Model Answers to Exercise 7 of April 10, 2018

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Problem 1

Scaling a SP

Since

\[(\alpha X) \ast h = \alpha (X \ast h),\]

we conclude that

\[Y \ast h = \alpha (X \ast h),\]

so

\[
\text{Power of } Y \ast h = \text{Power of } \alpha (X \ast h) = \alpha^2 \text{Power of } X \ast h = \int_{-\infty}^{\infty} \alpha^2 S_{XX}(f) |\hat{h}(f)|^2 \, df.
\]

Since \(S_{XX}\) is symmetric so is \(\alpha^2 S_{XX}(\cdot)\), and we can conclude that

\[S_{YY}(f) = \alpha^2 S_{XX}(f), \quad f \in \mathbb{R}.\]

Problem 2

The Operational PSD of a Sum of Independent SPs

Feeding \(X + Y\) to a stable filter of impulse response \(h\) produces the signal \(X \ast h + Y \ast h\). And, since \(X\) and \(Y\) are independent, so are \(X \ast h\) and \(Y \ast h\). Consequently,

\[
\text{Power in } X \ast h + Y \ast h = \lim_{T \to \infty} \frac{1}{2T} \mathbb{E} \left[ \int_{-T}^{T} ((X \ast h)(t) + (Y \ast h)(t))^2 \, dt \right]
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \left( \mathbb{E} \left[ \int_{-T}^{T} (X \ast h)(t)^2 \, dt \right] + \mathbb{E} \left[ \int_{-T}^{T} (Y \ast h)(t)^2 \, dt \right] \right)
\]

\[
= \text{Power in } X \ast h + \text{Power in } Y \ast h
\]

\[
= \int_{-\infty}^{\infty} S_{XX}(f) |\hat{h}(f)|^2 \, df + \int_{-\infty}^{\infty} S_{YY}(f) |\hat{h}(f)|^2 \, df
\]

\[
= \int_{-\infty}^{\infty} (S_{XX}(f) + S_{YY}(f)) |\hat{h}(f)|^2 \, df.
\]
Since this holds for every stable filter $h$, and since the symmetry of $S_{XX}$ and $S_{YY}$ implies the symmetry of $S_{XX} + S_{YY}$, the operational PSD of $X + Y$ is the sum of the operational PSDs of $X$ and $Y$.

**Problem 3**  
*Operational PSD of a Deterministic SP*

The operational PSD of a deterministic energy-limited signal $x$ is zero. This can be argued as follows. If $x \in L_2$ and $h \in L_1$, then $x \ast h$ is energy limited, i.e.

$$
\int_{-\infty}^{\infty} (x \ast h)^2(t) \, dt < \infty
$$

(substitute $p = 2$ in (5.11)). Consequently

$$
\text{Power in } X \ast h = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (x \ast h)^2(t) \, dt
$$

$$
\leq \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} (x \ast h)^2(t) \, dt
$$

$$
= 0
$$

$$
= \int_{-\infty}^{\infty} 0 \, |\hat{h}(f)|^2 \, df, \quad h \in L_1.
$$

Since this holds for every $h \in L_1$, and since the all-zero function is symmetric, we conclude that the operational PSD of $x$ is the all-zero function (of frequency).

**Problem 4**  
*Stretching Time*

For any stable $h$,

$$(Y \ast h)(t) = \int_{-\infty}^{\infty} Y(t - \tau) h(\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} X \left( \frac{t - \tau}{a} \right) h(\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} X \left( \frac{t - \tau}{a} \right) h(\tau) \, d\tau$$

$$= a \int_{-\infty}^{\infty} X \left( \frac{t}{a} - \sigma \right) h(a\sigma) \, d\sigma$$

$$= a \int_{-\infty}^{\infty} X \left( \frac{t}{a} - \sigma \right) h'(\sigma) \, d\sigma$$

$$= a \cdot (X \ast h')(\frac{t}{a}),$$

where

$$h': \sigma \mapsto h(a\sigma)$$

and is thus of FT

$$\hat{h}'(f) = \frac{1}{a} \hat{h}\left(\frac{f}{a}\right), \quad f \in \mathbb{R}.$$
Thus

\[
\text{Power of } Y \ast h = \lim_{T \to \infty} \frac{1}{2T} \mathbb{E} \left[ \int_{-T}^{T} a^2 (X \ast h')^2 \left( \frac{t}{a} \right) \, dt \right]
\]

\[
= \lim_{T \to \infty} \frac{1}{2T} \mathbb{E} \left[ \int_{-T/a}^{T/a} a^3 (X \ast h')^2 (\tau) \, d\tau \right]
\]

\[
= \lim_{T \to \infty} \frac{1}{2T/a} \mathbb{E} \left[ \int_{-T/a}^{T/a} a^2 (X \ast h')^2 (\tau) \, d\tau \right]
\]

\[
= a^2 \text{ Power of } X \ast h'
\]

\[
= a^2 \int_{-\infty}^{\infty} S_{XX}(f) |\hat{h}'(f)|^2 \, df
\]

\[
= \int_{-\infty}^{\infty} S_{XX}(f) |\hat{h} (\frac{f}{a})|^2 \, df
\]

\[
= \int_{-\infty}^{\infty} a S_{XX}(a\tilde{f}) |\hat{h}(\tilde{f})|^2 \, d\tilde{f}.
\]

