Communication and Detection Theory

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Model Answers to Exercise 8 of April 17, 2018

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 *How General is QAM?*

Since a QAM signal is real, A must be real

$$\mathsf{A} \in \mathbb{R}.$$

Consequently, A can be written as $\mathsf{A} = |\mathsf{A}|\, e^{i\psi}$ for some ψ, which is either zero or π. As for W, our requirement is that

$$\mathsf{W} \geq 0$$

(although, since the $\operatorname{sinc}(\cdot)$ function is symmetric, this is not necessary). As to $\mathsf{T_s}$, our requirement is that

$$\mathsf{T_s} > 0$$

(although, technically speaking, this is not necessary: if $\mathsf{T_s}$ is negative, we can define $\tilde{\mathsf{T}}_\mathrm{s}$ to be $-\mathsf{T_s}$ and, by defining $\tilde{\ell} \triangleq n - \ell + 1$, write $\sum_\ell C_\ell \operatorname{sinc}\bigl(\mathsf{W}(t - \ell \mathsf{T_s})\bigr)$ as

$$\sum_{\tilde{\ell}=1}^{n} C_{n-\tilde{\ell}+1} \operatorname{sinc}\Bigl(\mathsf{W}\bigl(t - \tilde{\ell}\tilde{\mathsf{T}}_\mathrm{s} + (n+1)\tilde{\mathsf{T}}_\mathrm{s}\bigr)\Bigr),$$

which, can in turn be written as

$$\sum_{\tilde{\ell}=1}^{n} \tilde{C}_{\tilde{\ell}}\, g(t - \tilde{\ell}\tilde{\mathsf{T}}_\mathrm{s}),$$

with $\mathbf{g}\colon t \mapsto \operatorname{sinc}\bigl(\mathsf{W}(t + n\tilde{\mathsf{T}}_\mathrm{s})\bigr)$ and with $\tilde{C}_{\tilde{\ell}} \triangleq C_{n-\tilde{\ell}+1}$ for $\ell \in \{1, \ldots, n\}$).
For the signal to be a passband signal we require

$$f_\mathrm{c} > \frac{\mathsf{W}}{2}.$$

Finally,

$$\phi \in \mathbb{R}$$

can be arbitrary.
If A, f_c, ϕ, W, and $\mathsf{T_s}$ meet these condition, then the signal can be written in the QAM form

$$2\operatorname{Re}\Biggl(|\mathsf{A}| \sum_{\ell=1}^{n} \tilde{C}_\ell\, g(t - \ell \mathsf{T_s})\, e^{i2\pi f_\mathrm{c} t}\Biggr),$$

where

$$\tilde{C}_\ell = \frac{1}{2}\, e^{\mathrm{i}(\phi+\psi)},$$

$$\mathbf{g}\colon t \mapsto \operatorname{sinc}(Wt),$$

and ψ is 0 or π depending on whether A is positive or negative.

Problem 2 — *Transmission Rate, Encoder Rate, and Bandwidth*

(i) If the time shifts of ϕ by integer multiples of $\mathsf{T_s}$ are orthonormal, then the bandwidth of ϕ must be at least $1/(2\mathsf{T_s})$. Consequently, since the bandwidth W of the QAM signal is typically twice the bandwidth of the pulse shape (see for example Theorem 15.4.1)

$$W \geq \frac{1}{\mathsf{T_s}}, \tag{1}$$

where $\mathsf{T_s}$ denotes the baud period.

