Communication and Detection Theory

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Model Answers to Exercise 4 of March 14, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *A Specific Signal*

Denote the baseband representation of $\mathbf{x}$ by $\mathbf{x}_{\mathrm{BB}}$. We are given that

$$\mathbf{x}_{\mathrm{BB}}\Big(\frac{\ell}{W}\Big) = \begin{cases} 1+\mathrm{i} & \text{if } \ell = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \ell \in \mathbb{Z}.$$

Thus, the passband signal $\mathbf{x}$ is given by

$$\begin{aligned} x(t) &= 2\sum_{\ell=-\infty}^{\infty} \mathrm{Re}\Big(e^{\mathrm{i}2\pi f_{\mathrm{c}} t} x_{\mathrm{BB}}\Big(\frac{\ell}{W}\Big)\Big)\,\mathrm{sinc}(Wt-\ell) \\ &= 2\,\mathrm{Re}\Big(e^{\mathrm{i}2\pi f_{\mathrm{c}} t}(1+\mathrm{i})\Big)\,\mathrm{sinc}(Wt) \\ &= 2\Big(\cos(2\pi f_{\mathrm{c}} t) - \sin(2\pi f_{\mathrm{c}} t)\Big)\,\mathrm{sinc}(Wt), \quad t \in \mathbb{R}. \end{aligned}$$

Problem 2 — *Multiplying by a Carrier*

Define $\mathbf{y}\colon t \mapsto x(t)\cos(2\pi f_{\mathrm{c}} t)$ so

$$\hat{y}(f) = \frac{1}{2}\Big(\hat{x}(f-f_{\mathrm{c}}) + \hat{x}(f+f_{\mathrm{c}})\Big), \quad f \in \mathbb{R},$$

from which it follows that

$$\hat{y}_{\mathrm{BB}}(f) = \frac{1}{2}\,\hat{x}(f), \quad f \in \mathbb{R}$$

and hence

$$y_{\mathrm{BB}}(t) = \frac{1}{2}\,x(t), \quad t \in \mathbb{R}.$$

Thus, the complex samples of $\mathbf{y}$ are given by

$$y_{\mathrm{BB}}\Big(\frac{\ell}{W}\Big) = \frac{1}{2}\,x\Big(\frac{\ell}{W}\Big), \quad \ell \in \mathbb{Z}.$$

Next, define $\mathbf{u}\colon t \mapsto x(t)\sin(2\pi f_{\mathrm{c}} t)$, so

$$\hat{u}(f) = -\frac{\mathrm{i}}{2}\Big(\hat{x}(f-f_{\mathrm{c}}) - \hat{x}(f+f_{\mathrm{c}})\Big), \quad f \in \mathbb{R},$$

from which it follows that

$$\hat{u}_{\mathrm{BB}}(f) = -\frac{\mathrm{i}}{2}\,\hat{x}(f), \quad f \in \mathbb{R},$$

and hence

$$u_{\mathrm{BB}}(t) = -\frac{\mathrm{i}}{2}\,x(t), \quad t \in \mathbb{R}.$$

Thus, the complex samples of $\mathbf{u}$ are given by

$$u_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}}\Bigr) = -\frac{\mathrm{i}}{2}x\Bigl(\frac{\ell}{\mathsf{W}}\Bigr), \quad \ell \in \mathbb{Z}.$$

Problem 3 — *Orthogonal Passband Signals*

Since the inner product between $\mathbf{x}_{\mathrm{PB}}$ and $\mathbf{y}_{\mathrm{PB}}$ is given by

$$\langle \mathbf{x}_{\mathrm{PB}}, \mathbf{y}_{\mathrm{PB}} \rangle = \frac{2}{\mathsf{W}}\,\mathrm{Re}\Biggl(\sum_{\ell=-\infty}^{\infty} x_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}}\Bigr)\,y^*_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}}\Bigr)\Biggr),$$

it follows that the two passband signals $\mathbf{x}_{\mathrm{PB}}$ and $\mathbf{y}_{\mathrm{PB}}$ are orthogonal if, and only if,

$$\mathrm{Re}\Biggl(\sum_{\ell=-\infty}^{\infty} x_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}}\Bigr)\,y^*_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}}\Bigr)\Biggr) = 0.$$

