Model Answers to Exercise 9 of April 21, 2015
http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1

\(\pi/4\)-QPSK

Let \(K_{CC}(\cdot)\) be the autocovariance function of \((C_{\ell})\). Since \((C_{\ell})\) is WSS and \(\alpha\) is of unit magnitude, \((\tilde{C}_{\ell})\) is also WSS and its autocovariance function \(K_{\tilde{C}\tilde{C}}(\cdot)\) is

\[
K_{\tilde{C}\tilde{C}}(m) = E\left[\tilde{C}_{\ell}^* C_{\ell+m}\right] = E\left[e^{-i\frac{m}{4}\pi} C_{\ell}^* e^{i(\ell+m)\frac{\pi}{4}} C_{\ell+m}\right] = e^{im\frac{\pi}{4}} E[C_{\ell}^* C_{\ell+m}] = e^{im\frac{\pi}{4}} K_{CC}(m), \quad m \in \mathbb{Z},
\]

(Exercise 17.12). By substituting the autocovariance function in (18.33) we obtain that the power in \(\pi/4\)-QPSK is

\[
\frac{2\lambda^2}{T_s} \sum_{m=-\infty}^{\infty} K_{CC}(m) e^{im\frac{\pi}{4}} R_{gg}(mT_s).
\]

Substituting the autocovariance function in (18.50) yields the operational PSD

\[
\frac{\lambda^2}{T_s} \sum_{m=-\infty}^{\infty} K_{CC}(m) e^{im\frac{\pi}{4}} e^{i2\pi(|f| - f_c)mT_s} |\hat{g}(|f| - f_c)|^2.
\]

Problem 2

**Operational PSD of Differential PSK**

Since \((D_\ell)\) are IID random bits,

\[(4D_{3\ell} + 2D_{3\ell+1} + D_{3\ell+2}, \ \ell \in \mathbb{Z}) \sim \text{IID } U(\{0,1,\ldots,7\}).\]

We next show that

\[(C_{\ell}, \ \ell \in \mathbb{Z}) \sim \text{IID } U(\{1,e^{i2\pi\frac{1}{8}},e^{i2\pi\frac{2}{8}},\ldots,e^{i2\pi\frac{7}{8}}\}),\]

from which it follows (by substituting \(I\{m = 0\}\) for \(K_{CC}(m)\) in Theorem 18.4.3) that

\[
S_{XX}(f) = \frac{\lambda^2}{T_s} |\hat{g}(|f| - f_c)|^2, \quad f \in \mathbb{R}.
\]
To see that \((C_t)\) are indeed distributed as we claim, first note that, by construction, the data tuple \((D_{3t}, D_{3t+1}, D_{3t+2})\) is independent of the past symbols \(C_t, C_{t-1}, \ldots\) Consequently,

\[
\Pr[C_{t+1} = c_{t+1} \mid C_t = c_t, C_{t-1} = c_{t-1}, \ldots] = \Pr[C_{t+1} = c_{t+1} \mid C_t = c_t] = \frac{1}{8},
\]

where the last equality follows because the tuple \((D_{3t}, D_{3t+1}, D_{3t+2})\) produces \(C_{t+1}\) by rotating \(C_t\) at random (cf. Proposition 24.2.4). Thus, the conditional distribution of \(C_{t+1}\) given the infinite past \(C_t, C_{t-1}, \ldots\) is uniform, so \((C_t)\) is IID and uniform.

**Problem 3**

**Sums of Independent Gaussians**

Remember that the sum \(Z = X_1 + X_2\) of two independent random variables \(X_1, X_2\) with densities \(f_{X_1}(\cdot)\) and \(f_{X_2}(\cdot)\), respectively, has a density given by

\[
f_Z(z) = (f_{X_1} \ast f_{X_2})(z).
\]

Compute now the density of \(Z\) when \(X_1 \sim \mathcal{N}(0, \sigma_1^2)\) and \(X_2 \sim \mathcal{N}(0, \sigma_2^2)\):

\[
f_Z(z) = (f_{X_1} \ast f_{X_2})(z)
= \int_{-\infty}^{\infty} f_{X_1}(\xi) f_{X_2}(z - \xi) \, d\xi
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{\xi^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(z-\xi)^2}{2\sigma_2^2}} \, d\xi
= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2} \exp\left( -\frac{\sigma_1^2 + \sigma_2^2}{2\sigma_1^2\sigma_2^2} \left( \xi^2 - \frac{2z\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \xi + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right) \right) \, d\xi
= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2} \exp\left( -\frac{\sigma_1^2 + \sigma_2^2}{2\sigma_1^2\sigma_2^2} \left( z^2 - \frac{2z\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \xi + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right) \right) \, d\xi
= \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp\left( -\frac{\sigma_1^2 + \sigma_2^2}{2\sigma_1^2\sigma_2^2} \left( \xi^2 - \frac{2z\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \xi + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right) \right) \, d\xi
= \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} \exp\left( -\frac{\sigma_1^2 + \sigma_2^2}{2\sigma_1^2\sigma_2^2} \left( \xi^2 - \frac{2z\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \xi + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right) \right),
\]

where in the last step we used the fact that the Gaussian density integrates to one. Therefore, \(Z \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)\).

