Model Answers to Exercise 5 of March 17, 2015

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1

Passband Signaling

(i) Define $y: t \mapsto e^{j2\pi f_0 t}x(t)$. Then

$$y(0) = x(0),$$  \hspace{1cm} (1)

and, since $e^{j2\pi f_0 t} \neq 0$ for every $t \in \mathbb{R}$, it follows that for every $\ell \in \mathbb{Z}$

$$\left( y(\ell T_s) = 0 \right) \iff \left( x(\ell T_s) = 0 \right).$$  \hspace{1cm} (2)

Consequently, by (1) and (2) $y$ is a Nyquist pulse of parameter $T_s$ if, and only if, $x$ is a Nyquist pulse of parameter $T_s$.

(ii) If $x$ is a Nyquist pulse of parameter $T_s$, then so is $t \mapsto \cos(2\pi f_0 t) x(t)$ because the value of $

\cos(2\pi f_0 t) x(t)$ at $t = 0$ is that of $x(t)$ and $t = 0$, and for every $\ell \in \mathbb{Z}$

$$\left( x(\ell T_s) = 0 \right) \implies \left( \cos(2\pi f_0 \ell T_s) x(\ell T_s) = 0 \right).$$  \hspace{1cm} (3)

(iii) Even if $t \mapsto \cos(2\pi f_0 t) x(t)$ is a Nyquist pulse of parameter $T_s$, the signal $x$ need not be one. For example, suppose that

$$f_0 = \frac{1}{4T_s}$$

and that $x$ is a Nyquist pulse of parameter $2T_s$ but not of parameter $T_s$. We show that although $x$ is not a Nyquist pulse of parameter $T_s$, the pulse $t \mapsto \cos(2\pi f_0 t) x(t)$ is. Clearly the values at zero of $x$ and $t \mapsto \cos(2\pi f_0 t) x(t)$ are equal, so the fact that $x$ is a Nyquist pulse (of parameter $2T_s$) implies that the value of $t \mapsto \cos(2\pi f_0 t) x(t)$ at $t = 0$ is 1. If $\ell$ is even but not zero, then the value of $x$ at $t = \ell T_s$ must be zero (because $x$ is a Nyquist pulse of parameter $2T_s$) so the same must be true of $t \mapsto \cos(2\pi f_0 t) x(t)$. And if $\ell$ is odd, then the value of $t \mapsto \cos(2\pi f_0 t) x(t)$ at $t = \ell T_s$ must be zero because $\cos(2\pi f_0 \ell T_s)$ is then zero.
Problem 2

The Self-Similarity Function of a Delayed Signal

\[ R_{vv}(\tau) = \int_{-\infty}^{\infty} v(t + \tau) v^*(t) \, dt \]
\[ = \int_{-\infty}^{\infty} u(t + \tau - t_0) u^*(t - t_0) \, dt \]
\[ = \int_{-\infty}^{\infty} u(\tilde{t} + \tau) u^*(\tilde{t}) \, d\tilde{t} \]
\[ = R_{uu}(\tau), \quad \tau \in \mathbb{R}, \]

where we have substituted \( \tilde{t} \) for \( t - t_0 \).

Problem 3

The Self-Similarity Function of a Frequency Shifted Signal

\[ R_{vv}(\tau) = \int_{-\infty}^{\infty} v(t + \tau) v^*(t) \, dt \]
\[ = \int_{-\infty}^{\infty} u(t + \tau) e^{i2\pi f_0(t + \tau)} u^*(t) e^{-i2\pi f_0 t} \, dt \]
\[ = e^{i2\pi f_0 \tau} \int_{-\infty}^{\infty} u(t + \tau) u^*(t) \, dt \]
\[ = e^{i2\pi f_0 \tau} R_{uu}(\tau), \quad \tau \in \mathbb{R}. \]

Problem 4

Relaxing the Orthonormality Condition

The condition that the time-shifts of a signal \( \phi \) by even multiples of \( T_s \) are orthonormal, is equivalent to the condition that the time-shifts of \( \phi \) by all integer multiples of twice \( T_s \) are orthonormal. Thus, by Corollary 11.3.5, the minimum bandwidth of a signal whose time shifts by even multiples of \( T_s \) are orthonormal is \( 1/(2(2T_s)) \), i.e.,

\[ \frac{1}{4T_s}. \]

The condition that the time-shifts of a signal \( \phi \) by odd multiples of \( T_s \) are orthonormal, is equivalent to the condition that the time-shifts of \( t \mapsto \phi(t - T_s) \) by even multiples of \( T_s \) are orthonormal. Thus, for this condition to hold, the bandwidth of \( t \mapsto \phi(t - T_s) \) must be at least \( 1/(4T_s) \). And since the bandwidth of \( \phi \) is the same as the bandwidth of \( t \mapsto \phi(t - T_s) \), the bandwidth \( \phi \) must be at least \( 1/(4T_s) \).

Problem 5

A Specific Signal

(i) See Figure 0.1.