Problem 4 — *The Convolution Revisited*

Let $\mathbf{z} = \mathbf{x} \star \mathbf{y}$. By Proposition 7.6.12, $\mathbf{z}$ is a real integrable passband signal that is bandlimited to W Hz around the carrier frequency f_c and whose baseband representation is

$$\mathbf{z}_{\mathrm{BB}} = \mathbf{x}_{\mathrm{BB}} \star \mathbf{y}_{\mathrm{BB}},$$

where $\mathbf{x}_{\mathrm{BB}}$ and $\mathbf{y}_{\mathrm{BB}}$ are the baseband representations of $\mathbf{x}$ and $\mathbf{y}$. The complex samples of $\mathbf{z}$ can now be written as

$$\begin{aligned} \mathbf{z}_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}}\Bigr) &= \int_{-\infty}^{\infty} x_{\mathrm{BB}}(\tau)\,y_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}} - \tau\Bigr)\,\mathrm{d}\tau \\ &= \Bigl\langle \mathbf{x}_{\mathrm{BB}}, t \mapsto y^*_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}} - t\Bigr)\Bigr\rangle. \end{aligned} \tag{1}$$

Since $\mathbf{x}_{\mathrm{BB}}$ and $\mathbf{y}_{\mathrm{BB}}$ are integrable signals bandlimited to $\mathsf{W}/2$ Hz they are also energy-limited (see Note 6.4.12). Moreover, if $\mathbf{y}_{\mathrm{BB}}$ is energy-limited and bandlimited to $\mathsf{W}/2$ Hz, then so is the signal $t \mapsto y^*_{\mathrm{BB}}(\ell/\mathsf{W} - t)$. Consequently, we can use Part (iii) of the Sampling Theorem (Theorem 8.4.3) to express the inner product in (1) in terms of the complex samples of $\mathbf{x}$ and $\mathbf{y}$:

$$\Bigl\langle \mathbf{x}_{\mathrm{BB}}, t \mapsto y^*_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}} - t\Bigr)\Bigr\rangle = \frac{1}{\mathsf{W}}\sum_{\ell'=-\infty}^{\infty} x_{\mathrm{BB}}\Bigl(\frac{\ell'}{\mathsf{W}}\Bigr)\,y_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}} - \frac{\ell'}{\mathsf{W}}\Bigr).$$

We conclude that the complex samples of $\mathbf{z}$ are obtained by convolving the complex samples of $\mathbf{x}$ with the complex samples of $\mathbf{y}$ and scaling the result by $1/\mathsf{W}$:

$$\mathbf{z}_{\mathrm{BB}}\Bigl(\frac{\ell}{\mathsf{W}}\Bigr) = \frac{1}{\mathsf{W}}\sum_{\ell'=-\infty}^{\infty} x_{\mathrm{BB}}\Bigl(\frac{\ell'}{\mathsf{W}}\Bigr)\,y_{\mathrm{BB}}\Bigl(\frac{\ell - \ell'}{\mathsf{W}}\Bigr).$$

Problem 5 *Exploiting Orthogonality*

The numbers $X^{(1)}$ and $X^{(2)}$ can be recovered from $\mathbf{X}$ by computing linear combinations of the inner products $\langle \mathbf{X}, \boldsymbol{\phi}_1 \rangle$ and $\langle \mathbf{X}, \boldsymbol{\phi}_2 \rangle$. To see this, first note that

$$\langle \mathbf{X}, \boldsymbol{\phi}_1 \rangle = \left(\mathsf{A}^{(1)} X^{(1)} + \mathsf{A}^{(2)} X^{(2)} \right) \|\boldsymbol{\phi}_1\|_2^2 \, ,$$

and that

$$\langle \mathbf{X}, \boldsymbol{\phi}_2 \rangle = \left(\mathsf{A}^{(1)} X^{(1)} - \mathsf{A}^{(2)} X^{(2)} \right) \|\boldsymbol{\phi}_2\|_2^2 \, ,$$

where in the computation of the inner products we have used that $\langle \boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \rangle = 0$. It thus follows that

$$\begin{aligned} X^{(1)} &= \frac{1}{2\mathsf{A}^{(1)}} \left(\frac{\langle \mathbf{X}, \boldsymbol{\phi}_1 \rangle}{\|\boldsymbol{\phi}_1\|_2^2} + \frac{\langle \mathbf{X}, \boldsymbol{\phi}_2 \rangle}{\|\boldsymbol{\phi}_2\|_2^2} \right) \\ X^{(2)} &= \frac{1}{2\mathsf{A}^{(2)}} \left(\frac{\langle \mathbf{X}, \boldsymbol{\phi}_1 \rangle}{\|\boldsymbol{\phi}_1\|_2^2} - \frac{\langle \mathbf{X}, \boldsymbol{\phi}_2 \rangle}{\|\boldsymbol{\phi}_2\|_2^2} \right). \end{aligned}$$