Since this holds for every \( h \in \mathcal{L}_t \), and since the mapping 

\[
\tilde{f} \mapsto a S_{XX}(a\tilde{f})
\]

is symmetric, this mapping must be the operational PSD of \( (Y(t)) \).

**Problem 5**

**The Operational PSD of PAM**

(i) A sample function of \( (X_1(t), t \in \mathbb{R}) \) might look as follows:

![Graph of X1(t)]

(ii) The symbols \( (X_\ell) \) are uncorrelated and of unit variance, i.e.,

\[
K_{XX}(m) = I\{m = 0\}, \quad m \in \mathbb{Z}.
\]

Therefore, by (15.24) the operational PSD of \( (X_1(t), t \in \mathbb{R}) \) is

\[
S_{X_1X_1}(f) = \frac{A^2}{T_s} |\hat{g}(f)|^2 = A^2 T_s |\text{sinc}(T_sf)|^2, \quad f \in \mathbb{R}.
\]

(iii) A sample function of \( X_2(\cdot) \) is for instance

![Graph of X2(t)]
The autocovariance function $K_{XX}$ of $(X_t)$ is the same as above but the baud period is double. Therefore the operational PSD of $(X_2(t), t \in \mathbb{R})$ is
\[ S_{X_2X_2}(f) = \frac{A^2}{2T_s} |\hat{g}(f)|^2 = \frac{A^2T_s}{2} |\text{sinc}(T_s f)|^2, \quad f \in \mathbb{R}. \]

(iv) The operational PSD of $(X_2(t))$ is half that of $(X_1(t))$.

Problem 6

The Operational PSD and Block Codes

Here we consider a $(1, 2)$ binary-to-reals block encoder, so $N = 2$. By direct computation we obtain
\[ E[X_\ell X_{\ell'}] = \begin{cases} 1 & \ell = \ell', \\ -1 & \ell \neq \ell', \quad \ell, \ell' \in \{1, 2\}. \end{cases} \]

Using this and (14.37) we obtain that the power in bi-infinite block mode is
\[
P = \frac{1}{NT_s} E \left[ \int_{-\infty}^{\infty} \left( A \sum_{\ell=1}^{N} X_\ell g(t - \ell T_s) \right)^2 dt \right] \\
= \frac{A^2}{2T_s} \int_{-\infty}^{\infty} E \left[ \left( A \sum_{\ell=1}^{N} X_\ell g(t - \ell T_s) \right)^2 \right] dt \\
= \frac{A^2}{2T_s} \int_{-\infty}^{\infty} \left[ A^2 \sum_{\ell=1}^{N} X_\ell^2 g^2(t - \ell T_s) + 2X_1X_2 g(t - T_s) g(t - 2T_s) + X_2^2 g^2(t - 2T_s) \right] dt \\
= \frac{A^2}{2T_s} \int_{-\infty}^{\infty} \left[ g^2(t - T_s) - 2X_1X_2 g(t - T_s) g(t - 2T_s) + g^2(t - 2T_s) \right] dt \\
= \frac{A^2}{T_s} \left( \|g\|_2^2 - R_{gg}(T_s) \right) \\
= \frac{A^2}{T_s} \int_{-\infty}^{\infty} \left( |\hat{g}(f)|^2 - |\hat{g}(f)|^2 e^{i2\pi fT_s} \right) df \\
= \int_{-\infty}^{\infty} \frac{A^2}{T_s} |\hat{g}(f)|^2 \left( 1 - e^{i2\pi fT_s} \right) df.
\]

By (15.26), the operational PSD is
\[
S_{XX}(f) = \frac{A^2}{NT_s} \sum_{\ell=1}^{N} \sum_{\ell'=1}^{N} E[X_\ell X_{\ell'}] e^{i2\pi f(\ell - \ell')T_s} |\hat{g}(f)|^2 \\
= \frac{A^2}{2T_s} (2 - e^{i2\pi fT_s} - e^{-i2\pi fT_s}) |\hat{g}(f)|^2 \\
= \frac{A^2}{T_s} (1 - \cos(2\pi fT_s)) |\hat{g}(f)|^2, \quad f \in \mathbb{R}.
\]