Model Answers to Exercise 6 of March 27, 2018

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Problem 1

Superimposing Independent Transmissions

Since the data streams \((D_j^{(1)})\) and \((D_j^{(1)})\) are independent, it follows that the sequences of symbols \((X_\ell^{(1)})\) and \((X_\ell^{(2)})\) are independent. Consequently, the stochastic processes \((X^{(1)}(t))\) and \((X^{(2)}(t))\) are independent. Therefore, a fortiori, at every epoch \(t \in \mathbb{R}\) the random variables \(X^{(1)}(t)\) and \(X^{(2)}(t)\) are independent and hence uncorrelated. Since they are both of zero mean,

\[
E[X^{(1)}(t)X^{(2)}(t)] = 0, \quad t \in \mathbb{R}.
\]

The power \(P\) in the sum of the processes is thus

\[
P = \lim_{T \to \infty} \frac{1}{2T} E \left[ \int_{-T}^{T} (X^{(1)}(t) + X^{(2)}(t))^2 \, dt \right]
\]
\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E \left[ (X^{(1)}(t) + X^{(2)}(t))^2 \right] \, dt
\]
\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left( E \left[ (X^{(1)}(t))^2 \right] + E \left[ (X^{(2)}(t))^2 \right] \right) \, dt
\]
\[
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E \left[ (X^{(1)}(t))^2 \right] \, dt + \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E \left[ (X^{(2)}(t))^2 \right] \, dt
\]
\[
= P^{(1)} + P^{(2)}.
\]

Problem 2

The Minimum Distance of a Constellation and Power

Using Theorem 14.5.2 we obtain

\[
P = \frac{A^2}{T_s} \lim_{L \to \infty} \frac{1}{2L + 1} \sum_{\ell=-L}^{L} E[X_\ell^2]
\]
\[
= \frac{A^2}{T_s} E[X_0^2],
\]
where the second equality follows from the hypothesis that the symbols \((X_\ell)\) are IID and \textit{a fortiori} of the same second moment. It thus remains to compute this second moment:

\[
\mathbb{E}[X_0^2] = \frac{1}{\nu} \frac{d^2}{4} \left( 1^2 + 3^2 + \cdots + (2\nu - 1)^2 \right)
= \frac{1}{\nu} \frac{d^2}{4} \frac{(2\nu + 1)(2\nu - 1)}{3}
= \frac{1}{12} d^2 (2\nu + 1)(2\nu - 1).
\]

Hence,

\[
P = \frac{A^2}{T_s} \frac{d^2}{2} \frac{(2\nu + 1)(2\nu - 1)}{12}.
\]

**Problem 3**

\textit{PAM with Nonorthogonal Pulses}

Using (14.37) and noting that for the repetition code \(X_1 = X_2\), we obtain

\[
P = \frac{1}{2T_s} \mathbb{E} \left[ \int_{-\infty}^{\infty} A^2 \sum_{\ell=1}^{2} X_\ell g(t - \ell T_s) \right]^2 dt
= \frac{1}{2T_s} \mathbb{E} \left[ \int_{-\infty}^{\infty} A^2 X_1^2 \left( g(t - T_s) + g(t - 2T_s) \right)^2 dt \right]
= \frac{1}{2T_s} \mathbb{E} \left[ \int_{-\infty}^{\infty} A^2 X_1^2 \left( g(t) + g(t - T_s) \right)^2 dt \right]
= \frac{1}{2T_s} A^2 \mathbb{E}[X_1^2] \int_{-\infty}^{\infty} \left( I\{|t| \leq T_s\} + I\{|t - T_s| \leq T_s\} \right)^2 dt
= \frac{A^2}{2T_s} \int_{-\infty}^{\infty} \left( I\{|t| \leq T_s\} + I\{|t - T_s| \leq T_s\} \right)^2 dt
= \frac{A^2}{2T_s} 6T_s
= 3A^2.
\]

**Problem 4**

\textit{Non-IID Data Bits}

The following example demonstrates that if the data bits are not IID then (14.37) need not hold. Suppose that the symbols \((X_\ell)\) are produced from the data bits \((D_j)\) according to the antipodal rule

\[
0 \mapsto \rightarrow 1, \quad 1 \mapsto \rightarrow -1.
\]

Further assume that the data bits are not IID but that instead the odd data bits are IID random bits but \(D_{2\ell} = D_{2\ell-1}\) for all \(\ell \in \mathbb{Z}\). Thus,

\[
\ldots, D_3, D_{-1}, D_1, D_3, D_5, \ldots,
\]

are IID random bits but

\[
\ldots, D_0 = D_{-1}, \quad D_2 = D_1, \quad D_4 = D_3, \ldots
\]

Further assume the pulse shape \(g: t \mapsto I\{|t| \leq T_s\}\), where \(T_s\) is the baud period. This scenario can be viewed as though the IID random bits \((D_{2j+1})\) where fed to a blocklength \(N = 2\) repetition code as in Exercise 14.3. From that exercise we conclude that the transmitted power is

\[
P = 3A^2.
\]
However, if we ignored the dependence and computed the RHS of (14.37) (with N = 1) we would get $2A^2$, which is clearly not the transmitted power.

**Problem 5**  
*The Power in Nonorthogonal PAM*

(i) By (14.33) the power in this case is given by

\[
P = \frac{1}{T_s} A^2 \|g\|^2\]

\[= \frac{1}{T_s} A^2 2T_s \]

\[= 2A^2.\]

(ii) Using (14.31) the power in this case is given by

\[
P = \frac{1}{T_s} A^2 \left( K_{XX} (−1) R_{gg} (−T_s) + K_{XX} (0) R_{gg} (0) + K_{XX} (1) R_{gg} (T_s) \right)\]

\[= \frac{1}{T_s} A^2 \left( \frac{1}{2} T_s + 1(2T_s) + \frac{1}{2} T_s \right) \]

\[= 3A^2.\]

**Problem 6**  
*Pre-Encoding*

This does not change the transmitted power. Indeed, since the original data bits $D_1, \ldots, D_K$ are IID random bits, it follows that the K-tuple $(D_1, \ldots, D_K)$ is uniformly distributed over the set $\{0, 1\}^K$ of all binary K-tuples. Since $\phi$ is a one-to-one mapping, it follows that the K-tuple $(D'_1, \ldots, D'_K)$ is also uniformly distributed over $\{0, 1\}^K$, so $D'_1, \ldots, D'_K$ are also IID random bits. Thus, the statistics of the bits fed to the encoder in the two cases are the same, and the power is thus the same.