Communication and Detection Theory

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Model Answers to Exercise 8 of April 11, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 ***How General is QAM?***

Since a QAM signal is real, A must be real

$$\mathsf{A} \in \mathbb{R}.$$

Consequently, A can be written as $\mathsf{A} = |\mathsf{A}|\, e^{i\psi}$ for some ψ, which is either zero or π. As for W, our requirement is that

$$\mathsf{W} \geq 0$$

(although, since the $\operatorname{sinc}(\cdot)$ function is symmetric, this is not necessary). As to $\mathsf{T_s}$, our requirement is that

$$\mathsf{T_s} > 0$$

(although, technically speaking, this is not necessary: if $\mathsf{T_s}$ is negative, we can define $\tilde{\mathsf{T}}_\mathsf{s}$ to be $-\mathsf{T_s}$ and, by defining $\tilde{\ell} \triangleq n - \ell + 1$, write $\sum_\ell C_\ell \operatorname{sinc}\bigl(\mathsf{W}(t - \ell \mathsf{T_s})\bigr)$ as

$$\sum_{\tilde{\ell}=1}^{n} C_{n-\tilde{\ell}+1} \operatorname{sinc}\Bigl(\mathsf{W}\bigl(t - \tilde{\ell}\tilde{\mathsf{T}}_\mathsf{s} + (n+1)\tilde{\mathsf{T}}_\mathsf{s}\bigr)\Bigr),$$

which, can in turn be written as

$$\sum_{\tilde{\ell}=1}^{n} \tilde{C}_{\tilde{\ell}}\, g(t - \tilde{\ell}\tilde{\mathsf{T}}_\mathsf{s}),$$

with $\mathbf{g}\colon t \mapsto \operatorname{sinc}\bigl(\mathsf{W}(t + n\tilde{\mathsf{T}}_\mathsf{s})\bigr)$ and with $\tilde{C}_{\tilde{\ell}} \triangleq C_{n-\tilde{\ell}+1}$ for $\ell \in \{1, \ldots, n\}$).
For the signal to be a passband signal we require

$$f_\mathrm{c} > \frac{\mathsf{W}}{2}.$$

Finally,

$$\phi \in \mathbb{R}$$

can be arbitrary.
If A, f_c, ϕ, W, and $\mathsf{T_s}$ meet these condition, then the signal can be written in the QAM form

$$2\,\mathrm{Re}\Biggl(|\mathsf{A}| \sum_{\ell=1}^{n} \tilde{C}_\ell\, g(t - \ell \mathsf{T_s})\, e^{i2\pi f_\mathrm{c} t}\Biggr),$$

where

$$\tilde{C}_\ell = \frac{1}{2}\, e^{\mathrm{i}(\phi+\psi)},$$

$$\mathbf{g}\colon t \mapsto \operatorname{sinc}(Wt),$$

and ψ is 0 or π depending on whether A is positive or negative.

Problem 2 — *Transmission Rate, Encoder Rate, and Bandwidth*

(i) If the time shifts of $\boldsymbol{\phi}$ by integer multiples of T_s are orthonormal, then the bandwidth of $\boldsymbol{\phi}$ must be at least $1/(2\mathsf{T}_\mathrm{s})$. Consequently, since the bandwidth W of the QAM signal is typically twice the bandwidth of the pulse shape (see for example Theorem 15.4.1)

$$W \geq \frac{1}{\mathsf{T}_\mathrm{s}}, \tag{1}$$

where T_s denotes the baud period.

If a QAM encoder $\varphi\colon \{0,1\}^k \to \mathbb{C}^n$ is of constellation $\mathcal{C}$, then all the complex n-tuples it produces are in $\mathcal{C}^n$, and hence it cannot produce more than $\#\mathcal{C}^n$ different sequences. And since, by definition, every encoder is one-to-one, it cannot map two different data k-tuples to the same complex n-tuple. Since there are 2^k binary k-tuples, this is only possible if

$$2^k \leq \#\mathcal{C}^n.$$

Consequently, the rate of the encoder can be bounded by

$$\begin{aligned} \frac{k}{n} &= \frac{1}{n}\log_2 2^k \\ &\leq \frac{1}{n}\log_2\bigl(\#\mathcal{C}^n\bigr) \\ &= \log_2 \#\mathcal{C} \quad \left[\frac{\text{bits}}{\text{complex symbol}}\right]. \end{aligned} \tag{2}$$

The bit rate R_b can be thus bounded by

$$\begin{aligned} \mathsf{R}_\mathrm{b} &= \frac{k}{n}\left[\frac{\text{bits}}{\text{complex symbol}}\right] \frac{1}{\mathsf{T}_\mathrm{s}}\left[\frac{\text{complex symbols}}{\text{sec}}\right] \\ &\leq \frac{k}{n} W \\ &\leq W \log_2 \#\mathcal{C} \left[\frac{\text{bits}}{\text{sec}}\right]. \end{aligned}$$

