Communication and Detection Theory
Spring Semester 2018
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Model Answers to Exercise 5 of March 20, 2018

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1
Passband Signaling

(i) Define \( y: t \mapsto e^{j2\pi f_0 t} x(t) \). Then
\[
y(0) = x(0),
\]
and, since \( e^{j2\pi f_0 t} \neq 0 \) for every \( t \in \mathbb{R} \), it follows that for every \( \ell \in \mathbb{Z} \)
\[
(y(\ell T_s) = 0) \iff (x(\ell T_s) = 0).
\]
Consequently, by (1) and (2) \( y \) is a Nyquist Pulse of parameter \( T_s \) if, and only if, \( x \) is a Nyquist Pulse of parameter \( T_s \).

(ii) If \( x \) is a Nyquist Pulse of parameter \( T_s \), then so is \( t \mapsto \cos(2\pi f_0 t) x(t) \) because the value of \( \cos(2\pi f_0 t) x(t) \) at \( t = 0 \) is that of \( x(t) \) and \( t = 0 \), and for every \( \ell \in \mathbb{Z} \)
\[
(x(\ell T_s) = 0) \implies (\cos(2\pi f_0 \ell T_s) x(\ell T_s) = 0).
\]

(iii) Even if \( t \mapsto \cos(2\pi f_0 t) x(t) \) is a Nyquist Pulse of parameter \( T_s \), the signal \( x \) need not be one. For example, suppose that
\[
f_0 = \frac{1}{4T_s}
\]
and that \( x \) is a Nyquist Pulse of parameter \( 2T_s \) but not of parameter \( T_s \). We show that although \( x \) is not a Nyquist Pulse of parameter \( T_s \), the pulse \( t \mapsto \cos(2\pi f_0 t) x(t) \) is. Clearly the values at zero of \( x \) and \( t \mapsto \cos(2\pi f_0 t) x(t) \) are equal, so the fact that \( x \) is a Nyquist Pulse (of parameter \( 2T_s \)) implies that the value of \( t \mapsto \cos(2\pi f_0 t) x(t) \) at \( t = 0 \) is 1. If \( \ell \) is even but not zero, then the value of \( x \) at \( t = \ell T_s \) must be zero (because \( x \) is a Nyquist Pulse of parameter \( 2T_s \)), so the same must be true of \( t \mapsto \cos(2\pi f_0 t) x(t) \). And if \( \ell \) is odd, then the value of \( t \mapsto \cos(2\pi f_0 t) x(t) \) at \( t = \ell T_s \) must be zero because \( \cos(2\pi f_0 \ell T_s) \) is then zero.

Problem 2
The Self-Similarity Function of a Delayed Signal

\[
R_{vv}(\tau) = \int_{-\infty}^{\infty} v(t + \tau) v^*(t) \, dt
\]
\[
= \int_{-\infty}^{\infty} u(t + \tau - t_0) u^*(t - t_0) \, dt
\]
\[
= \int_{-\infty}^{\infty} u(\tilde{t} + \tau) u^*(\tilde{t}) \, d\tilde{t}
\]
\[
= R_{uu}(\tau), \quad \tau \in \mathbb{R},
\]

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Problem 3  

**The Self-Similarity Function of a Frequency Shifted Signal**

\[
R_{vv}(\tau) = \int_{-\infty}^{\infty} u(t + \tau) v^*(t) \, dt \\
= \int_{-\infty}^{\infty} u(t + \tau) e^{i2\pi f_0(t+\tau)} u^*(t) e^{-i2\pi f_0 t} \, dt \\
= e^{i2\pi f_0 \tau} \int_{-\infty}^{\infty} u(t + \tau) u^*(t) \, dt \\
= e^{i2\pi f_0 \tau} R_{uu}(\tau), \quad \tau \in \mathbb{R}.
\]

Problem 4  

**Relaxing the Orthonormality Condition**

The condition that the time shifts of a signal \( \phi \) by *even* multiples of \( T_s \) are orthonormal, is equivalent to the condition that the time shifts of \( \phi \) by *all* integer multiples of *twice* \( T_s \) are orthonormal. Thus, by Corollary 11.3.5, the minimum bandwidth of a signal whose time shifts by even multiples of \( T_s \) are orthonormal is \( 1/(2(2T_s)) \), i.e.,

\[
\frac{1}{4T_s}
\]

The condition that the time shifts of a signal \( \phi \) by *odd* multiples of \( T_s \) are orthonormal, is equivalent to the condition that the time-shifts of \( t \mapsto \phi(t - T_s) \) by *even* multiples of \( T_s \) are orthonormal. Thus, for this condition to hold, the bandwidth of \( t \mapsto \phi(t - T_s) \) must be at least \( 1/(4T_s) \). And since the bandwidth of \( \phi \) is the same as the bandwidth of \( t \mapsto \phi(t - T_s) \), the bandwidth of \( \phi \) must be at least \( 1/(4T_s) \).

Problem 5  

**A Specific Signal**

(i) See Figure 0.1.