If a QAM encoder $\varphi\colon \{0,1\}^k \to \mathbb{C}^n$ is of constellation $\mathcal{C}$, then all the complex n-tuples it produces are in $\mathcal{C}^n$, and hence it cannot produce more than $\#\mathcal{C}^n$ different sequences. And since, by definition, every encoder is one-to-one, it cannot map two different data k-tuples to the same complex n-tuple. Since there are 2^k binary k-tuples, this is only possible if

$$2^k \leq \#\mathcal{C}^n.$$

Consequently, the rate of the encoder can be bounded by

$$\begin{aligned}\frac{k}{n} &= \frac{1}{n}\log_2 2^k \\ &\leq \frac{1}{n}\log_2\bigl(\#\mathcal{C}^n\bigr) \\ &= \log_2 \#\mathcal{C} \quad \left[\frac{\text{bits}}{\text{complex symbol}}\right].\end{aligned} \tag{2}$$

The bit rate $\mathsf{R_b}$ can be thus bounded by

$$\begin{aligned}\mathsf{R_b} &= \frac{k}{n}\left[\frac{\text{bits}}{\text{complex symbol}}\right]\frac{1}{\mathsf{T_s}}\left[\frac{\text{complex symbols}}{\text{sec}}\right] \\ &\leq \frac{k}{n} W \\ &\leq W \log_2 \#\mathcal{C}\left[\frac{\text{bits}}{\text{sec}}\right].\end{aligned}$$

We conclude that the constellation size $\#\mathcal{C}$ must satisfy

$$\#\mathcal{C} \geq \left\lceil 2^{\frac{\mathsf{R_b}}{W}} \right\rceil. \tag{3}$$

(ii) Since we are asked to use pulse shapes of excess-bandwidth at least 15%

$$W \geq \frac{1.15}{\mathsf{T_s}}.$$

Proceeding as in the first part we conclude that now

$$\#\mathcal{C} \geq \left\lceil 2^{\frac{1.15\mathsf{R_b}}{W}} \right\rceil.$$

Problem 3 *Synthesis of 16-QAM*

Since

$$X_\nu(t) = 2\mathsf{A}\,\mathrm{Re}\biggl(\sum_{\ell=1}^{n} C_\ell^{(\nu)}\, g(t-\ell\mathsf{T}_\mathrm{s})\, e^{\mathrm{i}2\pi f_\mathrm{c} t}\biggr), \quad \nu = 1,2,$$

and since α is real

$$\begin{aligned}
X(t) &= \alpha X_1(t) + X_2(t) \\
&= \alpha 2\mathsf{A}\,\mathrm{Re}\biggl(\sum_{\ell=1}^{n} C_\ell^{(1)}\, g(t-\ell\mathsf{T}_\mathrm{s})\, e^{\mathrm{i}2\pi f_\mathrm{c} t}\biggr) + 2\mathsf{A}\,\mathrm{Re}\biggl(\sum_{\ell=1}^{n} C_\ell^{(2)}\, g(t-\ell\mathsf{T}_\mathrm{s})\, e^{\mathrm{i}2\pi f_\mathrm{c} t}\biggr) \\
&= 2\mathsf{A}\,\mathrm{Re}\biggl(\sum_{\ell=1}^{n} \bigl(\alpha C_\ell^{(1)} + C_\ell^{(2)}\bigr)\, g(t-\ell\mathsf{T}_\mathrm{s})\, e^{\mathrm{i}2\pi f_\mathrm{c} t}\biggr), \quad t \in \mathbb{R},
\end{aligned}$$

which has the form of a QAM signal with the complex symbols

$$C_\ell = \alpha C_\ell^{(1)} + C_\ell^{(2)}, \quad \ell \in \{1, \dots, n\}.$$

For the choice $\alpha = 2$ or $\alpha = 1/2$ one easily checks that the square 16-QAM constellation of Figure 0.1 is achieved. Note that each value of $C_\ell^{(2)}$ is "transformed" to four possible places by the addition of $\alpha C_\ell^{(1)}$.

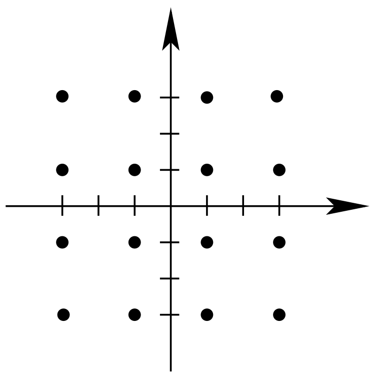

Figure 0.1: 16-QAM constellation.

