}}$ has a purely imaginary baseband representation, namely $\mathbf{w}/(2i)$, and by Exercise 7.1

$$\hat{x}_{\mathrm{PB}}(f_{\mathrm{c}} + \delta) = -\hat{x}^*_{\mathrm{PB}}(f_{\mathrm{c}} - \delta), \quad |\delta| \leq \frac{W}{2}. \tag{12}$$

Conversely, if (12) holds, then by Exercise 7.1 the baseband representation of $\mathbf{x}_{\text{PB}}$ is purely imaginary, so

$$\begin{aligned} x_{\text{PB}}(t) &= 2\,\text{Re}\Bigl(x_{\text{BB}}(t)e^{i2\pi f_{\text{c}}t}\Bigr) \\ &= 2i x_{\text{BB}}(t)\sin(2\pi f_{\text{c}}t), \end{aligned}$$

and $\mathbf{x}_{\text{PB}}$ has the desired representation with $\mathbf{w} = 2i\mathbf{x}_{\text{BB}}$.