Communication and Detection Theory

Signal and Information Processing Laboratory
Institut für Signal- und Informationsverarbeitung

Spring Semester 2017

Prof. Dr. A. Lapidoth

Model Answers to Exercise 7 of April 4, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *Scaling a SP*

Since

$$(\alpha\mathbf{X}) \star \mathbf{h} = \alpha(\mathbf{X} \star \mathbf{h}),$$

we conclude that

$$\mathbf{Y} \star \mathbf{h} = \alpha(\mathbf{X} \star \mathbf{h}),$$

so

$$\begin{aligned}\text{Power of } \mathbf{Y} \star \mathbf{h} &= \text{Power of } \alpha(\mathbf{X} \star \mathbf{h}) \\ &= \alpha^2 \text{ Power of } \mathbf{X} \star \mathbf{h} \\ &= \int_{-\infty}^{\infty} \alpha^2\, \mathsf{S}_{XX}(f) \left|\hat{h}(f)\right|^2 \mathrm{d}f.\end{aligned}$$

Since S_{XX} is symmetric so is $\alpha^2\, \mathsf{S}_{XX}(\cdot)$, and we can conclude that

$$\mathsf{S}_{YY}(f) = \alpha^2\, \mathsf{S}_{XX}(f), \quad f \in \mathbb{R}.$$

Problem 2 — *The Operational PSD of a Sum of Independent SPs*

Feeding $\mathbf{X} + \mathbf{Y}$ to a stable filter of impulse response $\mathbf{h}$ produces the signal $\mathbf{X} \star \mathbf{h} + \mathbf{Y} \star \mathbf{h}$. And, since $\mathbf{X}$ and $\mathbf{Y}$ are independent, so are $\mathbf{X} \star \mathbf{h}$ and $\mathbf{Y} \star \mathbf{h}$. Consequently,

$$\begin{aligned}&\text{Power in } \mathbf{X} \star \mathbf{h} + \mathbf{Y} \star \mathbf{h} \\ &\quad = \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \mathsf{E}\left[\int_{-\mathsf{T}}^{\mathsf{T}} \big((\mathbf{X} \star \mathbf{h})(t) + (\mathbf{Y} \star \mathbf{h})(t)\big)^2 \mathrm{d}t\right] \\ &\quad = \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \left(\mathsf{E}\left[\int_{-\mathsf{T}}^{\mathsf{T}} \big((\mathbf{X} \star \mathbf{h})(t)\big)^2 \mathrm{d}t\right] + \mathsf{E}\left[\int_{-\mathsf{T}}^{\mathsf{T}} \big((\mathbf{Y} \star \mathbf{h})(t)\big)^2 \mathrm{d}t\right]\right) \\ &\quad = \text{Power in } \mathbf{X} \star \mathbf{h} \; + \; \text{Power in } \mathbf{Y} \star \mathbf{h} \\ &\quad = \int_{-\infty}^{\infty} \mathsf{S}_{XX}(f) \left|\hat{h}(f)\right|^2 \mathrm{d}f + \int_{-\infty}^{\infty} \mathsf{S}_{YY}(f) \left|\hat{h}(f)\right|^2 \mathrm{d}f \\ &\quad = \int_{-\infty}^{\infty} \big(\mathsf{S}_{XX}(f) + \mathsf{S}_{YY}(f)\big) \left|\hat{h}(f)\right|^2 \mathrm{d}f.\end{aligned}$$

Since this holds for every stable filter $\mathbf{h}$, and since the symmetry of S_{XX} and S_{YY} implies the symmetry of $\mathsf{S}_{XX} + \mathsf{S}_{YY}$, the operational PSD of $\mathbf{X} + \mathbf{Y}$ is the sum of the operational PSDs of $\mathbf{X}$ and $\mathbf{Y}$.