We conclude that the constellation size $\#\mathcal{C}$ must satisfy

$$\#\mathcal{C} \geq \left\lceil 2^{\frac{\mathsf{R}_\mathrm{b}}{W}} \right\rceil. \tag{3}$$

(ii) Since we are asked to use pulse shapes of excess-bandwidth at least 15%

$$W \geq \frac{1.15}{\mathsf{T}_\mathrm{s}}.$$

Proceeding as in the first part we conclude that now

$$\#\mathcal{C} \geq \left\lceil 2^{\frac{1.15\mathsf{R}_\mathrm{b}}{W}} \right\rceil.$$

Problem 3 *Synthesis of 16-QAM*

Since

$$X_\nu(t) = 2\mathsf{A}\,\mathrm{Re}\biggl(\sum_{\ell=1}^{n} C_\ell^{(\nu)}\, g(t-\ell \mathsf{T}_\mathrm{s})\, e^{i2\pi f_\mathrm{c} t}\biggr), \quad \nu = 1,2,$$

and since α is real

$$\begin{aligned} X(t) &= \alpha X_1(t) + X_2(t) \\ &= \alpha 2\mathsf{A}\,\mathrm{Re}\biggl(\sum_{\ell=1}^{n} C_\ell^{(1)}\, g(t-\ell \mathsf{T}_\mathrm{s})\, e^{i2\pi f_\mathrm{c} t}\biggr) + 2\mathsf{A}\,\mathrm{Re}\biggl(\sum_{\ell=1}^{n} C_\ell^{(2)}\, g(t-\ell \mathsf{T}_\mathrm{s})\, e^{i2\pi f_\mathrm{c} t}\biggr) \\ &= 2\mathsf{A}\,\mathrm{Re}\biggl(\sum_{\ell=1}^{n} \bigl(\alpha C_\ell^{(1)} + C_\ell^{(2)}\bigr)\, g(t-\ell \mathsf{T}_\mathrm{s})\, e^{i2\pi f_\mathrm{c} t}\biggr), \quad t \in \mathbb{R}, \end{aligned}$$

which has the form of a QAM signal with the complex symbols

$$C_\ell = \alpha C_\ell^{(1)} + C_\ell^{(2)}, \quad \ell \in \{1,\ldots,n\}.$$

For the choice $\alpha = 2$ or $\alpha = 1/2$ one easily checks that the square 16-QAM constellation of Figure 0.1 is achieved. Note that each value of $C_\ell^{(2)}$ is "transformed" to four possible places by the addition of $\alpha C_\ell^{(1)}$.

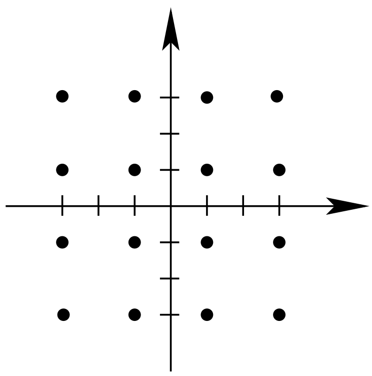

Figure 0.1: 16-QAM constellation.

Problem 4 *Phase Imprecision*

Let $\tilde{x}_\mathrm{BB}(\cdot)$ denote the output of the passband-to-baseband converter employing the imprecise local oscillator phase. Then by (7.35a) (with the wrong phase)

$$\begin{aligned} \tilde{\mathbf{x}}_\mathrm{BB} &= \bigl(\tau \mapsto x_\mathrm{PB}(\tau)\, e^{-i(2\pi f_\mathrm{c}\tau - \Delta\phi)}\bigr) \star \mathrm{LPF}_{W_\mathrm{c}} \\ &= e^{i\Delta\phi}\bigl(\tau \mapsto x_\mathrm{PB}(\tau)\, e^{-i2\pi f_\mathrm{c}\tau}\bigr) \star \mathrm{LPF}_{W_\mathrm{c}} \\ &= e^{i\Delta\phi}\mathbf{x}_\mathrm{BB}. \end{aligned}$$

If $\mathbf{x}_\mathrm{PB}$ is the QAM signal (16.6), then $\mathbf{x}_\mathrm{BB}$ is as in (16.5a) and

$$\tilde{x}_\mathrm{BB}(t) = \sum \tilde{C}_\ell\, g(t - \ell\mathsf{T}_\mathrm{s})$$

where the symbol $\tilde{C}_\ell$ is the rotation of C_ℓ by $\Delta\phi$, i.e.