Problem 4 *Phase Imprecision*

Let $\tilde{x}_\mathrm{BB}(\cdot)$ denote the output of the passband-to-baseband converter employing the imprecise local oscillator phase. Then by (7.35a) (with the wrong phase)

$$\begin{aligned}
\tilde{\mathbf{x}}_\mathrm{BB} &= \bigl(\tau \mapsto x_\mathrm{PB}(\tau)\, e^{-\mathrm{i}(2\pi f_\mathrm{c}\tau - \Delta\phi)}\bigr) \star \mathrm{LPF}_{\mathsf{W}_\mathrm{c}} \\
&= e^{\mathrm{i}\Delta\phi}\bigl(\tau \mapsto x_\mathrm{PB}(\tau)\, e^{-\mathrm{i}2\pi f_\mathrm{c}\tau}\bigr) \star \mathrm{LPF}_{\mathsf{W}_\mathrm{c}} \\
&= e^{\mathrm{i}\Delta\phi}\mathbf{x}_\mathrm{BB}.
\end{aligned}$$

If $\mathbf{x}_\mathrm{PB}$ is the QAM signal (16.6), then $\mathbf{x}_\mathrm{BB}$ is as in (16.5a) and

$$\tilde{x}_\mathrm{BB}(t) = \sum \tilde{C}_\ell\, g(t-\ell\mathsf{T}_\mathrm{s})$$

where the symbol $\tilde{C}_\ell$ is the rotation of C_ℓ by $\Delta\phi$, i.e.

$$\tilde{C}_\ell = C_\ell\, e^{\mathrm{i}\Delta\phi}.$$

Problem 5 — *The Distribution of* $\mathrm{Re}(Z)$ *and* $|Z|$

(i) Let the real random variables X and Y be the real and imaginary parts of Z. Since Z is uniformly distributed on the unit disc, the joint density function $f_{X,Y}(\cdot,\cdot)$ of (X,Y) is

$$f_{X,Y}(x,y) = \frac{1}{\pi}\,\mathrm{I}\{x^2+y^2 \le 1\}, \quad x,y \in \mathbb{R}.$$

Integrating over y we obtain the density of X:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y \\ &= \begin{cases} \frac{1}{\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\mathrm{d}y & \text{if } |x|\le 1, \\ 0 & \text{if } |x|>1 \end{cases} \\ &= \frac{2}{\pi}\sqrt{1-x^2}\,\mathrm{I}\{|x|\le 1\}, \quad x\in\mathbb{R}. \end{aligned}$$

(ii) Let R and Θ be the magnitude and argument of Z. Then, by Lemma 17.3.5, their joint density $f_{R,\Theta}(\cdot,\cdot)$ is

$$\begin{aligned} f_{R,\Theta}(r,\theta) &= r\,f_Z\bigl(r\,e^{\mathrm{i}\theta}\bigr) \\ &= \frac{r}{\pi}\,\mathrm{I}\{0\le r\le 1, -\pi\le\theta<\pi\}. \end{aligned}$$

By integrating over θ we obtain that the density of R $(=|Z|)$ is

$$\begin{aligned} f_R(r) &= \int_{-\pi}^{\pi}\frac{r}{\pi}\,\mathrm{I}\{0\le r\le 1\}\,\mathrm{d}\theta \\ &= 2r\,\mathrm{I}\{0\le r\le 1\}, \quad r\in\mathbb{R}. \end{aligned}$$