(ii) Define

\[
g(f) = T_s (1 - |T_s f - 1|) 1 \{ 0 \leq f \leq \frac{2}{T_s} \}, \quad f \in \mathbb{R}.
\]
Then $g \in \mathcal{L}_1 \cap \mathcal{L}_2$ and, by Proposition 6.4.5 Part (i),

$$p(t) = \hat{g}(t), \quad t \in \mathbb{R}.$$  

The assumptions of Theorem 11.3.2 are thus satisfied and $p(\cdot)$ is a Nyquist Pulse of parameter $T_s$ if, and only if,

$$\lim_{J \to \infty} \int_{-1/(2T_s)}^{1/(2T_s)} \left| T_s - \sum_{j=-J}^{J} g \left( f + \frac{j}{T_s} \right) \right| df = 0. \quad (4)$$

Since $g$ is symmetric around $1/T_s$, this condition is equivalent to

$$\lim_{J \to \infty} \int_{0}^{1/(2T_s)} \left| T_s - \sum_{j=-J}^{J} g \left( f + \frac{j}{T_s} \right) \right| df = 0. \quad (5)$$

On the interval $(0, 1/(2T_s))$ only three terms contribute to the sum: the terms corresponding to $j = 0$ and $j = \pm 1$. Thus, to verify that $\hat{g}$ is a Nyquist Pulse of parameter $T_s$ we only need to show that

$$g \left( f - \frac{1}{T_s} \right) + g(f) + g \left( f + \frac{1}{T_s} \right) = T_s, \quad 0 < t < \frac{1}{2T_s}.$$  

This can be done graphically or algebraically.

(iii) Since $p(\cdot)$ is a Nyquist Pulse of parameter $T_s$, so is $\text{Re}(p(\cdot))$ because

$$(p(0) = 1) \implies (\text{Re}(p(0)) = 1)$$

and

$$(p(\ell T_s) = 0, \ \ell \in \mathbb{Z} \setminus \{0\}) \implies (\text{Re}(p(\ell T_s)) = 0, \ \ell \in \mathbb{Z} \setminus \{0\}).$$

(iv) The imaginary part of $p(\cdot)$ is not a Nyquist pulse because

$$\text{Im}(p(0)) = \text{Im}(1) = 0 \neq 1.$$  

**Problem 6**

**Mapping a Discrete-Time Stationary SP**

We need to show that for every positive integer $n$ and all choices of $\eta, \eta' \in \mathbb{Z}$

$$(Y_\eta, \ldots, Y_{\eta+n-1}) \not\leq (Y_{\eta'}, \ldots, Y_{\eta'+n-1}). \quad (6)$$

Since $(X_\nu)$ is stationary,

$$(X_\eta, \ldots, X_{\eta+n-1}) \not\leq (X_{\eta'}, \ldots, X_{\eta'+n-1})$$

and consequently,

$$(g(X_\eta), \ldots, g(X_{\eta+n-1})) \not\leq (g(X_{\eta'}), \ldots, g(X_{\eta'+n-1})) \quad (7)$$

because applying a deterministic mapping (in this case the componentwise mapping $g(\cdot)$) to random vectors of equal law, results in random vectors of equal law. Since $Y_\nu$ is defined as $g(X_\nu)$ we see that (7) establishes (6).
Problem 7 \hspace{1cm} \textit{Mapping a Discrete-Time WSS SP}

No, \((Y_\nu)\) need not be WSS, as the following example shows. Suppose that
\[ \ldots, X_{-3}, X_{-1}, X_{1}, X_3, \ldots \]  
are drawn IID each taking on the values \(\pm 1\) equiprobably. Suppose that
\[ \ldots, X_{-4}, X_{-2}, X_0, X_2, X_4, \ldots \]
are drawn independently of \((8)\) and IID each taking on the values \(-\sqrt{3/2}, 0, \sqrt{3/2}\) equiprobably. We note that \((X_\nu)\) is WSS because — irrespective of whether \(\nu\) is odd or even — \(E[X_\nu] = 0\) and \(\text{Var}[X_\nu] = 1\), and as to the covariance, \(\text{Cov}[X_\nu, X_{\nu+\eta}]\) is zero whenever \(\eta\) is not zero, so
\[ \text{Cov}[X_\nu, X_{\nu+\eta}] = 1\{\eta = 0\}, \quad \eta \in \mathbb{Z}. \]

Consider now the mapping \(g: \xi \mapsto \xi^4\). We claim that the SP \((Y_\nu)\) defined by
\[ Y_\nu = g(X_\nu), \quad \nu \in \mathbb{Z} \]
is not WSS. Indeed, \((Y_\nu)\) is deterministically 1 for \(\nu\) odd and takes on the values \(9/4\) and 0 with probabilities 2/3 and 1/3, respectively, when \(\eta\) is even. Thus, \(E[Y_\nu]\) equals 1 when \(\nu\) is odd and equals \(3/2\) when \(\nu\) is even, i.e., it \textit{does} depend on \(\nu\), and \((Y_\nu)\) is not WSS.