Problem 1  

Passband Signaling

(i) Define \( y : t \mapsto e^{i2\pi f_0 t} x(t) \). Then

\[ y(0) = x(0), \]

(1)

and, since \( e^{i2\pi f_0 t} \neq 0 \) for every \( t \in \mathbb{R} \), it follows that for every \( \ell \in \mathbb{Z} \)

\[ (y(\ell T_s) = 0) \iff (x(\ell T_s) = 0). \]

(2)

Consequently, by (1) and (2) \( y \) is a Nyquist pulse of parameter \( T_s \) if, and only if, \( x \) is a Nyquist pulse of parameter \( T_s \).

(ii) If \( x \) is a Nyquist pulse of parameter \( T_s \), then so is \( t \mapsto \cos(2\pi f_0 t) x(t) \) because the value of \( \cos(2\pi f_0 t) x(t) \) at \( t = 0 \) is that of \( x(t) \) and \( t = 0 \), and for every \( \ell \in \mathbb{Z} \)

\[ x(\ell T_s) = 0 \implies \cos(2\pi f_0 \ell T_s) x(\ell T_s) = 0. \]

(3)

(iii) Even if \( t \mapsto \cos(2\pi f_0 t) x(t) \) is a Nyquist pulse of parameter \( T_s \), the signal \( x \) need not be one. For example, suppose that \( f_0 = \frac{1}{4T_s} \)

and that \( x \) is a Nyquist pulse of parameter \( 2T_s \) but not of parameter \( T_s \). We show that although \( x \) is not a Nyquist pulse of parameter \( T_s \), the pulse \( t \mapsto \cos(2\pi f_0 t) x(t) \) is. Clearly the values at zero of \( x \) and \( t \mapsto \cos(2\pi f_0 t) x(t) \) are equal, so the fact that \( x \) is a Nyquist pulse (of parameter \( 2T_s \)) implies that the value of \( t \mapsto \cos(2\pi f_0 t) x(t) \) at \( t = 0 \) is 1. If \( \ell \) is even but not zero, then the value of \( x \) at \( t = \ell T_s \) must be zero (because \( x \) is a Nyquist pulse of parameter \( 2T_s \)) so the same must be true of \( t \mapsto \cos(2\pi f_0 t) x(t) \). And if \( \ell \) is odd, then the value of \( t \mapsto \cos(2\pi f_0 t) x(t) \) at \( t = \ell T_s \) must be zero because \( \cos(2\pi f_0 \ell T_s) \) is then zero.
Problem 2
The Self-Similarity Function of a Delayed Signal

\[ R_{vv}(\tau) = \int_{-\infty}^{\infty} v(t + \tau) v^*(t) \, dt \]
\[ = \int_{-\infty}^{\infty} u(t + \tau - t_0) u^*(t - t_0) \, dt \]
\[ = \int_{-\infty}^{\infty} u(\tilde{t} + \tau) u^*(\tilde{t}) \, d\tilde{t} \]
\[ = R_{uu}(\tau), \quad \tau \in \mathbb{R}, \]
where we have substituted \( \tilde{t} \) for \( t - t_0 \).

Problem 3
The Self-Similarity Function of a Frequency Shifted Signal

\[ R_{vv}(\tau) = \int_{-\infty}^{\infty} v(t + \tau) v^*(t) \, dt \]
\[ = \int_{-\infty}^{\infty} u(t + \tau) e^{i2\pi f_0(t+\tau)} u^*(t) \, dt \]
\[ = e^{i2\pi f_0 \tau} \int_{-\infty}^{\infty} u(t + \tau) u^*(t) \, dt \]
\[ = e^{i2\pi f_0 \tau} R_{uu}(\tau), \quad \tau \in \mathbb{R}. \]

Problem 4
Relaxing the Orthonormality Condition

The condition that the times-shifts of a signal \( \phi \) by \( \text{even} \) multiples of \( T_s \) are orthonormal, is equivalent to the condition that the time-shifts of \( \phi \) by \( \text{all integer multiples of twice} \cdot T_s \) are orthonormal. Thus, by Corollary 11.3.5, the minimum bandwidth of a signal whose time shifts by even multiples of \( T_s \) are orthonormal is \( 1/(2(2T_s)) \), i.e.,

\[ \frac{1}{4T_s}. \]

The condition that the times-shifts of a signal \( \phi \) by \( \text{odd} \) multiples of \( T_s \) are orthonormal, is equivalent to the condition that the time-shifts of \( t \mapsto \phi(t - T_s) \) by \( \text{even} \) multiples of \( T_s \) are orthonormal. Thus, for this condition to hold, the bandwidth of \( t \mapsto \phi(t - T_s) \) must be at least \( 1/(4T_s) \). And since the bandwidth of \( \phi \) is the same as the bandwidth of \( t \mapsto \phi(t - T_s) \), the bandwidth of \( \phi \) must be at least \( 1/(4T_s) \).

Problem 5
A Specific Signal

(i) See Figure 0.1.