Problem 6 — *Product of Proper CRVs*

Let Z and W be independent and proper CRVs. Then the expectation of their product must be zero because

$$\begin{aligned} \mathsf{E}[ZW] &= \mathsf{E}[Z]\,\mathsf{E}[W] \\ &= 0, \end{aligned}$$

where the first equality follows because Z and W are independent, and where the second equality follows because Z and W are proper so $\mathsf{E}[Z]=\mathsf{E}[W]=0$. Similarly,

$$\begin{aligned} \mathsf{E}\bigl[(ZW)^2\bigr] &= \mathsf{E}\bigl[Z^2W^2\bigr] \\ &= \mathsf{E}\bigl[Z^2\bigr]\,\mathsf{E}\bigl[W^2\bigr] \\ &= 0, \end{aligned}$$

where the second equality follows because the independence of Z and W implies the independence of Z^2 and W^2, and where the third equality follows because Z and W are proper so

$$\mathsf{E}\bigl[Z^2\bigr] = \mathsf{E}\bigl[W^2\bigr] = 0.$$

Finally, ZW is of finite variance because

$$\begin{aligned} \mathsf{E}\left[|ZW|^2\right] &= \mathsf{E}\left[|Z|^2|W|^2\right] \\ &= \mathsf{E}\left[|Z|^2\right] \mathsf{E}\left[|W|^2\right] \\ &< \infty, \end{aligned}$$

where the second equality follows because the independence of Z and W implies the independence of $|Z|^2$ and $|W|^2$, and the inequality follows because Z and W are proper.
The assumption that W and Z are independent is essential. For example, if Z is proper then so is Z^* (Exercise 17.6) and yet their product ZZ^* is not proper (unless Z is zero) because its mean is not zero:

$$\begin{aligned} \mathsf{E}[ZZ^*] &= \mathsf{E}\left[|Z|^2\right] \\ &\neq 0. \end{aligned}$$

Problem 7 — *Reversing the Direction of Time*

Since (Z_ν) is of finite variance, so is (Y_ν). And since the mean of (Z_ν) does not depend on ν, nor does that of (Y_ν). We now compute $\mathsf{Cov}[Y_{\nu+\eta}, Y_\nu]$ and show that it does not depend on ν:

$$\begin{aligned} \mathsf{Cov}[Y_{\nu+\eta}, Y_\nu] &= \mathsf{Cov}[Z_{-\nu-\eta}, Z_{-\nu}] \\ &= \mathsf{Cov}\left[Z_{-\nu+(-\eta)}, Z_{-\nu}\right] \\ &= \mathsf{K}_{ZZ}(-\eta), \quad \nu, \eta \in \mathbb{Z}. \end{aligned}$$

We conclude that (Y_ν) is WSS and

$$\begin{aligned} \mathsf{K}_{YY}(\eta) &= \mathsf{K}_{ZZ}(-\eta) \\ &= \mathsf{K}^*_{ZZ}(\eta), \quad \eta \in \mathbb{Z}. \end{aligned}$$

Problem 8 — $\pi/4$-*QPSK*

Let $\mathsf{K}_{CC}(\cdot)$ be the autocovariance function of (C_ℓ). Since (C_ℓ) is WSS and α is of unit magnitude, $(\tilde{C}_\ell)$ is also WSS and its autocovariance function $\mathsf{K}_{\tilde{C}\tilde{C}}(\cdot)$ is

$$\begin{aligned} \mathsf{K}_{\tilde{C}\tilde{C}}(m) &= \mathsf{E}\left[\tilde{C}^*_\ell C_{\ell+m}\right] \\ &= \mathsf{E}\left[e^{-\mathrm{i}\ell\frac{4}{\pi}} C^*_\ell e^{\mathrm{i}(\ell+m)\frac{\pi}{4}} C_{\ell+m}\right] \\ &= e^{\mathrm{i}m\frac{\pi}{4}} \mathsf{E}[C^*_\ell C_{\ell+m}] \\ &= e^{\mathrm{i}m\frac{\pi}{4}} \mathsf{K}_{CC}(m), \quad m \in \mathbb{Z}, \end{aligned}$$

