Model Answers to Exercise 3 of March 3, 2015

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Problem 1

Separation between Signals

(i) Since the norm is nonnegative, the condition $\|v - u_1\|_2 = \|v - u_2\|_2$ is equivalent to the condition

$$\|v - u_1\|_2^2 = \|v - u_2\|_2^2. \tag{1}$$

Moreover,

$$\|v - u_1\|_2^2 = \|v\|_2^2 + \|u_1\|_2^2 - 2 \text{Re}(\langle v, u_1 \rangle), \tag{2}$$

$$\|v - u_2\|_2^2 = \|v\|_2^2 + \|u_2\|_2^2 - 2 \text{Re}(\langle v, u_2 \rangle). \tag{3}$$

Taking the difference of (3) and (2) we see that the relation defining the subspace (1) can be written equivalently as

$$0 = \|u_2\|_2^2 - \|u_1\|_2^2 - 2 \text{Re}(\langle v, u_2 - u_1 \rangle)$$

or

$$\text{Re}(\langle v, u_2 - u_1 \rangle) = \frac{\|u_2\|_2^2 - \|u_1\|_2^2}{2}.$$

(ii) In general, $\mathcal{V}$ is not a linear subspace. Indeed, if $\|u_1\|_2 \neq \|u_2\|_2$ then the all-zero signal is not in $\mathcal{V}$.

(iii) Let $w = (u_1 + u_2)/2$. We have that

$$\|w - u_1\|_2 = \|u_2 - u_1\|_2/2,$$

$$\|w - u_2\|_2 = \|u_1 - u_2\|_2/2.$$

Since $\|(u_2 - u_1)/2\|_2 = \|(u_1 - u_2)/2\|_2$, we have that $\|w - u_1\|_2 = \|w - u_2\|_2$ and thus $w = (u_1 + u_2)/2 \in \mathcal{V}$.

Problem 2

Orthogonal Subspace

Let $\mathcal{U}$ denote the set of all signals $u \in L_2$ which are orthogonal to the signals $v_1, \ldots, v_n$. By definition, the set $\mathcal{U}$ is a subset of $L_2$. We have to show that it is closed under linear combinations.
For arbitrary $u_1$ and $u_2$ in $\mathcal{U}$ and arbitrary complex numbers $\alpha_1$ and $\alpha_2$, by the linearity of the inner product:

$$\langle \alpha_1 u_1 + \alpha_2 u_2, v_\ell \rangle = \alpha_1 \langle u_1, v_\ell \rangle + \alpha_2 \langle u_2, v_\ell \rangle = 0, \quad \ell \in \{1, \ldots, n\},$$

since

$$\langle u_j, v_\ell \rangle = 0, \quad j \in \{1, 2\}, \ell \in \{1, \ldots, n\}.$$  

Thus, any sum of the form $\alpha_1 u_1 + \alpha_2 u_2$ is an element of $\mathcal{U}$. This proves that $\mathcal{U}$ is closed under linear combinations and hence forms a subspace of $\mathcal{L}_2$.

**Problem 3**

*Constructing an Orthonormal Basis*

(i) See Figure 0.1

(ii) It can be verified that the signals

$$\phi_1 \triangleq \frac{s_1}{\sqrt{T_s}}, \quad \phi_2 \triangleq \frac{s_2}{\sqrt{T_s}}, \quad \phi_3 \triangleq \frac{2s_3 - s_2}{\sqrt{T_s}}, \quad \phi_4 \triangleq \frac{2s_4 - s_1}{\sqrt{T_s}},$$

form an orthonormal basis for $\text{span} (s_1, s_2, s_3, s_4)$.

