Problem 1  
**Scaling a SP**

Since 
\[(αX) ∗ h = α(X ∗ h),\]
we conclude that 
\[Y ∗ h = α(X ∗ h),\]
so

\[
\text{Power of } Y ∗ h = \text{Power of } α(X ∗ h) \\
= α^2 \text{Power of } X ∗ h \\
= \int_{−∞}^{∞} α^2 S_{XX}(f) |\hat{h}(f)|^2 df.
\]

Since \(S_{XX}\) is symmetric so is \(α^2 S_{XX}(\cdot)\), and we can conclude that 
\[S_{YY}(f) = α^2 S_{XX}(f), \quad f ∈ ℝ.\]

Problem 2  
**The Operational PSD of a Sum of Independent SPs**

Feeding \(X + Y\) to a stable filter of impulse response \(h\) produces the signal \(X ∗ h + Y ∗ h\). And, since \(X\) and \(Y\) are independent and centered, so are \(X ∗ h\) and \(Y ∗ h\). Consequently,

\[
\text{Power in } X ∗ h + Y ∗ h = \lim_{T \to ∞} \frac{1}{2T} E \left[ \int_{-T}^{T} (X(t) ∗ h(t) + (Y ∗ h)(t))^2 dt \right] \\
= \lim_{T \to ∞} \frac{1}{2T} \left( E \left[ \int_{-T}^{T} (X(t) ∗ h(t))^2 dt \right] + E \left[ \int_{-T}^{T} ((Y ∗ h)(t))^2 dt \right] \right) \\
= \text{Power in } X ∗ h + \text{Power in } Y ∗ h \\
= \int_{−∞}^{∞} S_{XX}(f) |\hat{h}(f)|^2 df + \int_{−∞}^{∞} S_{YY}(f) |\hat{h}(f)|^2 df \\
= \int_{−∞}^{∞} (S_{XX}(f) + S_{YY}(f)) |\hat{h}(f)|^2 df.
\]
Since this holds for every stable filter \( h \), and since the symmetry of \( S_{XX} \) and \( S_{YY} \) implies the symmetry of \( S_{XX} + S_{YY} \), the operational PSD of \( X + Y \) is the sum of the operational PSDs of \( X \) and \( Y \).

Problem 3  
**Operational PSD of a Deterministic SP**

The operational PSD of a deterministic energy-limited signal \( x \) is zero. This can be argued as follows. If \( x \in L_2 \) and \( h \in L_1 \), then \( x \ast h \) is energy limited, i.e.

\[
\int_{-\infty}^{\infty} (x \ast h)^2(t) \, dt < \infty
\]

(substitute \( p = 2 \) in (5.11)). Consequently

\[
\text{Power in } X \ast h = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (x \ast h)^2(t) \, dt
\]

\[
\leq \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{\infty} (x \ast h)^2(t) \, dt
\]

\[
= 0
\]

\[
= \int_{-\infty}^{\infty} 0 \left| \hat{h}(f) \right|^2 \, df, \quad h \in L_1.
\]

Since this holds for every \( h \in L_1 \), and since the all-zero function is symmetric, we conclude that the operational PSD of \( x \) is the all-zero function (of frequency).

Problem 4  
**Operational PSD of a Deterministic SP**

For any stable \( h \),

\[
(Y \ast h)(t) = \int_{-\infty}^{\infty} Y(t - \tau) \, h(\tau) \, d\tau
\]

\[
= \int_{-\infty}^{\infty} X \left( \frac{t - \tau}{a} \right) \, h(\tau) \, d\tau
\]

\[
= \int_{-\infty}^{\infty} X \left( \frac{t - \tau}{a} \right) \, h(\tau) \, d\tau
\]

\[
= a \int_{-\infty}^{\infty} X \left( \frac{t - \sigma}{a} \right) \, h(a\sigma) \, d\sigma
\]

\[
= a \int_{-\infty}^{\infty} X \left( \frac{t - \sigma}{a} \right) \, h'(\sigma) \, d\sigma
\]

\[
= a \cdot \left( X \ast h' \right) \left( \frac{t}{a} \right),
\]

where

\[
h': \sigma \mapsto h(a\sigma)
\]

and is thus of FT

\[
\hat{h}'(f) = \frac{1}{a} \hat{h} \left( \frac{f}{a} \right), \quad f \in \mathbb{R}.
\]