(ii) Define

\[ g(f) = T_s(1 - |T_s f - 1|) 1 \{0 \leq f \leq \frac{2}{T_s}\}, \quad f \in \mathbb{R}. \]

Then \( g \in \mathcal{L}_1 \cap \mathcal{L}_2 \) and, by Proposition 6.4.5 Part (i),

\[ p(t) = g(t), \quad t \in \mathbb{R}. \]

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The assumptions of Theorem 11.3.2 are thus satisfied and \( p(\cdot) \) is a Nyquist Pulse of parameter \( T_s \), if, and only if,

\[
\lim_{J \to \infty} \int_{-1/(2T_s)}^{1/(2T_s)} \left| T_s - \sum_{j=-J}^{1} g(f + j T_s) \right| df = 0.
\]

(4)

Since \( g \) is symmetric around \( 1/T_s \), this condition is equivalent to

\[
\lim_{J \to \infty} \int_{0}^{1/(2T_s)} \left| T_s - \sum_{j=-J}^{1} g(f + j T_s) \right| df = 0.
\]

(5)

On the interval \((0, 1/(2T_s))\) only three terms contribute to the sum: the terms corresponding to \( j = 0 \) and \( j = \pm 1 \). Thus, to verify that \( \hat{g} \) is a Nyquist Pulse of parameter \( T_s \) we only need to show that

\[
g\left( f - \frac{1}{T_s} \right) + g(f) + g\left( f + \frac{1}{T_s} \right) = T_s, \quad 0 < t < \frac{1}{2T_s}.
\]

This can be done graphically or algebraically.

(iii) Since \( p(\cdot) \) is a Nyquist Pulse of parameter \( T_s \), so is \( \text{Re}(p(\cdot)) \) because

\[
(p(0) = 1) \Rightarrow (\text{Re}(p(0)) = 1)
\]

and

\[
(p(\ell T_s) = 0, \ \ell \in \mathbb{Z} \setminus \{0\}) \Rightarrow (\text{Re}(p(\ell T_s)) = 0, \ \ell \in \mathbb{Z} \setminus \{0\}).
\]

(iv) The imaginary part of \( p(\cdot) \) is not a Nyquist pulse because

\[
\text{Im}(p(0)) = \text{Im}(1)
\]

\[
= 0
\]

\[
\neq 1.
\]

Problem 6

**Mapping a Discrete-Time Stationary SP**

We need to show that for every positive integer \( n \) and all choices of \( \eta, \eta' \in \mathbb{Z} \)

\[
(Y_\eta, \ldots, Y_{\eta+n-1}) \overset{d}{=} (Y_{\eta'}, \ldots, Y_{\eta'+n-1}).
\]

(6)
Since \((X_\nu)\) is stationary,
\[(X_\eta, \ldots, X_{\eta+n-1}) \overset{d}{=} (X_{\eta'}, \ldots, X_{\eta'+n-1})\]
and consequently,
\[
(g(X_\eta), \ldots, g(X_{\eta+n-1})) \overset{d}{=} (g(X_{\eta'}), \ldots, g(X_{\eta'+n-1}))
\]
because applying a deterministic mapping (in this case the componentwise mapping \(g(\cdot)\)) to random vectors of equal law, results in random vectors of equal law. Since \((Y_\nu)\) is defined as \(g(X_\nu)\) we see that (7) establishes (6).

**Problem 7**

**Mapping a Discrete-Time WSS SP**

No, \((Y_\nu)\) need not be WSS, as the following example shows. Suppose that
\[
\ldots, X_{-3}, X_{-1}, X_1, X_3, \ldots
\]
are drawn IID each taking on the values ±1 equiprobably. Suppose that
\[
\ldots, X_{-4}, X_{-2}, X_0, X_2, X_4, \ldots
\]
are drawn independently of (8) and IID each taking on the values \(-\sqrt{3}/2, 0, \sqrt{3}/2\) equiprobably. We note that \((X_\nu)\) is WSS because — irrespective of whether \(\nu\) is odd or even — \(E[X_\nu] = 0\) and \(\Var[X_\nu] = 1\), and as to the covariance, \(\Cov[X_\nu, X_{\nu+\eta}]\) is zero whenever \(\eta\) is not zero, so
\[
\Cov[X_\nu, X_{\nu+\eta}] = 1\{\eta = 0\}, \quad \eta \in \mathbb{Z}.
\]
Consider now the mapping \(\xi \mapsto \xi^4\). We claim that the SP \((Y_\nu)\) defined by
\[
Y_\nu = g(X_\nu), \quad \nu \in \mathbb{Z}
\]
is not WSS. Indeed, \((Y_\nu)\) is deterministically 1 for \(\nu\) odd and takes on the values 9/4 and 0 with probabilities 2/3 and 1/3, respectively, when \(\eta\) is even. Thus, \(E[X_\nu]\) is 1 when \(\nu\) is odd and is 3/2 when \(\nu\) is even, i.e., it does depend on \(\nu\), and \((Y_\nu)\) is not WSS.