(iii) The signals $s_1, s_2, s_3, s_4$ can be expressed as:

$$s_1 = \sqrt{T_s} \phi_1, \quad s_2 = \sqrt{T_s} \phi_2, \quad s_3 = \frac{2}{T_s} (\phi_3 + \phi_2), \quad s_4 = \frac{2}{T_s} (\phi_4 + \phi_1).$$

**Problem 4**

*Expansion of a Function*

Using (6.33) and (6.34) we obtain that

$$t \mapsto \operatorname{sinc}\left(\frac{t}{2}\right)$$

is of $\text{FT}$ $f \mapsto 21\{|f| \leq \frac{1}{4}\}$.
Consequently, by Proposition 6.6.1,
\[ t \mapsto \text{sinc}^2\left(\frac{t}{2}\right) \] is of FT \( f \mapsto (2 - 4|f|) I\{ |f| \leq \frac{1}{2} \} \), i.e., of bandwidth \( 1/2 \) Hz. Consequently, if we define \( T \) as the reciprocal of twice the bandwidth, then \( T = 1 \). By Proposition 8.4.2 we thus obtain the orthonormal expansion of \( t \mapsto \text{sinc}^2\left(\frac{t}{2}\right) \)

\[
\sum_{\ell=-\infty}^{\infty} \text{sinc}^2\left(-\frac{\ell}{2}\right) \text{sinc}(t + \ell).
\]

Since
\[
\text{sinc}^2\left(-\frac{\ell}{2}\right) = \begin{cases} 
1 & \text{if } \ell = 0, \\
0 & \text{if } \ell \text{ is nonzero and even} \\
\frac{4}{\pi^2\ell^2} & \text{if } \ell \text{ is odd},
\end{cases}
\]

the expansion can also be written as
\[
\text{sinc}(t) + \sum_{\ell \text{ odd}} \frac{4}{\pi^2\ell^2} \text{sinc}(t + \ell).
\]

**Problem 5**  
**Inner Product with a Bandlimited Signal**

The signal \( y \) can be written as

\[
y = y \ast \text{LPFW} + (y - y \ast \text{LPFW})
\]

\[= y_{\text{LPF}} + (y - y_{\text{LPF}}),\]

where \( y_{\text{LPF}} \) is defined as the convolution \( y \ast \text{LPFW} \) and, by Proposition 6.4.7, is an energy-limited signal that is bandlimited to \( W \) Hz. Consequently, by Theorem 8.6.1 Part (ii),

\[
\langle x, y_{\text{LPF}} \rangle = T_s \sum_{\ell=-\infty}^{\infty} x(\ell T_s) y^*_{\text{LPF}}(\ell T_s).
\]

Since the FT of \( y - y_{\text{LPF}} \) vanishes in the band \([-W, W]\), and since \( x \) is bandlimited to \( W \) Hz, it follows from Parseval’s Theorem that

\[
\langle x, y - y_{\text{LPF}} \rangle = 0.
\]

Hence

\[
\langle x, y \rangle = \langle x, y_{\text{LPF}} + y - y_{\text{LPF}} \rangle
\]

\[= \langle x, y_{\text{LPF}} \rangle + \langle x, y - y_{\text{LPF}} \rangle
\]

\[= \langle x, y_{\text{LPF}} \rangle
\]

\[= T_s \sum_{\ell=-\infty}^{\infty} x(\ell T_s) y^*_{\text{LPF}}(\ell T_s).
\]

An alternative solution goes as follows. Let \( T_s = 1/(2W) \). From the \( L_2 \)-Sampling Theorem

\[
\lim_{L \to \infty} \int_{-\infty}^{\infty} \left| x(t) - \sum_{\ell=-L}^{L} x(-\ell T_s) \text{sinc}\left(\frac{t}{T_s} + \ell\right) \right|^2 dt = 0.
\]
For every $L \in \mathbb{N}$ define the signal
\[
    u_L(t) = x(t) - \sum_{\ell=-L}^{L} x(\ell T_s \mathrm{sinc}\left(\frac{t}{T_s} + \ell\right), \quad t \in \mathbb{R}.
\]