(ii) Define

\[ g(f) = T_s(1 - |T_s f - 1|)I\{0 \leq f \leq \frac{2}{T_s}\}, \quad f \in \mathbb{R}. \]

Then \( g \in L_1 \cap L_2 \) and, by Proposition 6.4.5 Part (i),

\[ p(t) = \hat{g}(t), \quad t \in \mathbb{R}. \]
The assumptions of Theorem 11.3.2 are thus satisfied and \( p(\cdot) \) is a Nyquist Pulse of parameter \( T_s \) if, and only if,

\[
\lim_{J \to \infty} \int_{-1/(2T_s)}^{1/(2T_s)} \left| T_s - \sum_{j=-J}^{J} g \left( f + \frac{j}{T_s} \right) \right| df = 0. \tag{4}
\]

Since \( g \) is symmetric around \( 1/T_s \), this condition is equivalent to

\[
\lim_{J \to \infty} \int_{0}^{1/(2T_s)} \left| T_s - \sum_{j=-J}^{J} g \left( f + \frac{j}{T_s} \right) \right| df = 0. \tag{5}
\]

On the interval \((0,1/(2T_s))\) only three terms contribute to the sum: the terms corresponding to \( j = 0 \) and \( j = \pm 1 \). Thus, to verify that \( \tilde{g} \) is a Nyquist Pulse of parameter \( T_s \) we only need to show that

\[
g \left( f - \frac{1}{T_s} \right) + g(f) + g \left( f + \frac{1}{T_s} \right) = T_s, \quad 0 < t < \frac{1}{2T_s}.
\]

This can be done graphically or algebraically.

(iii) Since \( p(\cdot) \) is a Nyquist Pulse of parameter \( T_s \), so is \( \text{Re}(p(\cdot)) \) because

\[
p(0) = 1 \implies \text{Re}(p(0)) = 1
\]

and

\[
p(\ell T_s) = 0, \; \ell \in \mathbb{Z} \setminus \{0\} \implies \text{Re}(p(\ell T_s)) = 0, \; \ell \in \mathbb{Z} \setminus \{0\}.
\]

(iv) The imaginary part of \( p(\cdot) \) is not a Nyquist pulse because

\[
\text{Im}(p(0)) = \text{Im}(1) = 0 \neq 1.
\]

Problem 6 \hspace{1cm} Mapping a Discrete-Time Stationary SP

We need to show that for every positive integer \( n \) and all choices of \( \eta, \eta' \in \mathbb{Z} \)

\[
(Y_\eta, \ldots, Y_{\eta+n-1}) \overset{D}{=} (Y_{\eta'}, \ldots, Y_{\eta'+n-1}). \tag{6}
\]

Since \((X_\nu)\) is stationary,

\[
(X_\eta, \ldots, X_{\eta+n-1}) \overset{D}{=} (X_{\eta'}, \ldots, X_{\eta'+n-1})
\]
and consequently,

$$(g(X_\eta), \ldots, g(X_{\eta+n-1})) \overset{\text{d}}{=} (g(X_{\eta'}), \ldots, g(X_{\eta'+n-1}))$$

(7)

due to a deterministic mapping (in this case the componentwise mapping $g(\cdot)$) to random vectors of equal law, results in random vectors of equal law. Since $(Y_\nu)$ is defined as $g(X_\nu)$ we see that (7) establishes (6).

**Problem 7**

**Mapping a Discrete-Time WSS SP**

No, $(Y_\nu)$ need not be WSS, as the following example shows. Suppose that

$$\ldots, X_{-3}, X_{-1}, X_1, X_3, \ldots$$

(8)

are drawn IID each taking on the values $\pm 1$ equiprobably. Suppose that

$$\ldots, X_{-4}, X_{-2}, X_0, X_2, X_4, \ldots$$

are drawn independently of (8) and IID each taking on the values $-\sqrt{3}/2, 0, \sqrt{3}/2$ equiprobably.

We note that $(X_\nu)$ is WSS because — irrespective of whether $\nu$ is odd or even — $E[X_\nu] = 0$ and $\text{Var}[X_\nu] = 1$, and as to the covariance, $\text{Cov}[X_\nu, X_{\nu+\eta}]$ is zero whenever $\eta$ is not zero, so

$$\text{Cov}[X_\nu, X_{\nu+\eta}] = 1\{\eta = 0\}, \quad \eta \in \mathbb{Z}.$$  

Consider now the mapping $\xi \mapsto \xi^4$. We claim that the SP $(Y_\nu)$ defined by

$$Y_\nu = g(X_\nu), \quad \nu \in \mathbb{Z}$$

is not WSS. Indeed, $(Y_\nu)$ is deterministically $1$ for $\nu$ odd and takes on the values $9/4$ and $0$ with probabilities $2/3$ and $1/3$, respectively, when $\eta$ is even. Thus, $E[Y_\nu]$ is $1$ when $\nu$ is odd and is $3/2$ when $\nu$ is even, i.e., it does depend on $\nu$, and $(Y_\nu)$ is not WSS.