(Exercise 17.22). By substituting this autocovariance in (18.50) we obtain the power in $\frac{\pi}{4}$-QPSK is

$$\frac{2\mathsf{A}^2}{\mathsf{T}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{CC}(m)\, e^{\mathrm{i}m\frac{\pi}{4}}\, \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T}_\mathrm{s}).$$

and substituting it in (18.50) yields the operational PSD

$$\frac{\mathsf{A}^2}{\mathsf{T}_\mathrm{s}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{CC}(m)\, e^{\mathrm{i}m\frac{\pi}{4}} e^{\mathrm{i}2\pi(|f|-f_\mathrm{c})m\mathsf{T}_\mathrm{s}} \left|\hat{g}\big(|f| - f_\mathrm{c}\big)\right|^2.$$

To compute the power P_{I} in

$$t \mapsto 2\mathsf{A} \sum_{\ell=-\infty}^{\infty} \mathrm{Re}(C_\ell)\, g(t - \ell \mathsf{T}_{\mathrm{s}}) \cos(2\pi f_{\mathrm{c}} t),$$

we view this signal as a QAM signal corresponding to the symbols $C'_\ell = \mathrm{Re}(C_\ell) + \mathrm{i}0$ to obtain from Theorem 18.3.1 that

$$\mathsf{P}_{\mathrm{I}} = \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{C'C'}(m)\, \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T}_{\mathrm{s}}),$$

where $\mathsf{K}_{C'C'}$ is the autocovariance function of $\bigl(C'_\ell\bigr)$, i.e., the autocovariance function of the real part of $\bigl(C_\ell\bigr)$.
Similarly, to compute the power P_{Q} in

$$t \mapsto -2\mathsf{A} \sum_{\ell=-\infty}^{\infty} \mathrm{Im}(C_\ell)\, g(t - \ell \mathsf{T}_{\mathrm{s}}) \sin(2\pi f_{\mathrm{c}} t),$$

we view this signal as a QAM signal corresponding to the symbols $C''_\ell = 0 + \mathrm{i}\,\mathrm{Im}(C_\ell)$ to obtain from Theorem 18.3.1 that

$$\mathsf{P}_{\mathrm{Q}} = \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{C''C''}(m)\, \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T}_{\mathrm{s}}),$$

where $\mathsf{K}_{C''C''}$ is the autocovariance function of $\bigl(C''_\ell\bigr)$. Since the autocovariance function of $\bigl(C''_\ell\bigr)$ is equal to the autocovariance of the imaginary part of $\bigl(C_\ell,\ \ell \in \mathbb{Z}\bigr)$; and since the autocovariance functions of the real part and the imaginary part of $\bigl(C_\ell,\ \ell \in \mathbb{Z}\bigr)$ add up to the real part of the autocovariance of $\bigl(C_\ell,\ \ell \in \mathbb{Z}\bigr)$ (Exercise 17.17), it follows that

$$\begin{aligned}
\mathsf{P}_{\mathrm{I}} + \mathsf{P}_{\mathrm{Q}} &= \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \sum_{m=-\infty}^{\infty} \mathrm{Re}\bigl(\mathsf{K}_{CC}(m)\bigr) \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T}_{\mathrm{s}}) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \mathsf{K}_{CC}(0)\, \|\mathbf{g}\|_2^2 + \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \sum_{m=1}^{\infty} \Bigl(\mathrm{Re}\bigl(\mathsf{K}_{CC}(m)\bigr) \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T}_{\mathrm{s}}) \\
&\qquad\qquad + \mathrm{Re}\bigl(\mathsf{K}_{CC}(-m)\bigr) \mathsf{R}^*_{\mathbf{gg}}(-m\mathsf{T}_{\mathrm{s}}) \Bigr) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \mathsf{K}_{CC}(0)\, \|\mathbf{g}\|_2^2 + \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \sum_{m=1}^{\infty} \Bigl(\mathrm{Re}\bigl(\mathsf{K}_{CC}(m)\bigr) + \mathrm{Re}\bigl(\mathsf{K}_{CC}(-m)\bigr) \Bigr)\, \mathsf{R}_{\mathbf{gg}}(m\mathsf{T}_{\mathrm{s}}) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \mathsf{K}_{CC}(0)\, \|\mathbf{g}\|_2^2 + \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \sum_{m=1}^{\infty} \Bigl(\mathsf{K}_{CC}(m) + \mathsf{K}_{CC}(-m) \Bigr)\, \mathsf{R}_{\mathbf{gg}}(m\mathsf{T}_{\mathrm{s}}) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \mathsf{K}_{CC}(0)\, \|\mathbf{g}\|_2^2 + \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \sum_{m=1}^{\infty} \Bigl(\mathsf{K}_{CC}(m)\, \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T}_{\mathrm{s}}) + \mathsf{K}_{CC}(-m)\, \mathsf{R}^*_{\mathbf{gg}}(-m\mathsf{T}_{\mathrm{s}}) \Bigr) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T}_{\mathrm{s}}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{CC}(m)\, \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T}_{\mathrm{s}}),
\end{aligned}$$

