Model Answers to Exercise 3 of March 8, 2016

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1

(i) Since the norm is nonnegative, the condition $\|v - u_1\|_2 = \|v - u_2\|_2$ is equivalent to the condition

$$\|v - u_1\|_2^2 = \|v - u_2\|_2^2. \tag{1}$$

Moreover,

$$\|v - u_1\|_2^2 = \|v\|_2^2 + \|u_1\|_2^2 - 2\Re(\langle v, u_1 \rangle), \tag{2}$$

$$\|v - u_2\|_2^2 = \|v\|_2^2 + \|u_2\|_2^2 - 2\Re(\langle v, u_2 \rangle). \tag{3}$$

Taking the difference of (3) and (2) we see that the relation defining the subspace (1) can be written equivalently as

$$0 = \|u_2\|_2^2 - \|u_1\|_2^2 - 2\Re(\langle v, u_2 - u_1 \rangle)$$

or

$$\Re(\langle v, u_2 - u_1 \rangle) = \frac{\|u_2\|_2^2 - \|u_1\|_2^2}{2}.$$ 

(ii) In general, $V$ is not a linear subspace. Indeed, if $\|u_1\|_2 \neq \|u_2\|_2$ then the all-zero signal is not in $V$.

(iii) Let $w = (u_1 + u_2)/2$. We have that

$$\|w - u_1\|_2 = \|(u_2 - u_1)/2\|_2,$$

$$\|w - u_2\|_2 = \|(u_1 - u_2)/2\|_2.$$

Since $\|(u_2 - u_1)/2\|_2 = \|(u_1 - u_2)/2\|_2$, we have that $\|w - u_1\|_2 = \|w - u_2\|_2$ and thus $w = (u_1 + u_2)/2 \in V$.

Problem 2

Orthogonal Subspace

Let $U$ denote the set of all signals $u \in L_2$ which are orthogonal to the signals $v_1, \ldots, v_n$. By definition, the set $U$ is a subset of $L_2$. We have to show that it is closed under linear combinations.
For arbitrary $u_1$ and $u_2$ in $\mathcal{U}$ and arbitrary complex numbers $\alpha_1$ and $\alpha_2$, by the linearity of the inner product:

$$\langle \alpha_1 u_1 + \alpha_2 u_2, v_\ell \rangle = \alpha_1 \langle u_1, v_\ell \rangle + \alpha_2 \langle u_2, v_\ell \rangle = 0, \quad \ell \in \{1, \ldots, n\},$$

since

$$\langle u_j, v_\ell \rangle = 0, \quad j \in \{1, 2\}, \ell \in \{1, \ldots, n\}.$$

Thus, any sum of the form $\alpha_1 u_1 + \alpha_2 u_2$ is an element of $\mathcal{U}$. This proves that $\mathcal{U}$ is closed under linear combinations and hence forms a subspace of $L_2$.

**Problem 3**

**Constructing an Orthonormal Basis**

(i) See Figure 0.1

![Figure 0.1](image)

Figure 0.1:

(ii) It can be verified that the signals

$$\phi_1 \triangleq \frac{s_1}{\sqrt{T_s}},$$

$$\phi_2 \triangleq \frac{s_2}{\sqrt{T_s}},$$

$$\phi_3 \triangleq \frac{2s_3 - s_2}{\sqrt{T_s}},$$

$$\phi_4 \triangleq \frac{2s_4 - s_1}{\sqrt{T_s}},$$

form an orthonormal basis for $\text{span}(s_1, s_2, s_3, s_4)$.

(iii) The signals $s_1, s_2, s_3, s_4$ can be expressed as:

$$s_1 = \sqrt{T_s} \phi_1,$$

$$s_2 = \sqrt{T_s} \phi_2,$$

$$s_3 = \frac{1}{2} (\phi_3 + \phi_2),$$

$$s_4 = \frac{1}{2} (\phi_4 + \phi_1).$$

**Problem 4**

**Constructing an Orthonormal Basis**

Using (6.33) and (6.34) we obtain that

$$t \mapsto \text{sinc}\left(\frac{t}{2}\right)$$

is of FT $f \mapsto 21\{|f| \leq \frac{1}{4}\}$. 

