Communication and Detection Theory

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Model Answers to Exercise 3 of March 7, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *Separation between Signals*

(i) Since the norm is nonnegative, the condition $\|\mathbf{v}-\mathbf{u}_1\|_2 = \|\mathbf{v}-\mathbf{u}_2\|_2$ is equivalent to the condition

$$\|\mathbf{v}-\mathbf{u}_1\|_2^2 = \|\mathbf{v}-\mathbf{u}_2\|_2^2\,. \tag{1}$$

Moreover,

$$\|\mathbf{v}-\mathbf{u}_1\|_2^2 = \|\mathbf{v}\|_2^2 + \|\mathbf{u}_1\|_2^2 - 2\,\mathrm{Re}(\langle \mathbf{v}, \mathbf{u}_1\rangle), \tag{2}$$

$$\|\mathbf{v}-\mathbf{u}_2\|_2^2 = \|\mathbf{v}\|_2^2 + \|\mathbf{u}_2\|_2^2 - 2\,\mathrm{Re}(\langle \mathbf{v}, \mathbf{u}_2\rangle). \tag{3}$$

Taking the difference of (3) and (2) we see that the relation defining the subspace (1) can be written equivalently as

$$0 = \|\mathbf{u}_2\|_2^2 - \|\mathbf{u}_1\|_2^2 - 2\,\mathrm{Re}(\langle \mathbf{v}, \mathbf{u}_2 - \mathbf{u}_1\rangle)$$

or

$$\mathrm{Re}(\langle \mathbf{v}, \mathbf{u}_2 - \mathbf{u}_1\rangle) = \frac{\|\mathbf{u}_2\|_2^2 - \|\mathbf{u}_1\|_2^2}{2}.$$

(ii) In general, $\mathcal{V}$ is *not* a linear subspace. Indeed, if $\|\mathbf{u}_1\|_2 \neq \|\mathbf{u}_2\|_2$ then the all-zero signal is not in $\mathcal{V}$.

(iii) Let $\mathbf{w} = (\mathbf{u}_1 + \mathbf{u}_2)/2$. We have that

$$\|\mathbf{w}-\mathbf{u}_1\|_2 = \|(\mathbf{u}_2-\mathbf{u}_1)/2\|_2\,,$$
$$\|\mathbf{w}-\mathbf{u}_2\|_2 = \|(\mathbf{u}_1-\mathbf{u}_2)/2\|_2\,.$$

Since $\|(\mathbf{u}_2-\mathbf{u}_1)/2\|_2 = \|(\mathbf{u}_1-\mathbf{u}_2)/2\|_2$, we have that $\|\mathbf{w}-\mathbf{u}_1\|_2 = \|\mathbf{w}-\mathbf{u}_2\|_2$ and thus $\mathbf{w} = (\mathbf{u}_1+\mathbf{u}_2)/2 \in \mathcal{V}$.

Problem 2 — *Orthogonal Subspace*

Let $\mathcal{U}$ denote the set of all signals $\mathbf{u} \in \mathcal{L}_2$ which are orthogonal to the signals $\mathbf{v}_1, \ldots, \mathbf{v}_n$. By definition, the set $\mathcal{U}$ is a subset of $\mathcal{L}_2$. We have to show that it is closed under linear combinations.

For arbitrary $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathcal{U}$ and arbitrary complex numbers α_1 and α_2, by the linearity of the inner product:

$$\langle \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2, \mathbf{v}_\ell \rangle = \alpha_1 \langle \mathbf{u}_1, \mathbf{v}_\ell \rangle + \alpha_2 \langle \mathbf{u}_2, \mathbf{v}_\ell \rangle = 0, \quad \ell \in \{1, \ldots, n\},$$

since

$$\langle \mathbf{u}_j, \mathbf{v}_\ell \rangle = 0, \quad j \in \{1, 2\}, \ell \in \{1, \ldots, n\}.$$

Thus, any sum of the form $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2$ is an element of $\mathcal{U}$. This proves that $\mathcal{U}$ is closed under linear combinations and hence forms a subspace of $\mathcal{L}_2$.