where the second line follows because $\mathsf{K}_{CC}(0)$ is the variance of C_ℓ and is thus real; the third line because $\mathbf{g}$ is real, so its self-similarity is real and symmetric; the fourth line because the autocovariance function of a WSS complex SP is conjugate symmetric by (17.60); the fifth line again because the self-similarity of $\mathbf{g}$ is real and symmetric; and the final line by trivial algebra.

The intuitive explanation for this result is that, ignoring some mathematical technicalities, the two signals are "essentially" orthogonal and add up to the QAM signal. Since they are "essentially" orthogonal, the power in their sum should be the sum of their powers.
One would not expect a similar result for the operational PSD because the latter is related to the power in the result of filtering the QAM signal, and filtering is equivalent to replacing the pulse shape with a new pulse shape, which is in general not real. This destroys the orthogonality.
Moreover, filtering a QAM signal cannot be viewed as separately filtering each of the quadrature signals: there are also cross terms.

Problem 10 — *Sums of Independent Gaussians*

Recall that if Z is the sum of two independent random variables X_1 and X_2 with densities $f_{X_1}(\cdot)$ and $f_{X_2}(\cdot)$, then the density of Z is

$$f_Z(z) = (f_{X_1} \star f_{X_2})(z).$$

Compute now the density of Z when $X_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(0, \sigma_2^2)$:

$$\begin{aligned}
f_Z(z) &= (f_{X_1} \star f_{X_2})(z) \\
&= \int_{-\infty}^{\infty} f_{X_1}(\xi) f_{X_2}(z-\xi)\,\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{\xi^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(z-\xi)^2}{2\sigma_2^2}}\,\mathrm{d}\xi \\
&= \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left(-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\xi^2 - \frac{2z\sigma_1^2}{\sigma_1^2+\sigma_2^2}\xi + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}z^2\right)\right)\mathrm{d}\xi \\
&= \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left(-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\left(\xi - \frac{z\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 + \frac{z^2\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2}\right)\right)\mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}} \\
&\qquad \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}} \exp\left(-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\xi - \frac{z\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2\right)\mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}},
\end{aligned}$$

where in the last step we used the fact that the Gaussian density integrates to one.
Therefore, $Z \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.

Problem 11 — *Computing Probabilities*

The solution hinges on (19.8), i.e., that for $\sigma \neq 0$

$$\Bigl(X \sim \mathcal{N}(\mu, \sigma^2)\Bigr) \implies \left(\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)\right).$$

(i) By simple transformation

$$\begin{aligned}
\Pr[X \leq 2] &= 1 - \Pr[X > 2] \\
&= 1 - \mathcal{Q}\left(\frac{2-1}{\sqrt{3}}\right) \\
&= 1 - \mathcal{Q}\left(\frac{1}{\sqrt{3}}\right).
\end{aligned}$$