$$\tilde{C}_\ell = C_\ell\, e^{i\Delta\phi}.$$

Problem 5 ***The Distribution of*** $\mathrm{Re}(Z)$ ***and*** $|Z|$

(i) Let the real random variables X and Y be the real and imaginary parts of Z. Since Z is uniformly distributed on the unit disc, the joint density function $f_{X,Y}(\cdot,\cdot)$ of (X,Y) is

$$f_{X,Y}(x,y) = \frac{1}{\pi}\,\mathrm{I}\{x^2+y^2 \le 1\}, \quad x,y \in \mathbb{R}.$$

Integrating over y we obtain the density of X:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}y \\ &= \begin{cases} \frac{1}{\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1 \end{cases} \\ &= \frac{2}{\pi}\sqrt{1-x^2}\,\mathrm{I}\{|x|\le 1\}, \quad x \in \mathbb{R}. \end{aligned}$$

(ii) Let R and Θ be the magnitude and argument of Z. Then, by Lemma 17.3.5, their joint density $f_{R,\Theta}(\cdot,\cdot)$ is

$$\begin{aligned} f_{R,\Theta}(r,\theta) &= r\,f_Z\big(r\,e^{\mathrm{i}\theta}\big) \\ &= \frac{r}{\pi}\,\mathrm{I}\{0 \le r \le 1, -\pi \le \theta < \pi\}. \end{aligned}$$

By integrating over θ we obtain that the density of R $(= |Z|)$ is

$$\begin{aligned} f_R(r) &= \int_{-\pi}^{\pi} \frac{r}{\pi}\,\mathrm{I}\{0 \le r \le 1\}\,\mathrm{d}\theta \\ &= 2r\,\mathrm{I}\{0 \le r \le 1\}, \quad r \in \mathbb{R}. \end{aligned}$$

Problem 6 ***Product of Proper CRVs***

Let Z and W be independent and proper CRVs. Then the expectation of their product must be zero because

$$\begin{aligned} \mathsf{E}[ZW] &= \mathsf{E}[Z]\,\mathsf{E}[W] \\ &= 0, \end{aligned}$$

where the first equality follows because Z and W are independent, and where the second equality follows because Z and W are proper so $\mathsf{E}[Z] = \mathsf{E}[W] = 0$. Similarly,

$$\begin{aligned} \mathsf{E}\big[(ZW)^2\big] &= \mathsf{E}\big[Z^2W^2\big] \\ &= \mathsf{E}\big[Z^2\big]\,\mathsf{E}\big[W^2\big] \\ &= 0, \end{aligned}$$

where the second equality follows because the independence of Z and W implies the independence of Z^2 and W^2, and where the third equality follows because Z and W are proper so

$$\mathsf{E}\big[Z^2\big] = \mathsf{E}\big[W^2\big] = 0.$$

Finally, ZW is of finite variance because

$$\begin{aligned}
\mathsf{E}\bigl[|ZW|^2\bigr] &= \mathsf{E}\bigl[|Z|^2|W|^2\bigr] \\
&= \mathsf{E}\bigl[|Z|^2\bigr]\,\mathsf{E}\bigl[|W|^2\bigr] \\
&< \infty,
\end{aligned}$$

where the second equality follows because the independence of Z and W implies the independence of $|Z|^2$ and $|W|^2$, and the inequality follows because Z and W are proper.
The assumption that W and Z are independent is essential. For example, if Z is proper then so is Z^* (Exercise 17.6) and yet their product ZZ^* is not proper (unless Z is zero) because its mean is not zero:

$$\begin{aligned}
\mathsf{E}[ZZ^*] &= \mathsf{E}\bigl[|Z|^2\bigr] \\
&\neq 0.
\end{aligned}$$

Problem 7 *Reversing the Direction of Time*

Since (Z_ν) is of finite variance, so is (Y_ν). And since the mean of (Z_ν) does not depend on ν, nor does that of (Y_ν). We now compute $\mathsf{Cov}[Y_{\nu+\eta}, Y_\nu]$ and show that it does not depend on ν:

$$\begin{aligned}
\mathsf{Cov}[Y_{\nu+\eta}, Y_\nu] &= \mathsf{Cov}[Z_{-\nu-\eta}, Z_{-\nu}] \\
&= \mathsf{Cov}\bigl[Z_{-\nu+(-\eta)}, Z_{-\nu}\bigr] \\
&= \mathsf{K}_{ZZ}(-\eta), \quad \nu, \eta \in \mathbb{Z}.
\end{aligned}$$

We conclude that (Y_ν) is WSS and

$$\begin{aligned}
\mathsf{K}_{YY}(\eta) &= \mathsf{K}_{ZZ}(-\eta) \\
&= \mathsf{K}^*_{ZZ}(\eta), \quad \eta \in \mathbb{Z}.
\end{aligned}$$