Problem 3 *Constructing an Orthonormal Basis*

(i) See Figure 0.1

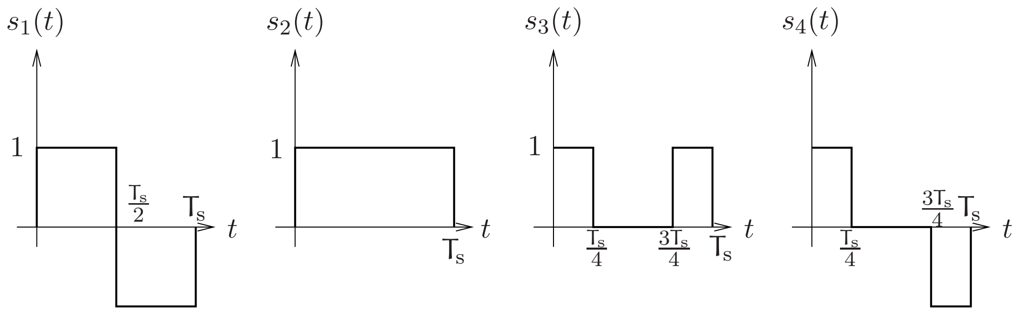

Figure 0.1:

(ii) It can be verified that the signals

$$\boldsymbol{\phi}_1 \triangleq \frac{\mathbf{s}_1}{\sqrt{\mathsf{T_s}}},$$
$$\boldsymbol{\phi}_2 \triangleq \frac{\mathbf{s}_2}{\sqrt{\mathsf{T_s}}},$$
$$\boldsymbol{\phi}_3 \triangleq \frac{2\mathbf{s}_3 - \mathbf{s}_2}{\sqrt{\mathsf{T_s}}},$$
$$\boldsymbol{\phi}_4 \triangleq \frac{2\mathbf{s}_4 - \mathbf{s}_1}{\sqrt{\mathsf{T_s}}},$$

form an orthonormal basis for $\text{span}(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4)$.

(iii) The signals $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ can be expressed as:

$$\mathbf{s}_1 = \sqrt{\mathsf{T_s}}\boldsymbol{\phi}_1,$$
$$\mathbf{s}_2 = \sqrt{\mathsf{T_s}}\boldsymbol{\phi}_2,$$
$$\mathbf{s}_3 = \frac{\sqrt{\mathsf{T_s}}}{2}(\boldsymbol{\phi}_3 + \boldsymbol{\phi}_2),$$
$$\mathbf{s}_4 = \frac{\sqrt{\mathsf{T_s}}}{2}(\boldsymbol{\phi}_4 + \boldsymbol{\phi}_1).$$

Problem 4 *Constructing an Orthonormal Basis*

Using (6.33) and (6.34) we obtain that

$$t \mapsto \operatorname{sinc}\left(\frac{t}{2}\right) \quad \text{is of FT} \quad f \mapsto 2\,\mathrm{I}\left\{|f| \leq \frac{1}{4}\right\}.$$

Consequently, by Proposition 6.6.1,

$$t \mapsto \operatorname{sinc}^2\Big(\frac{t}{2}\Big) \quad \text{is of FT} \quad f \mapsto \big(2 - 4|f|\big)\, \mathrm{I}\Big\{|f| \le \frac{1}{2}\Big\},$$

i.e., of bandwidth 1/2 Hz. Consequently, if we define T as the reciprocal of twice the bandwidth, then $\mathsf{T} = 1$. By Theorem 8.4.3 (i) we thus obtain the orthonormal expansion of $t \mapsto \operatorname{sinc}^2(t/2)$

$$\sum_{\ell=-\infty}^{\infty} \operatorname{sinc}^2\Big(-\frac{\ell}{2}\Big)\operatorname{sinc}(t+\ell).$$

Since

$$\operatorname{sinc}^2\Big(-\frac{\ell}{2}\Big) = \begin{cases} 1 & \text{if } \ell = 0, \\ 0 & \text{if } \ell \text{ is nonzero and even}, \\ 4/(\pi^2\ell^2) & \text{if } \ell \text{ is odd}, \end{cases}$$

the expansion can also be written as

$$\operatorname{sinc}(t) + \sum_{\ell \text{ odd}} \frac{4}{\pi^2\ell^2}\operatorname{sinc}(t+\ell).$$

Problem 5 *Inner Product with a Bandlimited Signal*

The signal $\mathbf{y}$ can be written as

$$\begin{aligned} \mathbf{y} &= \mathbf{y} \star \mathrm{LPF}_W + (\mathbf{y} - \mathbf{y} \star \mathrm{LPF}_W) \\ &= \mathbf{y}_{\mathrm{LPF}} + (\mathbf{y} - \mathbf{y}_{\mathrm{LPF}}), \end{aligned}$$

where $\mathbf{y}_{\mathrm{LPF}}$ is defined as the convolution $\mathbf{y} \star \mathrm{LPF}_W$ and, by Proposition 6.4.7, is an energy-limited signal that is bandlimited to W Hz. Consequently, by Theorem 8.7.1 Part (ii),

$$\langle \mathbf{x}, \mathbf{y}_{\mathrm{LPF}} \rangle = \mathsf{T}_\mathrm{s} \sum_{\ell=-\infty}^{\infty} x(\ell \mathsf{T}_\mathrm{s}) y^*_{\mathrm{LPF}}(\ell \mathsf{T}_\mathrm{s}).$$