(ii) Since X and Y are independent, so are $2X$ and $3Y$. And since scaling Gaussians produces Gaussians, $2X \sim \mathcal{N}(2, 12)$ and $3Y \sim \mathcal{N}(-6, 36)$. (Note that if X is of variance σ^2 then aX is of variance $a^2\sigma^2$.) Since the sum of independent Gaussians is Gaussian, $2X+3Y \sim \mathcal{N}(-4, 48)$ and

$$\begin{aligned}\Pr\bigl[2X + 3Y > -2\bigr] &= \mathcal{Q}\Bigl(\frac{-2-(-4)}{\sqrt{48}}\Bigr)\\ &= \mathcal{Q}\Bigl(\frac{1}{2\sqrt{3}}\Bigr).\end{aligned}$$

Problem 12 — *Bounds on the $\mathcal{Q}$-Function*

a) Using a change of variable $\eta = z - \xi$, we can express $\mathcal{Q}(\xi)$ as

$$\begin{aligned}\mathcal{Q}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\xi}^{\infty} e^{-z^2/2}\,\mathrm{d}z\\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(\eta+\xi)^2/2}\,\mathrm{d}\eta\\ &= \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2} \int_{0}^{\infty} e^{-\eta^2/2-\xi\eta}\,\mathrm{d}\eta. \qquad (4)\end{aligned}$$

b) Since $e^{-x} \leq 1$ for $x \geq 0$, we can substitute $x = \eta^2/2$ to establish that $e^{-\eta^2/2} \leq 1$. Next we show that $e^{-x} \geq 1 - x$ for all $x \geq 0$. Note that the slope of $1-x$ is constant and is equal to -1. The slope of e^{-x} is $-e^{-x}$ which is greater than -1 for $x \geq 0$. Since e^{-x} and $1-x$ are both zero when $x = 0$, it follows that $1 - x$ always decreases faster than e^{-x} and thus $1 - x \leq e^{-x}$ for all $x \geq 0$. We can substitute $x = \eta^2/2$ to establish that $1 - \eta^2/2 \leq e^{-\eta^2/2}$. In summary, we have established that

$$1 - \eta^2/2 \leq e^{-\eta^2/2} \leq 1 \qquad \eta \in \mathbb{R}. \qquad (5)$$

c) We use the right inequality in (5) to upper bound $e^{-\eta^2/2}$ in (4):

$$\begin{aligned}\mathcal{Q}(\xi) &= \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2} \int_{0}^{\infty} e^{-\eta^2/2} e^{-\xi\eta}\,\mathrm{d}\eta\\ &\leq \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2} \int_{0}^{\infty} 1 \cdot e^{-\xi\eta}\,\mathrm{d}\eta\\ &= \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2} \cdot \left.\frac{e^{-\xi\eta}}{-\xi}\right|_{\eta=0}^{\infty}\\ &= \frac{1}{\sqrt{2\pi}\xi}\, e^{-\xi^2/2}. \qquad (6)\end{aligned}$$

In the same fashion, we use the left inequality in (5) to lower bound $e^{-\eta^2/2}$ in (4) as follows

$$\begin{aligned}\mathcal{Q}(\xi) &= \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2} \int_{0}^{\infty} e^{-\eta^2/2} e^{-\xi\eta}\,\mathrm{d}\eta\\ &\geq \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2} \int_{0}^{\infty} (1 - \eta^2/2) \cdot e^{-\xi\eta}\,\mathrm{d}\eta\\ &= \frac{1}{\sqrt{2\pi}}\, e^{-\xi^2/2} \cdot \left(\frac{1}{\xi} - \frac{1}{\xi^3}\right).\end{aligned}$$

Combining the two bounds we get the desired result:

$$\left(1 - \frac{1}{\xi^2}\right) \frac{1}{\sqrt{2\pi}\xi}\, e^{-\xi^2/2} \leq \mathcal{Q}(\xi) \leq \frac{1}{\sqrt{2\pi}\xi}\, e^{-\xi^2/2}.$$