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Thus

\[
\text{Power of } Y \star h = \lim_{T \to \infty} \frac{1}{2T} \mathbb{E} \left[ \int_{-T}^{T} a^2 (X \star h')^2 \left( \frac{t}{a} \right) \, dt \right] = \lim_{T \to \infty} \frac{1}{2T/a} \mathbb{E} \left[ \int_{-T/a}^{T/a} a^2 (X \star h')^2 (\tau) \, d\tau \right] = \lim_{T \to \infty} \frac{1}{2T/a} \mathbb{E} \left[ \int_{-T/a}^{T/a} a^2 (X \star h')^2 (\tau) \, d\tau \right] = a^2 \text{Power of } X \star h' = a^2 \int_{-\infty}^{\infty} S_{XX}(f) |\hat{h}'(f)|^2 \, df = \int_{-\infty}^{\infty} S_{XX}(f) \left| \hat{h} \left( \frac{f}{a} \right) \right|^2 \, df = \int_{-\infty}^{\infty} a S_{XX}(a\tilde{f}) |\hat{h} (\tilde{f})|^2 \, d\tilde{f}.
\]

Since this holds for every \( h \in \mathcal{L}_1 \), and since the mapping \( \tilde{f} \mapsto a S_{XX}(a\tilde{f}) \) is symmetric, this mapping must be the operational PSD of \((Y(t))\).

**Problem 5**

**The Operational PSD of PAM**

(i) A sample function of \((X_1(t), \ t \in \mathbb{R})\) might look as follows:

(ii) The symbols \((X_\ell)\) are uncorrelated and of unit variance, i.e.,

\[ K_{XX}(m) = I\{m = 0\}, \quad m \in \mathbb{Z}. \]

Therefore, by (15.21) the operational PSD of \((X_1(t), \ t \in \mathbb{R})\) is

\[ S_{X_2X_1}(f) = \frac{A^2}{T_s} |\hat{g}(f)|^2 = A^2 T_s |\text{sinc}(T_sf)|^2, \quad f \in \mathbb{R}. \]

(iii) A sample function of \(X_2(\cdot)\) is for instance
The autocovariance function $K_{XX}$ of $(X_t)$ is the same as above but the baud period is double. Therefore the operational PSD of $(X_2(t), t \in \mathbb{R})$ is

$$S_{X_2X_2}(f) = \frac{A^2}{2T_s} |\hat{g}(f)|^2 = \frac{A^2T_s}{2} |\text{sinc}(T_s f)|^2, \quad f \in \mathbb{R}.$$  

(iv) The operational PSD of $(X_2(t))$ is half that of $(X_1(t))$.

Problem 6

**The Operational PSD and Block Codes**

Here we consider a $(1, 2)$ binary-to-reals block encoder, so $N = 2$. By direct computation we obtain

$$E[X_\ell X_{\ell'}] = \begin{cases} 1 & \ell = \ell', \\ -1 & \ell \neq \ell', \quad \ell, \ell' \in \{1, 2\}. \end{cases}$$

Using this and (14.37) we obtain that the power in bi-infinite block mode is

$$P = \frac{1}{NT_s} E \left[ \int_{-\infty}^{\infty} \left( A \sum_{\ell=1}^{N} X_\ell g(t - \ell T_s) \right)^2 dt \right]$$

$$= \frac{A^2}{2T_s} \int_{-\infty}^{\infty} E \left[ \left( A \sum_{\ell=1}^{2} X_\ell g(t - \ell T_s) \right)^2 \right] dt$$

$$= \frac{A^2}{2T_s} \int_{-\infty}^{\infty} \left[ X_1^2 g^2(t - T_s) + 2X_1X_2 g(t - T_s) g(t - 2T_s) + X_2^2 g^2(t - 2T_s) \right] dt$$

$$= \frac{A^2}{2T_s} \int_{-\infty}^{\infty} \left( g^2(t - T_s) - 2 g(t - T_s) g(t - 2T_s) + g^2(t - 2T_s) \right) dt$$

$$= \frac{A^2}{T_s} \left( \|g\|_2^2 - R_{gg}(T_s) \right)$$

$$= \frac{A^2}{T_s} \int_{-\infty}^{\infty} (|\hat{g}(f)|^2 - |\hat{g}(f)|^2 e^{i2\pi f T_s}) df$$

$$= \int_{-\infty}^{\infty} \frac{A^2}{T_s} |\hat{g}(f)|^2 \left( 1 - e^{i2\pi f T_s} \right) df.$$  

By (15.23), the operational PSD is

$$S_{XX}(f) = \frac{A^2}{NT_s} \sum_{\ell=1}^{N} \sum_{\ell'=1}^{N} E[X_\ell X_{\ell'}] e^{i2\pi f(\ell - \ell')T_s} |\hat{g}(f)|^2$$

$$= \frac{A^2}{2T_s} \left( 2 - e^{i2\pi f T_s} - e^{-i2\pi f T_s} \right) |\hat{g}(f)|^2$$

$$= \frac{A^2}{T_s} (1 - \cos(2\pi f T_s)) |\hat{g}(f)|^2, \quad f \in \mathbb{R}.$$  

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