Communication and Detection Theory

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Model Answers to Exercise 5 of March 21, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 ***Passband Signaling***

(i) Define $\mathbf{y}\colon t \mapsto e^{i2\pi f_0 t} x(t)$. Then

$$y(0) = x(0), \tag{1}$$

and, since $e^{i2\pi f_0 t} \neq 0$ for every $t \in \mathbb{R}$, it follows that for every $\ell \in \mathbb{Z}$

$$\Big(y(\ell \mathsf{T_s}) = 0\Big) \iff \Big(x(\ell \mathsf{T_s}) = 0\Big). \tag{2}$$

Consequently, by (1) and (2) $\mathbf{y}$ is a Nyquist pulse of parameter $\mathsf{T_s}$ if, and only if, $\mathbf{x}$ is a Nyquist pulse of parameter $\mathsf{T_s}$.

(ii) If $\mathbf{x}$ is a Nyquist pulse of parameter $\mathsf{T_s}$, then so is $t \mapsto \cos(2\pi f_0 t)\, x(t)$ because the value of $\cos(2\pi f_0 t)\, x(t)$ at $t = 0$ is that of $x(t)$ and $t = 0$, and for every $\ell \in \mathbb{Z}$

$$\Big(x(\ell \mathsf{T_s}) = 0\Big) \implies \Big(\cos(2\pi f_0 \ell \mathsf{T_s})\, x(\ell \mathsf{T_s}) = 0\Big). \tag{3}$$

(iii) Even if $t \mapsto \cos(2\pi f_0 t)\, x(t)$ is a Nyquist pulse of parameter $\mathsf{T_s}$, the signal $\mathbf{x}$ need not be one. For example, suppose that

$$f_0 = \frac{1}{4\mathsf{T_s}}$$

and that $\mathbf{x}$ is a Nyquist pulse of parameter $2\mathsf{T_s}$ but not of parameter $\mathsf{T_s}$. We show that although $\mathbf{x}$ is not a Nyquist pulse of parameter $\mathsf{T_s}$, the pulse $t \mapsto \cos(2\pi f_0 t)\, x(t)$ is. Clearly the values at zero of $\mathbf{x}$ and $t \mapsto \cos(2\pi f_0 t)\, x(t)$ are equal, so the fact that $\mathbf{x}$ is a Nyquist pulse (of parameter $2\mathsf{T_s}$) implies that the value of $t \mapsto \cos(2\pi f_0 t)\, x(t)$ at $t = 0$ is 1. If ℓ is even but not zero, then the value of $\mathbf{x}$ at $t = \ell \mathsf{T_s}$ must be zero (because $\mathbf{x}$ is a Nyquist pulse of parameter $2\mathsf{T_s}$) so the same must be true of $t \mapsto \cos(2\pi f_0 t)\, x(t)$. And if ℓ is odd, then the value of $t \mapsto \cos(2\pi f_0 t)\, x(t)$ at $t = \ell \mathsf{T_s}$ must be zero because $\cos(2\pi f_0 \ell \mathsf{T_s})$ is then zero.

Problem 2 ***The Self-Similarity Function of a Delayed Signal***

$$\begin{aligned}
\mathsf{R}_{\mathbf{vv}}(\tau) &= \int_{-\infty}^{\infty} v(t+\tau)\, v^*(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} u(t+\tau-t_0)\, u^*(t-t_0)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} u(\tilde{t}+\tau)\, u^*(\tilde{t})\,\mathrm{d}\tilde{t} \\
&= \mathsf{R}_{\mathbf{uu}}(\tau), \quad \tau \in \mathbb{R},
\end{aligned}$$

where we have substituted $\tilde{t}$ for $t - t_0$.

Problem 3 ***The Self-Similarity Function of a Frequency Shifted Signal***

$$\begin{aligned}
\mathsf{R}_{\mathbf{vv}}(\tau) &= \int_{-\infty}^{\infty} v(t+\tau)\, v^*(t)\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} u(t+\tau)\, e^{\mathrm{i}2\pi f_0 (t+\tau)} u^*(t)\, e^{-\mathrm{i}2\pi f_0 t}\,\mathrm{d}t \\
&= e^{\mathrm{i}2\pi f_0 \tau} \int_{-\infty}^{\infty} u(t+\tau) u^*(t)\,\mathrm{d}t \\
&= e^{\mathrm{i}2\pi f_0 \tau}\, \mathsf{R}_{\mathbf{uu}}(\tau), \quad \tau \in \mathbb{R}.
\end{aligned}$$