Since the FT of $\mathbf{y} - \mathbf{y}_{\mathrm{LPF}}$ vanishes in the band $[-W, W]$, and since $\mathbf{x}$ is bandlimited to W Hz, it follows from Parseval's Theorem that

$$\langle \mathbf{x}, \mathbf{y} - \mathbf{y}_{\mathrm{LPF}} \rangle = 0.$$

Hence

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{y}_{\mathrm{LPF}} + \mathbf{y} - \mathbf{y}_{\mathrm{LPF}} \rangle \\ &= \langle \mathbf{x}, \mathbf{y}_{\mathrm{LPF}} \rangle + \langle \mathbf{x}, \mathbf{y} - \mathbf{y}_{\mathrm{LPF}} \rangle \\ &= \langle \mathbf{x}, \mathbf{y}_{\mathrm{LPF}} \rangle \\ &= \mathsf{T}_\mathrm{s} \sum_{\ell=-\infty}^{\infty} x(\ell \mathsf{T}_\mathrm{s}) y^*_{\mathrm{LPF}}(\ell \mathsf{T}_\mathrm{s}). \end{aligned}$$

An alternative solution goes as follows. Let $\mathsf{T}_\mathrm{s} = 1/(2W)$. From the $\mathcal{L}_2$-Sampling Theorem

$$\lim_{\mathsf{L}\to\infty} \int_{-\infty}^{\infty} \bigg| x(t) - \sum_{\ell=-\mathsf{L}}^{\mathsf{L}} x(-\ell \mathsf{T}_\mathrm{s}) \operatorname{sinc}\Big(\frac{t}{\mathsf{T}_\mathrm{s}} + \ell\Big) \bigg|^2 \, \mathrm{d}t = 0.$$

For every $\mathsf{L} \in \mathbb{N}$ define the signal

$$u_{\mathsf{L}}(t) = x(t) - \sum_{\ell=-\mathsf{L}}^{\mathsf{L}} x(\ell \mathsf{T_s}) \operatorname{sinc}\Bigl(\frac{t}{\mathsf{T_s}} + \ell\Bigr), \quad t \in \mathbb{R}.$$

Then $\|\mathbf{u}_{\mathsf{L}}\|_2 \to 0$ as $\mathsf{L} \to \infty$. Fix some $\beta \in \mathbb{C}$. For every energy-limited $\mathbf{y}$ we have by (4.14)

$$\bigl(\|\mathbf{y}\|_2 - |\beta|\,\|\mathbf{u}_{\mathsf{L}}\|_2\bigr)^2 \leq \|\beta \mathbf{u}_{\mathsf{L}} + \mathbf{y}\|_2^2 \leq \bigl(\|\mathbf{y}\|_2 + |\beta|\,\|\mathbf{u}_{\mathsf{L}}\|_2\bigr)^2,$$

so $\|\beta \mathbf{u}_{\mathsf{L}} + \mathbf{y}\|_2^2 \to \|\mathbf{y}\|_2^2$ as $\mathsf{L} \to \infty$. We now expand $\|\beta \mathbf{u}_{\mathsf{L}} + \mathbf{y}\|_2^2$:

$$\begin{aligned}
\|\beta \mathbf{u}_{\mathsf{L}} + \mathbf{y}\|_2^2 &= \int_{-\infty}^{\infty} \left| \beta \left(x(t) - \sum_{\ell=-\mathsf{L}}^{\mathsf{L}} x(-\ell \mathsf{T_s}) \operatorname{sinc}\Bigl(\frac{t}{\mathsf{T_s}} + \ell\Bigr) \right) + y(t) \right|^2 \mathrm{d}t \\
&= |\beta|^2 \int_{-\infty}^{\infty} \left| x(t) - \sum_{\ell=-\mathsf{L}}^{\mathsf{L}} x(-\ell \mathsf{T_s}) \operatorname{sinc}\Bigl(\frac{t}{\mathsf{T_s}} + \ell\Bigr) \right|^2 \mathrm{d}t + \int_{-\infty}^{\infty} |y(t)|^2 \,\mathrm{d}t \\
&\quad + 2 \operatorname{Re}\left(\beta \int_{-\infty}^{\infty} x(t) y^*(t) \,\mathrm{d}t - \beta \int_{-\infty}^{\infty} \sum_{\ell=-\mathsf{L}}^{\mathsf{L}} x(-\ell \mathsf{T_s}) \operatorname{sinc}\Bigl(\frac{t}{\mathsf{T_s}} + \ell\Bigr) y^*(t) \,\mathrm{d}t \right) \\
&= |\beta|^2 \,\|\mathbf{u}_{\mathsf{L}}\|_2^2 + \|\mathbf{y}\|_2^2 \\
&\quad + 2 \operatorname{Re}\left(\beta \,\langle \mathbf{x}, \mathbf{y} \rangle - \beta \sum_{\ell=-\mathsf{L}}^{\mathsf{L}} x(-\ell \mathsf{T_s}) \int_{-\infty}^{\infty} \operatorname{sinc}\Bigl(\frac{t}{\mathsf{T_s}} + \ell\Bigr) y^*(t) \,\mathrm{d}t \right).
\end{aligned}$$