**Problem 4**

**Computing Probabilities**

The solution hinges on (19.8), i.e., that for \(\sigma \neq 0\)

\[
\left( X \sim \mathcal{N}(\mu, \sigma^2) \right) \Rightarrow \left( \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1) \right).
\]

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(i) By simple transformation

\[
\Pr[X \leq 2] = 1 - \Pr[X > 2] = 1 - Q\left(\frac{2 - 1}{\sqrt{3}}\right) = 1 - Q\left(\frac{1}{\sqrt{3}}\right).
\]

(ii) Since \(X\) and \(Y\) are independent, so are \(2X\) and \(3Y\). And since scaling Gaussians produces Gaussians, \(2X \sim \mathcal{N}(2, 12)\) and \(3Y \sim \mathcal{N}(-6, 36)\). (Note that if \(X\) is of variance \(\sigma^2\) then \(aX\) is of variance \(a^2\sigma^2\).) Since the sum of independent Gaussians is Gaussian, \(2X + 3Y \sim \mathcal{N}(-4, 48)\) and

\[
\Pr[2X + 3Y > -2] = Q\left(\frac{-2 - (-4)}{\sqrt{48}}\right) = Q\left(\frac{1}{2\sqrt{3}}\right).
\]

Problem 5

**Bounds on the \(Q\)-function**

a) Using a change of variable \(\eta = z - \xi\), we can express \(Q(\xi)\) as

\[
Q(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-z^2/2} \, dz = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(\eta + \xi)^2/2} \, d\eta = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\xi^2/2} \int_{-\infty}^{\infty} e^{-\eta^2/2 - \xi\eta} \, d\eta.
\]

(1)

b) Since \(e^{-x} \leq 1\) for \(x \geq 0\), we can substitute \(x = \eta^2/2\) to establish that \(e^{-\eta^2/2} \leq 1\). Next we show that \(e^{-x} \geq 1 - x\) for all \(x \geq 0\). Note that the slope of \(1 - x\) is constant and is equal to \(-1\). The slope of \(e^{-x}\) is \(-e^{-x}\) which is greater than \(-1\) for \(x \geq 0\). Since \(e^{-x}\) and \(1 - x\) are both zero when \(x = 0\), it follows that \(1 - x\) always decreases faster than \(e^{-x}\) and thus \(1 - x \leq e^{-x}\) for all \(x \geq 0\). We can substitute \(x = \eta^2/2\) to establish that \(1 - \eta^2/2 \leq e^{-\eta^2/2}\). In summary, we have established that

\[
1 - \eta^2/2 \leq e^{-\eta^2/2} \leq 1 \quad \eta \in \mathbb{R}.
\]

(2)

c) We use the right inequality in (2) to upper bound \(e^{-\eta^2/2}\) in (1):

\[
Q(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \int_{0}^{\infty} e^{-\eta^2/2} e^{-\xi\eta} \, d\eta \leq \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \int_{0}^{\infty} e^{-\eta\xi} \, d\eta = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \left. e^{-\eta\xi} \right|_{\eta=0}^{\infty} = \frac{1}{\sqrt{2\pi} \xi} e^{-\xi^2/2}.
\]
In the same fashion, we use the left inequality in (2) to lower bound $e^{-\eta^2/2}$ in (1) as follows

$$Q(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \int_0^\infty e^{-\eta^2/2} e^{-\xi\eta} \, d\eta \geq \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \int_0^\infty (1 - \eta^2/2) \cdot e^{-\xi\eta} \, d\eta = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \cdot \left( \frac{1}{\xi} - \frac{1}{\xi^3} \right).$$

Combining the two bounds we get the desired result:

$$\left(1 - \frac{1}{\xi^2}\right) \frac{1}{\sqrt{2\pi\xi}} e^{-\xi^2/2} \leq Q(\xi) \leq \frac{1}{\sqrt{2\pi\xi}} e^{-\xi^2/2}.$$