Problem 4 ***Relaxing the Orthonormality Condition***

The condition that the times-shifts of a signal $\boldsymbol{\phi}$ by *even* multiples of T_s are orthonormal, is equivalent to the condition that the time-shifts of $\boldsymbol{\phi}$ by *all* integer multiples of *twice*-T_s are orthonormal. Thus, by Corollary 11.3.5, the minimum bandwidth of a signal whose time shifts by even multiples of T_s are orthonormal is $1/(2(2\mathsf{T}_\mathrm{s}))$, i.e.,

$$\frac{1}{4\mathsf{T}_\mathrm{s}}.$$

The condition that the times-shifts of a signal $\boldsymbol{\phi}$ by *odd* multiples of T_s are orthonormal, is equivalent to the condition that the time-shifts of $t \mapsto \phi(t-\mathsf{T}_\mathrm{s})$ by *even* multiples of T_s are orthonormal. Thus, for this condition to hold, the bandwidth of $t \mapsto \phi(t-\mathsf{T}_\mathrm{s})$ must be at least $1/(4\mathsf{T}_\mathrm{s})$. And since the bandwidth of $\boldsymbol{\phi}$ is the same as the bandwidth of $t \mapsto \phi(t-\mathsf{T}_\mathrm{s})$, the bandwidth $\boldsymbol{\phi}$ must be at least $1/(4\mathsf{T}_\mathrm{s})$.

Problem 5 ***A Specific Signal***

(i) See Figure 0.1.

(ii) Define

$$g(f) = \mathsf{T}_\mathrm{s}\bigl(1 - |\mathsf{T}_\mathrm{s} f - 1|\bigr)\, \mathrm{I}\Bigl\{0 \le f \le \frac{2}{\mathsf{T}_\mathrm{s}}\Bigr\}, \quad f \in \mathbb{R}.$$

Then $\mathbf{g} \in \mathcal{L}_1 \cap \mathcal{L}_2$ and, by Proposition 6.4.5 Part (i),

$$p(t) = \check{g}(t), \quad t \in \mathbb{R}.$$

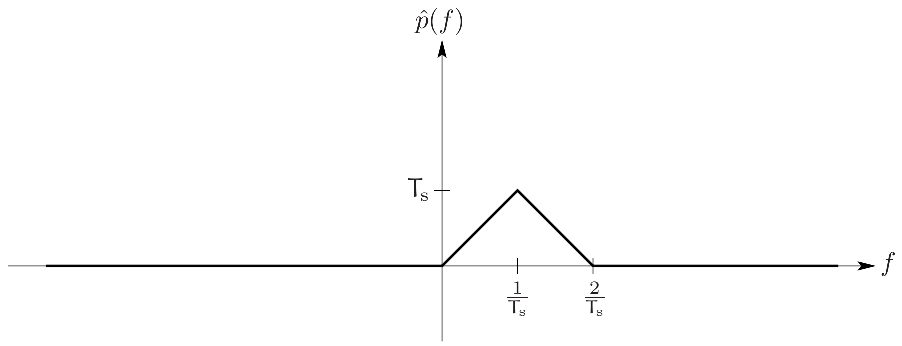

Figure 0.1: Plot of $\hat{p}(\cdot)$.

The assumptions of Theorem 11.3.2 are thus satisfied and $p(\cdot)$ is a Nyquist Pulse of parameter $\mathsf{T_s}$ if, and only if,

$$\lim_{\mathsf{J}\to\infty} \int_{-1/(2\mathsf{T_s})}^{1/(2\mathsf{T_s})} \left| \mathsf{T_s} - \sum_{j=-\mathsf{J}}^{\mathsf{J}} g\Big(f + \frac{j}{\mathsf{T_s}}\Big) \right| \mathrm{d}f = 0. \tag{4}$$

Since $\mathbf{g}$ is symmetric around $1/\mathsf{T_s}$, this condition is equivalent to

$$\lim_{\mathsf{J}\to\infty} \int_{0}^{1/(2\mathsf{T_s})} \left| \mathsf{T_s} - \sum_{j=-\mathsf{J}}^{\mathsf{J}} g\Big(f + \frac{j}{\mathsf{T_s}}\Big) \right| \mathrm{d}f = 0. \tag{5}$$

On the interval $(0, 1/(2\mathsf{T_s}))$ only three terms contribute to the sum: the terms corresponding to $j = 0$ and $j = \pm 1$. Thus, to verify that $\check{\mathbf{g}}$ is a Nyquist Pulse of parameter $\mathsf{T_s}$ we only need to show that

$$g\Big(f - \frac{1}{\mathsf{T_s}}\Big) + g(f) + g\Big(f + \frac{1}{\mathsf{T_s}}\Big) = \mathsf{T_s}, \quad 0 < t < \frac{1}{2\mathsf{T_s}}.$$

This can be done graphically or algebraically.