Problem 3 — *Operational PSD of a Deterministic SP*

The operational PSD of a deterministic energy-limited signal $\mathbf{x}$ is zero. This can be argued as follows. If $\mathbf{x} \in \mathcal{L}_2$ and $\mathbf{h} \in \mathcal{L}_1$, then $\mathbf{x} \star \mathbf{h}$ is energy limited, i.e.

$$\int_{-\infty}^{\infty} (\mathbf{x} \star \mathbf{h})^2(t)\,\mathrm{d}t < \infty$$

(substitute $p = 2$ in (5.11)). Consequently

$$\begin{aligned}
\text{Power in } \mathbf{X} \star \mathbf{h} &= \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \int_{-\mathsf{T}}^{\mathsf{T}} (\mathbf{x} \star \mathbf{h})^2(t)\,\mathrm{d}t \\
&\leq \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \int_{-\infty}^{\infty} (\mathbf{x} \star \mathbf{h})^2(t)\,\mathrm{d}t \\
&= 0 \\
&= \int_{-\infty}^{\infty} 0 \left|\hat{h}(f)\right|^2 \mathrm{d}f, \quad \mathbf{h} \in \mathcal{L}_1.
\end{aligned}$$

Since this holds for every $\mathbf{h} \in \mathcal{L}_1$, and since the all-zero function is symmetric, we conclude that the operational PSD of $\mathbf{x}$ is the all-zero function (of frequency).

Problem 4 — *Stretching Time*

For any stable $\mathbf{h}$,

$$\begin{aligned}
(\mathbf{Y} \star \mathbf{h})(t) &= \int_{-\infty}^{\infty} Y(t-\tau)\,h(\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} X\left(\frac{t-\tau}{a}\right) h(\tau)\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} X\left(\frac{t}{a} - \frac{\tau}{a}\right) h(\tau)\,\mathrm{d}\tau \\
&= a \int_{-\infty}^{\infty} X\left(\frac{t}{a} - \sigma\right) h(a\sigma)\,\mathrm{d}\sigma \\
&= a \int_{-\infty}^{\infty} X\left(\frac{t}{a} - \sigma\right) h'(\sigma)\,\mathrm{d}\sigma \\
&= a \cdot \bigl(\mathbf{X} \star \mathbf{h}'\bigr)\left(\frac{t}{a}\right),
\end{aligned}$$

where

$$\mathbf{h}' \colon \sigma \mapsto h(a\sigma)$$

and is thus of FT

$$\hat{h}'(f) = \frac{1}{a}\,\hat{h}\left(\frac{f}{a}\right), \quad f \in \mathbb{R}.$$

Thus

$$\begin{aligned}
\text{Power of } \mathbf{Y} \star \mathbf{h} &= \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \mathsf{E}\biggl[\int_{-\mathsf{T}}^{\mathsf{T}} a^2 \bigl(\mathbf{X} \star \mathbf{h}'\bigr)^2 \Bigl(\frac{t}{a}\Bigr) \,\mathrm{d}t\biggr] \\
&= \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}} \mathsf{E}\biggl[\int_{-\mathsf{T}/a}^{\mathsf{T}/a} a^3 \bigl(\mathbf{X} \star \mathbf{h}'\bigr)^2 (\tau) \,\mathrm{d}\tau\biggr] \\
&= \lim_{\mathsf{T}\to\infty} \frac{1}{2\mathsf{T}/a} \mathsf{E}\biggl[\int_{-\mathsf{T}/a}^{\mathsf{T}/a} a^2 \bigl(\mathbf{X} \star \mathbf{h}'\bigr)^2 (\tau) \,\mathrm{d}\tau\biggr] \\
&= a^2 \text{ Power of } \mathbf{X} \star \mathbf{h}' \\
&= a^2 \int_{-\infty}^{\infty} \mathsf{S}_{XX}(f) \left|\hat{h}'(f)\right|^2 \mathrm{d}f \\
&= \int_{-\infty}^{\infty} \mathsf{S}_{XX}(f) \left|\hat{h}\Bigl(\frac{f}{a}\Bigr)\right|^2 \mathrm{d}f \\
&= \int_{-\infty}^{\infty} a\, \mathsf{S}_{XX}\bigl(a\tilde{f}\bigr) \bigl|\hat{h}(\tilde{f})\bigr|^2 \,\mathrm{d}\tilde{f}.
\end{aligned}$$