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Consequently, by Proposition 6.6.1,

\[ t \mapsto \text{sinc}^2 \left( \frac{t}{2} \right) \] is of FT \( f \mapsto (2 - 4|f|) I\{|f| \leq \frac{1}{2} \}, \)
i.e., of bandwidth 1/2 Hz. Consequently, if we define \( T \) as the reciprocal of twice the bandwidth, then \( T = 1 \). By Theorem 8.4.3 (i) we thus obtain the orthonormal expansion of \( t \mapsto \text{sinc}^2 \left( \frac{t}{2} \right) \)
\[
\sum_{\ell=-\infty}^{\infty} \text{sinc}^2 \left( -\frac{\ell}{2} \right) \text{sinc}(t + \ell).
\]
Since
\[
\text{sinc}^2 \left( -\frac{\ell}{2} \right) = \begin{cases} 
1 & \text{if } \ell = 0, \\
0 & \text{if } \ell \text{ is nonzero and even,} \\
\frac{4}{\pi^2 \ell^2} & \text{if } \ell \text{ is odd,}
\end{cases}
\]
the expansion can also be written as
\[
\text{sinc}(t) + \sum_{\ell \text{ odd}} \frac{4}{\pi^2 \ell^2} \text{sinc}(t + \ell).
\]

Problem 5

**Inner Product with a Bandlimited Signal**

The signal \( y \) can be written as
\[
y = y \ast \text{LPF}_W + (y - y \ast \text{LPF}_W)
= y_{\text{LPF}} + (y - y_{\text{LPF}}),
\]
where \( y_{\text{LPF}} \) is defined as the convolution \( y \ast \text{LPF}_W \) and, by Proposition 6.4.7, is an energy-limited signal that is bandlimited to \( W \) Hz. Consequently, by Theorem 8.6.1 Part (ii),
\[
\langle x, y_{\text{LPF}} \rangle = T_s \sum_{\ell=-\infty}^{\infty} x(\ell T_s) y^*_{\text{LPF}}(\ell T_s).
\]
Since the FT of \( y - y_{\text{LPF}} \) vanishes in the band \([-W, W]\), and since \( x \) is bandlimited to \( W \) Hz, it follows from Parseval’s Theorem that
\[
\langle x, y - y_{\text{LPF}} \rangle = 0.
\]
Hence
\[
\langle x, y \rangle = \langle x, y_{\text{LPF}} + y - y_{\text{LPF}} \rangle
= \langle x, y_{\text{LPF}} \rangle + \langle x, y - y_{\text{LPF}} \rangle
= \langle x, y_{\text{LPF}} \rangle
= T_s \sum_{\ell=-\infty}^{\infty} x(\ell T_s) y^*_{\text{LPF}}(\ell T_s).
\]
An alternative solution goes as follows. Let \( T_s = 1/(2W) \). From the \( \mathcal{L}_2 \)-Sampling Theorem
\[
\lim_{L \to \infty} \int_{-\infty}^{\infty} \left| x(t) - \sum_{\ell=-L}^{L} x(-\ell T_s) \text{sinc} \left( \frac{t}{T_s} + \ell \right) \right|^2 dt = 0.
\]
For every $L \in \mathbb{N}$ define the signal
\[ u_L(t) = x(t) - \sum_{\ell=-L}^{L} x(\ell T_s) \text{sinc} \left( \frac{t}{T_s} + \ell \right), \quad t \in \mathbb{R}. \]