(iii) Since $p(\cdot)$ is a Nyquist Pulse of parameter $\mathsf{T_s}$, so is $\mathrm{Re}\big(p(\cdot)\big)$ because

$$\big(p(0) = 1\big) \Rightarrow \big(\mathrm{Re}\big(p(0)\big) = 1\big)$$

and

$$\big(p(\ell\mathsf{T_s}) = 0,\ \ell \in \mathbb{Z} \setminus \{0\}\big) \Rightarrow \Big(\mathrm{Re}\big(p(\ell\mathsf{T_s})\big) = 0,\ \ell \in \mathbb{Z} \setminus \{0\}\Big).$$

(iv) The imaginary part of $p(\cdot)$ is not a Nyquist pulse because

$$\begin{aligned} \mathrm{Im}\big(p(0)\big) &= \mathrm{Im}(1) \\ &= 0 \\ &\neq 1. \end{aligned}$$

Problem 6 *Mapping a Discrete-Time Stationary SP*

We need to show that for every positive integer n and all choices of $\eta, \eta' \in \mathbb{Z}$

$$(Y_\eta, \ldots, Y_{\eta+n-1}) \stackrel{\mathscr{L}}{=} (Y_{\eta'}, \ldots, Y_{\eta'+n-1}). \tag{6}$$

Since $\big(X_\nu\big)$ is stationary,

$$(X_\eta, \ldots, X_{\eta+n-1}) \stackrel{\mathscr{L}}{=} (X_{\eta'}, \ldots, X_{\eta'+n-1})$$

and consequently,

$$\big(g(X_\eta),\dots,g(X_{\eta+n-1})\big) \stackrel{\mathscr{L}}{=} \big(g(X_{\eta'}),\dots,g(X_{\eta'+n-1})\big) \tag{7}$$

because applying a deterministic mapping (in this case the componentwise mapping $g(\cdot)$) to random vectors of equal law, results in random vectors of equal law. Since $\big(Y_\nu\big)$ is defined as $g(X_\nu)$ we see that (7) establishes (6).

Problem 7 — ***Mapping a Discrete-Time WSS SP***

No, $\big(Y_\nu\big)$ need not be WSS, as the following example shows. Suppose that

$$\dots, X_{-3}, X_{-1}, X_1, X_3, \dots \tag{8}$$

are drawn IID each taking on the values ± 1 equiprobably. Suppose that

$$\dots, X_{-4}, X_{-2}, X_0, X_2, X_4, \dots$$

are drawn independently of (8) and IID each taking on the values $-\sqrt{3/2}$, 0, $\sqrt{3/2}$ equiprobably. We note that $\big(X_\nu\big)$ is WSS because — irrespective of whether ν is odd or even — $\mathsf{E}[X_\nu] = 0$ and $\mathsf{Var}[X_\nu] = 1$, and as to the covariance, $\mathsf{Cov}[X_\nu, X_{\nu+\eta}]$ is zero whenever η is not zero, so

$$\mathsf{Cov}[X_\nu, X_{\nu+\eta}] = \mathrm{I}\{\eta = 0\}, \quad \eta \in \mathbb{Z}.$$

Consider now the mapping $\xi \mapsto \xi^4$. We claim that the SP $\big(Y_\nu\big)$ defined by

$$Y_\nu = g(X_\nu), \quad \nu \in \mathbb{Z}$$

is not WSS. Indeed, $\big(Y_\nu\big)$ is deterministically 1 for ν odd and takes on the values 9/4 and 0 with probabilities 2/3 and 1/3, respectively, when η is even. Thus, $\mathsf{E}[X_\nu]$ is 1 when ν is odd and is 3/2 when ν is even, i.e., it *does* depend on ν, and $\big(Y_\nu\big)$ is not WSS.