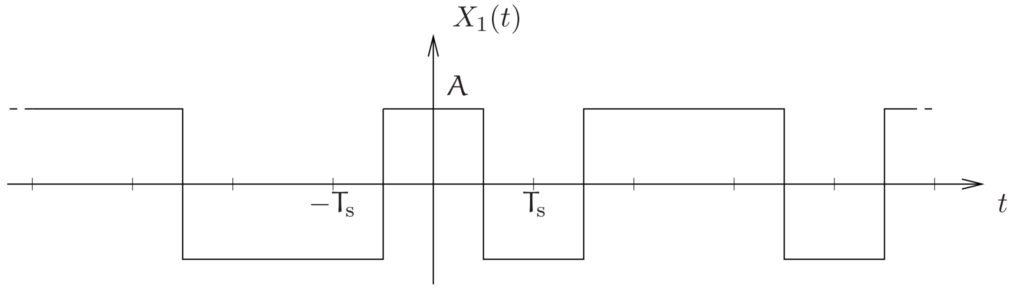
Since this holds for every $\mathbf{h} \in \mathcal{L}_1$, and since the mapping

$$\tilde{f} \mapsto a\, \mathsf{S}_{XX}\bigl(a\tilde{f}\bigr)$$

is symmetric, this mapping must be the operational PSD of $\bigl(Y(t)\bigr)$.

Problem 5 — *The Operational PSD of PAM*

(i) A sample function of $\bigl(X_1(t),\ t \in \mathbb{R}\bigr)$ might look as follows:



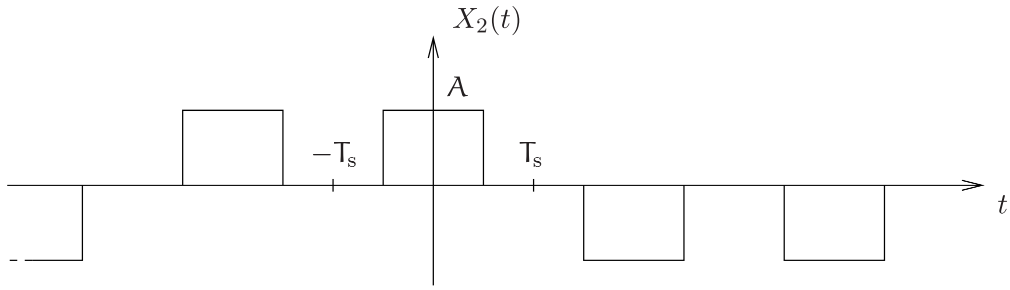
(ii) The symbols $\bigl(X_\ell\bigr)$ are uncorrelated and of unit variance, i.e.,

$$\mathsf{K}_{XX}(m) = \mathrm{I}\{m = 0\}, \quad m \in \mathbb{Z}.$$

Therefore, by (15.24) the operational PSD of $\bigl(X_1(t),\ t \in \mathbb{R}\bigr)$ is

$$\mathsf{S}_{X_1X_1}(f) = \frac{\mathsf{A}^2}{\mathsf{T}_\mathrm{s}} \bigl|\hat{g}(f)\bigr|^2 = \mathsf{A}^2 \mathsf{T}_\mathrm{s} \bigl|\mathrm{sinc}(\mathsf{T}_\mathrm{s} f)\bigr|^2, \quad f \in \mathbb{R}.$$

(iii) A sample function of $X_2(\cdot)$ is for instance



The autocovariance function K_{XX} of (X_ℓ) is the same as above but the baud period is double. Therefore the operational PSD of $(X_2(t),\ t \in \mathbb{R})$ is

$$\mathsf{S}_{X_2X_2}(f) = \frac{\mathsf{A}^2}{2\mathsf{T_s}}\big|\hat{g}(f)\big|^2 = \frac{\mathsf{A}^2\mathsf{T_s}}{2}\big|\mathrm{sinc}(\mathsf{T_s}f)\big|^2, \quad f \in \mathbb{R}.$$

(iv) The operational PSD of $(X_2(t))$ is half that of $(X_1(t))$.

