Communication and Detection Theory

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Model Answers to Exercise 6 of March 28, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *Superimposing Independent Transmissions*

Since the data streams $\big(D_j^{(1)}\big)$ and $\big(D_j^{(1)}\big)$ are independent, it follows that the sequences of symbols $\big(X_\ell^{(1)}\big)$ and $\big(X_\ell^{(2)}\big)$ are independent. Consequently, the stochastic processes $\big(X^{(1)}(t)\big)$ and $\big(X^{(2)}(t)\big)$ are independent. Therefore, *a fortiori*, at every epoch $t \in \mathbb{R}$ the random variables $X^{(1)}(t)$ and $X^{(2)}(t)$ are independent and hence uncorrelated. Since they are both of zero mean,

$$\mathsf{E}\big[X^{(1)}(t)\,X^{(2)}(t)\big] = 0, \quad t \in \mathbb{R}.$$

The power P in the sum of the processes is thus

$$\begin{aligned}
\mathsf{P} &= \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \mathsf{E}\bigg[\int_{-\mathsf{T}}^{\mathsf{T}} \big(X^{(1)}(t) + X^{(2)}(t)\big)^2 \,\mathrm{d}t\bigg] \\
&= \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \int_{-\mathsf{T}}^{\mathsf{T}} \mathsf{E}\Big[\big(X^{(1)}(t) + X^{(2)}(t)\big)^2\Big] \,\mathrm{d}t \\
&= \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \int_{-\mathsf{T}}^{\mathsf{T}} \Big(\mathsf{E}\Big[\big(X^{(1)}(t)\big)^2\Big] + \mathsf{E}\Big[\big(X^{(2)}(t)\big)^2\Big]\Big) \,\mathrm{d}t \\
&= \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \int_{-\mathsf{T}}^{\mathsf{T}} \mathsf{E}\Big[\big(X^{(1)}(t)\big)^2\Big] \,\mathrm{d}t + \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \int_{-\mathsf{T}}^{\mathsf{T}} \mathsf{E}\Big[\big(X^{(2)}(t)\big)^2\Big] \,\mathrm{d}t \\
&= \mathsf{P}^{(1)} + \mathsf{P}^{(2)}.
\end{aligned}$$

Problem 2 — *The Minimum Distance of a Constellation and Power*

Using Theorem 14.5.2 we obtain

$$\begin{aligned}
\mathsf{P} &= \frac{\mathsf{A}^2}{\mathsf{T_s}} \lim_{\mathsf{L}\to\infty} \frac{1}{2\mathsf{L}+1} \sum_{\ell=-\mathsf{L}}^{\mathsf{L}} \mathsf{E}\big[X_\ell^2\big] \\
&= \frac{\mathsf{A}^2}{\mathsf{T_s}} \mathsf{E}\big[X_0^2\big],
\end{aligned}$$

where the second equality follows from the hypothesis that the symbols $\bigl(X_\ell\bigr)$ are IID and *a fortiori* of the same second moment. It thus remains to compute this second moment:

$$\begin{aligned}
\mathsf{E}\bigl[X_0^2\bigr] &= \frac{1}{\nu}\frac{d^2}{4}\Bigl(1^2+3^2+\cdots+(2\nu-1)^2\Bigr)\\
&= \frac{1}{\nu}\frac{d^2}{4}\frac{(2\nu+1)(2\nu-1)\nu}{3}\\
&= \frac{1}{12}d^2(2\nu+1)(2\nu-1).
\end{aligned}$$

Hence,

$$\mathsf{P} = \frac{\mathsf{A}^2}{\mathsf{T_s}}d^2\frac{(2\nu+1)(2\nu-1)}{12}.$$