Letting L tend to infinity and recalling that $\|\mathbf{u}_{\mathsf{L}}\|_2 \to 0$ and $\|\beta \mathbf{u}_{\mathsf{L}} + \mathbf{y}\|_2^2 \to \|\mathbf{y}\|_2^2$ we conclude that

$$2 \operatorname{Re}\left(\beta \,\langle \mathbf{x}, \mathbf{y} \rangle - \beta \sum_{\ell=-\mathsf{L}}^{\mathsf{L}} x(-\ell \mathsf{T_s}) \int_{-\infty}^{\infty} \operatorname{sinc}\Bigl(\frac{t}{\mathsf{T_s}} + \ell\Bigr) y^*(t) \,\mathrm{d}t \right) = 0.$$

Since this has to hold for every $\beta \in \mathbb{C}$,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{\ell=-\mathsf{L}}^{\mathsf{L}} x(-\ell \mathsf{T_s}) \int_{-\infty}^{\infty} \operatorname{sinc}\Bigl(\frac{t}{\mathsf{T_s}} + \ell\Bigr) y^*(t) \,\mathrm{d}t. \tag{4}$$

We further have

$$\begin{aligned}
\int_{-\infty}^{\infty} \operatorname{sinc}\Bigl(\frac{t}{\mathsf{T_s}} + \ell\Bigr) y^*(t) \,\mathrm{d}t &= \int_{-\infty}^{\infty} \operatorname{sinc}\Bigl(\frac{1}{\mathsf{T_s}}(-\ell \mathsf{T_s} - t)\Bigr) y^*(t) \,\mathrm{d}t \\
&= \left(\mathbf{y}^* \star \left(t \mapsto \operatorname{sinc}\Bigl(\frac{t}{\mathsf{T_s}}\Bigr) \right) \right)(-\ell \mathsf{T_s}) \\
&= \mathsf{T_s}\, y_{\mathrm{LPF}}^*(-\ell \mathsf{T_s}),
\end{aligned} \tag{5}$$

where the first equality follows from the fact that $t \mapsto \operatorname{sinc}(t)$ is symmetric (even). Inserting (5) in the RHS of (4), we finally obtain

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathsf{T_s} \sum_{\ell=-\infty}^{\infty} x(-\ell \mathsf{T_s}) y_{\mathrm{LPF}}^*(-\ell \mathsf{T_s}) = \mathsf{T_s} \sum_{\ell=-\infty}^{\infty} x(\ell \mathsf{T_s}) y_{\mathrm{LPF}}^*(\ell \mathsf{T_s}).$$

Problem 6 — *Inner Product between Passband Signals*

By Theorem 7.7.12,

$$\langle \mathbf{x}_{\mathrm{PB}}, \mathbf{y}_{\mathrm{PB}} \rangle = 2 \operatorname{Re}\Bigl(\langle \mathbf{x}_{\mathrm{BB}}, \mathbf{y}_{\mathrm{BB}} \rangle \Bigr). \tag{6}$$

Also by Thorem 7.7.12, $\mathbf{x}_{\mathrm{BB}}$ and $\mathbf{y}_{\mathrm{BB}}$ are energy-limited signals that are bandlimited to $W/2$ Hz. Thus, by the Sampling Theorem,

$$\langle \mathbf{x}_{\mathrm{BB}}, \mathbf{y}_{\mathrm{BB}} \rangle = \mathsf{T} \sum_{\ell=-\infty}^{\infty} x_{\mathrm{BB}}(\ell \mathsf{T})\, y_{\mathrm{BB}}^{*}(\ell \mathsf{T}), \tag{7}$$

where $\mathsf{T} = 1/W$. Substituting (7) into (6) gives the desired equation.