Then $\|u_L\|_2 \to 0$ as $L \to \infty$. Fix some $\beta \in \mathbb{C}$. For every energy-limited $y$ we have by (4.14)
\[ (\|y\|_2 - |\beta| \|u_L\|_2)^2 \leq \|\beta u_L + y\|_2^2 \leq (\|y\|_2 + |\beta| \|u_L\|_2)^2, \]
so $\|\beta u_L + y\|_2^2 \to \|y\|_2^2$ as $L \to \infty$. We now expand $\|\beta u_L + y\|_2^2$:
\[
\|\beta u_L + y\|_2^2 = \int_{-\infty}^{\infty} \left| \beta \left( x(t) - \sum_{\ell=-L}^{L} x(-\ell T_s) \text{sinc} \left( \frac{t}{T_s} + \ell \right) \right) + y(t) \right|^2 dt \\
= |\beta|^2 \int_{-\infty}^{\infty} \left| x(t) - \sum_{\ell=-L}^{L} x(-\ell T_s) \text{sinc} \left( \frac{t}{T_s} + \ell \right) \right|^2 dt + \int_{-\infty}^{\infty} |y(t)|^2 dt \\
+ 2 \text{Re} \left( \beta \int_{-\infty}^{\infty} x(t)y^*(t) dt - \beta \int_{-\infty}^{\infty} \sum_{\ell=-L}^{L} x(-\ell T_s) \text{sinc} \left( \frac{t}{T_s} + \ell \right) y^*(t) dt \right) \\
= |\beta|^2 \|u_L\|_2^2 + \|y\|_2^2 \\
+ 2 \text{Re} \left( \beta \langle x, y \rangle - \beta \sum_{\ell=-L}^{L} x(-\ell T_s) \int_{-\infty}^{\infty} \text{sinc} \left( \frac{t}{T_s} + \ell \right) y^*(t) dt \right). 
\]

Letting $L$ tend to infinity and recalling that $\|u_L\|_2 \to 0$ and $\|\beta u_L + y\|_2^2 \to \|y\|_2^2$ we conclude that
\[ 2 \text{Re} \left( \beta \langle x, y \rangle - \beta \sum_{\ell=-L}^{L} x(-\ell T_s) \int_{-\infty}^{\infty} \text{sinc} \left( \frac{t}{T_s} + \ell \right) y^*(t) dt \right) = 0. \]

Since this has to hold for every $\beta \in \mathbb{C}$,
\[ \langle x, y \rangle = \sum_{\ell=-L}^{L} x(-\ell T_s) \int_{-\infty}^{\infty} \text{sinc} \left( \frac{t}{T_s} + \ell \right) y^*(t) dt. \]  

We further have
\[
\int_{-\infty}^{\infty} \text{sinc} \left( \frac{t}{T_s} + \ell \right) y^*(t) dt = \int_{-\infty}^{\infty} \text{sinc} \left( \frac{1}{T_s}(-\ell T_s - t) \right) y^*(t) dt \\
= \left( y^* \ast \left( t \mapsto \text{sinc} \left( \frac{t}{T_s} \right) \right) \right)(-\ell T_s) \\
= T_s y_{\text{LPF}}^*(-\ell T_s), 
\]
where the first equality follows from the fact that $t \mapsto \text{sinc}(t)$ is symmetric (even). Inserting (5) in the RHS of (4), we finally obtain
\[ \langle x, y \rangle = T_s \sum_{\ell=-\infty}^{\infty} x(-\ell T_s)y_{\text{LPF}}^*(-\ell T_s) = T_s \sum_{\ell=-\infty}^{\infty} x(\ell T_s)y_{\text{LPF}}^*(\ell T_s). \]

**Problem 6**

*Inner Product between Passband Signals*

By Theorem 7.7.12,
\[ \langle x_{\text{PB}}, y_{\text{PB}} \rangle = 2 \text{Re} \left( \langle x_{\text{BB}}, y_{\text{BB}} \rangle \right). \]  

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Also by Thorem 7.7.12, $x_{BB}$ and $y_{BB}$ are energy-limited signals that are bandlimited to $W/2$ Hz. Thus, by the Sampling Theorem,

$$\langle x_{BB}, y_{BB} \rangle = T \sum_{\ell=-\infty}^{\infty} x_{BB}(\ell T) y_{BB}^*(\ell T),$$

(7)

where $T = 1/W$. Substituting (7) into (6) gives the desired equation.