Problem 6 — *The Operational PSD and Block Codes*

Here we consider a $(1,2)$ binary-to-reals block encoder, so $\mathsf{N} = 2$. By direct computation we obtain

$$\mathsf{E}[X_\ell X_{\ell'}] = \begin{cases} 1 & \ell' = \ell, \\ -1 & \ell' \neq \ell, \end{cases} \quad \ell, \ell' \in \{1,2\}.$$

Using this and (14.37) we obtain that the power in bi-infinite block mode is

$$\begin{aligned}
\mathsf{P} &= \frac{1}{\mathsf{N}\mathsf{T_s}}\mathsf{E}\left[\int_{-\infty}^{\infty}\left(\mathsf{A}\sum_{\ell=1}^{\mathsf{N}} X_\ell\, g(t-\ell\mathsf{T_s})\right)^2 \mathrm{d}t\right] \\
&= \frac{1}{2\mathsf{T_s}}\int_{-\infty}^{\infty}\mathsf{E}\left[\left(\mathsf{A}\sum_{\ell=1}^{2} X_\ell\, g(t-\ell\mathsf{T_s})\right)^2\right]\mathrm{d}t \\
&= \frac{\mathsf{A}^2}{2\mathsf{T_s}}\int_{-\infty}^{\infty}\mathsf{E}\Big[X_1^2\, g^2(t-\mathsf{T_s}) + 2X_1X_2\, g(t-\mathsf{T_s})\, g(t-2\mathsf{T_s}) + X_2^2\, g^2(t-2\mathsf{T_s})\Big]\,\mathrm{d}t \\
&= \frac{\mathsf{A}^2}{2\mathsf{T_s}}\int_{-\infty}^{\infty}\big(g^2(t-\mathsf{T_s}) - 2\, g(t-\mathsf{T_s})\, g(t-2\mathsf{T_s}) + g^2(t-2\mathsf{T_s})\big)\,\mathrm{d}t \\
&= \frac{\mathsf{A}^2}{\mathsf{T_s}}\left(\|\mathbf{g}\|_2^2 - \mathsf{R}_{\mathbf{gg}}(\mathsf{T_s})\right) \\
&= \frac{\mathsf{A}^2}{\mathsf{T_s}}\int_{-\infty}^{\infty}\left(|\hat{g}(f)|^2 - |\hat{g}(f)|^2 e^{\mathrm{i}2\pi f\mathsf{T_s}}\right)\mathrm{d}f \\
&= \int_{-\infty}^{\infty}\frac{\mathsf{A}^2}{\mathsf{T_s}}|\hat{g}(f)|^2\left(1 - e^{\mathrm{i}2\pi f\mathsf{T_s}}\right)\mathrm{d}f.
\end{aligned}$$

By (15.26), the operational PSD is

$$\begin{aligned}
\mathsf{S}_{XX}(f) &= \frac{\mathsf{A}^2}{\mathsf{N}\mathsf{T_s}}\sum_{\ell=1}^{\mathsf{N}}\sum_{\ell'=1}^{\mathsf{N}}\mathsf{E}[X_\ell X_{\ell'}]\, e^{\mathrm{i}2\pi f(\ell-\ell')\mathsf{T_s}}|\hat{g}(f)|^2 \\
&= \frac{\mathsf{A}^2}{2\mathsf{T_s}}\big(2 - e^{\mathrm{i}2\pi f\mathsf{T_s}} - e^{-\mathrm{i}2\pi f\mathsf{T_s}}\big)\big|\hat{g}(f)\big|^2 \\
&= \frac{\mathsf{A}^2}{\mathsf{T_s}}\big(1 - \cos(2\pi f\mathsf{T_s})\big)\big|\hat{g}(f)\big|^2, \quad f \in \mathbb{R}.
\end{aligned}$$