Problem 3 — *PAM with Nonorthogonal Pulses*

Using (14.37) and noting that for the repetition code $X_1 = X_2$, we obtain

$$\begin{aligned}
\mathsf{P} &= \frac{1}{2\mathsf{T_s}}\mathsf{E}\Biggl[\int_{-\infty}^{\infty}\Biggl(\mathsf{A}\sum_{\ell=1}^{2}X_\ell\, g(t-\ell\mathsf{T_s})\Biggr)^2 \mathrm{d}t\Biggr]\\
&= \frac{1}{2\mathsf{T_s}}\mathsf{E}\Biggl[\int_{-\infty}^{\infty}\mathsf{A}^2X_1^2\Bigl(g(t-\mathsf{T_s})+g(t-2\mathsf{T_s})\Bigr)^2 \mathrm{d}t\Biggr]\\
&= \frac{1}{2\mathsf{T_s}}\mathsf{E}\Biggl[\int_{-\infty}^{\infty}\mathsf{A}^2X_1^2\Bigl(g(t)+g(t-\mathsf{T_s})\Bigr)^2 \mathrm{d}t\Biggr]\\
&= \frac{1}{2\mathsf{T_s}}\mathsf{A}^2\mathsf{E}\bigl[X_1^2\bigr]\int_{-\infty}^{\infty}\Bigl(\mathrm{I}\bigl\{|t|\le\mathsf{T_s}\bigr\}+\mathrm{I}\bigl\{|t-\mathsf{T_s}|\le\mathsf{T_s}\bigr\}\Bigr)^2 \mathrm{d}t\\
&= \frac{\mathsf{A}^2}{2\mathsf{T_s}}\int_{-\infty}^{\infty}\Bigl(\mathrm{I}\bigl\{|t|\le\mathsf{T_s}\bigr\}+\mathrm{I}\bigl\{|t-\mathsf{T_s}|\le\mathsf{T_s}\bigr\}\Bigr)^2 \mathrm{d}t\\
&= \frac{\mathsf{A}^2}{2\mathsf{T_s}}6\mathsf{T_s}\\
&= 3\mathsf{A}^2.
\end{aligned}$$

Problem 4 — *Non-IID Data Bits*

The following example demonstrates that if the data bits are not IID then (14.37) need not hold. Suppose that the symbols $\bigl(X_\ell\bigr)$ are produced from the data bits $\bigl(D_j\bigr)$ according to the antipodal rule

$$0 \mapsto 1, \quad 1 \mapsto -1.$$

Further assume that the data bits are not IID but that instead the odd data bits are IID random bits but $D_{2\ell} = D_{2\ell-1}$ for all $\ell \in \mathbb{Z}$. Thus,

$$\ldots, D_3, D_{-1}, D_1, D_3, D_5, \ldots,$$

are IID random bits but

$$\ldots, D_0 = D_{-1}, \quad D_2 = D_1, \quad D_4 = D_3, \ldots$$

Further assume the pulse shape $\mathbf{g}\colon t \mapsto \mathrm{I}\{|t| \le \mathsf{T_s}\}$, where $\mathsf{T_s}$ is the baud period. This scenario can be viewed as though the IID random bits $\bigl(D_{2j+1}\bigr)$ where fed to a blocklength $\mathsf{N} = 2$ repetition code as in Exercise 14.3. From that exercise we conclude that the transmitted power is

$$\mathsf{P} = 3\mathsf{A}^2.$$

However, if we ignored the dependence and computed the RHS of (14.37) (with $\mathsf{N} = 1$) we would get $2\mathsf{A}^2$, which is clearly not the transmitted power.

Problem 5 *The Power in Nonorthogonal PAM*

(i) By (14.33) the power in this case is given by

$$\begin{aligned}\mathsf{P} &= \frac{1}{\mathsf{T_s}}\mathsf{A}^2 \|\mathbf{g}\|_2^2 \\ &= \frac{1}{\mathsf{T_s}}\mathsf{A}^2 2\mathsf{T_s} \\ &= 2\mathsf{A}^2.\end{aligned}$$

(ii) Using (14.31) the power in this case is given by

$$\begin{aligned}\mathsf{P} &= \frac{1}{\mathsf{T_s}}\mathsf{A}^2\Big(\mathsf{K}_{XX}(-1)\,\mathsf{R}_{\mathbf{gg}}(-\mathsf{T_s}) + \mathsf{K}_{XX}(0)\,\mathsf{R}_{\mathbf{gg}}(0) + \mathsf{K}_{XX}(1)\,\mathsf{R}_{\mathbf{gg}}(\mathsf{T_s})\Big) \\ &= \frac{1}{\mathsf{T_s}}\mathsf{A}^2\Big(\frac{1}{2}\mathsf{T_s} + 1(2\mathsf{T_s}) + \frac{1}{2}\mathsf{T_s}\Big) \\ &= 3\mathsf{A}^2.\end{aligned}$$

Problem 6 *Pre-Encoding*

This does not change the transmitted power. Indeed, since the original data bits $D_1, \ldots, D_\mathsf{K}$ are IID random bits, it follows that the K-tuple $(D_1, \ldots, D_\mathsf{K})$ is uniformly distributed over the set $\{0,1\}^\mathsf{K}$ of all binary K-tuples. Since ϕ is a one-to-one mapping, it follows that the K-tuple $(D'_1, \ldots, D'_\mathsf{K})$ is also uniformly distributed over $\{0,1\}^\mathsf{K}$, so $D'_1, \ldots, D'_\mathsf{K}$ are also IID random bits. Thus, the statistics of the bits fed to the encoder in the two cases are the same, and the power is thus the same.