Then $\|u_L\|_2 \to 0$ as $L \to \infty$. Fix some $\beta \in \mathbb{C}$. For every energy-limited $y$ we have by (4.14)
\[
    (\|y\|_2 - |\beta| \|u_L\|_2)^2 \leq \|\beta u_L + y\|_2^2 \leq (\|y\|_2 + |\beta| \|u_L\|_2)^2,
\]
so $\|\beta u_L + y\|_2^2 \to \|y\|_2^2$ as $L \to \infty$. We now expand $\|\beta u_L + y\|_2^2$:
\[
    \|\beta u_L + y\|_2^2 = \int_{-\infty}^{\infty} \left| \beta \left( x(t) - \sum_{\ell=-L}^{L} x(-\ell T_s \mathrm{sinc}\left(\frac{t}{T_s} + \ell\right) \right) + y(t) \right|^2 dt \\
    = |\beta|^2 \int_{-\infty}^{\infty} \left| x(t) - \sum_{\ell=-L}^{L} x(-\ell T_s \mathrm{sinc}\left(\frac{t}{T_s} + \ell\right) \right|^2 dt + \int_{-\infty}^{\infty} |y(t)|^2 dt \\
    + 2 \Re \left( \beta \int_{-\infty}^{\infty} x(t) y^*(t) dt - \beta \int_{-\infty}^{\infty} \sum_{\ell=-L}^{L} x(-\ell T_s \mathrm{sinc}\left(\frac{t}{T_s} + \ell\right) y^*(t) dt \right) \\
    = |\beta|^2 \|u_L\|_2^2 + \|y\|_2^2 \\
    + 2 \Re \left( \beta \langle x, y \rangle - \beta \sum_{\ell=-L}^{L} x(-\ell T_s) \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{t}{T_s} + \ell\right) y^*(t) dt \right).
\]

Letting $L$ tend to infinity and recalling that $\|u_L\|_2 \to 0$ and $\|\beta u_L + y\|_2^2 \to \|y\|_2^2$ we conclude that
\[
    2 \Re \left( \beta \langle x, y \rangle - \beta \sum_{\ell=-L}^{L} x(-\ell T_s) \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{t}{T_s} + \ell\right) y^*(t) dt \right) = 0.
\]

Since this has to hold for every $\beta \in \mathbb{C}$,
\[
    \langle x, y \rangle = \sum_{\ell=-L}^{L} x(-\ell T_s) \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{t}{T_s} + \ell\right) y^*(t) dt. \tag{4}
\]

We further have
\[
    \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{t}{T_s} + \ell\right) y^*(t) dt = \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{1}{T_s} (-\ell T_s - t)\right) y^*(t) dt \\
    = \left( y^* \ast \left( t \mapsto \mathrm{sinc}\left(\frac{t}{T_s}\right) \right) \right) (-\ell T_s) \\
    = T_s y^*_{\mathrm{LPF}}(-\ell T_s), \tag{5}
\]
where the first equality follows from the fact that $t \mapsto \mathrm{sinc}(t)$ is symmetric (even). Inserting (5) in the RHS of (4), we finally obtain
\[
    \langle x, y \rangle = T_s \sum_{\ell=-\infty}^{\infty} x(-\ell T_s) y^*_{\mathrm{LPF}}(-\ell T_s) = T_s \sum_{\ell=-\infty}^{\infty} x(\ell T_s) y^*_{\mathrm{LPF}}(\ell T_s).
\]

**Problem 6**

**Inner Product between Passband Signals**

By Theorem 7.7.12,
\[
    \langle x_{\mathrm{PB}}, y_{\mathrm{PB}} \rangle = 2 \Re \left( \langle x_{\mathrm{BB}}, y_{\mathrm{BB}} \rangle \right). \tag{6}
\]

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Also by Theorem 7.7.12, $x_{BB}$ and $y_{BB}$ are energy-limited signals that are bandlimited to $W/2$ Hz. Thus, by the Sampling Theorem,

$$\langle x_{BB}, y_{BB} \rangle = T \sum_{\ell=-\infty}^{\infty} x_{BB}(\ell T) y_{BB}^*(\ell T),$$

where $T = 1/W$. Substituting (7) into (6) gives the desired equation.