Communication and Detection Theory

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Model Answers to Exercise 9 of April 25, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — $\pi/4$-***QPSK***

Let $\mathsf{K}_{CC}(\cdot)$ be the autocovariance function of (C_ℓ). Since (C_ℓ) is WSS and α is of unit magnitude, $(\tilde{C}_\ell)$ is also WSS and its autocovariance function $\mathsf{K}_{\tilde{C}\tilde{C}}(\cdot)$ is

$$\begin{aligned}
\mathsf{K}_{\tilde{C}\tilde{C}}(m) &= \mathsf{E}\left[\tilde{C}_\ell^* C_{\ell+m}\right] \\
&= \mathsf{E}\left[e^{-\mathrm{i}\ell\frac{4}{\pi}} C_\ell^* e^{\mathrm{i}(\ell+m)\frac{\pi}{4}} C_{\ell+m}\right] \\
&= e^{\mathrm{i}m\frac{\pi}{4}} \mathsf{E}[C_\ell^* C_{\ell+m}] \\
&= e^{\mathrm{i}m\frac{\pi}{4}} \mathsf{K}_{CC}(m), \quad m \in \mathbb{Z},
\end{aligned}$$

(Exercise 17.22). By substituting this autocovariance in (18.50) we obtain the power in $\frac{\pi}{4}$-QPSK is

$$\frac{2\mathsf{A}^2}{\mathsf{T}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{CC}(m)\, e^{\mathrm{i}m\frac{\pi}{4}}\, \mathsf{R}_{\mathbf{gg}}^*(m\mathsf{T}_\mathrm{s}).$$

and substituting it in (18.50) yields the operational PSD

$$\frac{\mathsf{A}^2}{\mathsf{T}_\mathrm{s}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{CC}(m)\, e^{\mathrm{i}m\frac{\pi}{4}} e^{\mathrm{i}2\pi(|f|-f_\mathrm{c})m\mathsf{T}_\mathrm{s}} \left|\hat{g}\big(|f|-f_\mathrm{c}\big)\right|^2.$$

Problem 2 — ***The Power in the In-Phase and Quadrature Components***

To compute the power P_I in

$$t \mapsto 2\mathsf{A} \sum_{\ell=-\infty}^{\infty} \mathrm{Re}(C_\ell)\, g(t-\ell\mathsf{T}_\mathrm{s}) \cos(2\pi f_\mathrm{c} t),$$

we view this signal as a QAM signal corresponding to the symbols $C'_\ell = \mathrm{Re}(C_\ell) + \mathrm{i}0$ to obtain from Theorem 18.3.1 that

$$\mathsf{P}_\mathrm{I} = \frac{2\mathsf{A}^2}{\mathsf{T}_\mathrm{s}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{C'C'}(m)\, \mathsf{R}_{\mathbf{gg}}^*(m\mathsf{T}_\mathrm{s}),$$

where $\mathsf{K}_{C'C'}$ is the autocovariance function of (C'_ℓ), i.e., the autocovariance function of the real part of (C_ℓ).

Similarly, to compute the power $\mathsf{P_Q}$ in

$$t \mapsto -2\mathsf{A} \sum_{\ell=-\infty}^{\infty} \mathrm{Im}(C_\ell)\, g(t-\ell\mathsf{T_s}) \sin(2\pi f_c t),$$

we view this signal as a QAM signal corresponding to the symbols $C''_\ell = 0 + \mathrm{i}\,\mathrm{Im}(C_\ell)$ to obtain from Theorem 18.3.1 that

$$\mathsf{P_Q} = \frac{2\mathsf{A}^2}{\mathsf{T_s}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{C''C''}(m)\, \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T_s}),$$

where $\mathsf{K}_{C''C''}$ is the autocovariance function of $\bigl(C''_\ell\bigr)$. Since the autocovariance function of $\bigl(C''_\ell\bigr)$ is equal to the autocovariance of the imaginary part of $\bigl(C_\ell,\ \ell \in \mathbb{Z}\bigr)$; and since the autocovariance functions of the real part and the imaginary part of $\bigl(C_\ell,\ \ell \in \mathbb{Z}\bigr)$ add up to the real part of the autocovariance of $\bigl(C_\ell,\ \ell \in \mathbb{Z}\bigr)$ (Exercise 17.17), it follows that

$$\begin{aligned}
\mathsf{P_I} + \mathsf{P_Q} &= \frac{2\mathsf{A}^2}{\mathsf{T_s}} \sum_{m=-\infty}^{\infty} \mathrm{Re}\bigl(\mathsf{K}_{CC}(m)\bigr) \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T_s}) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T_s}} \mathsf{K}_{CC}(0)\, \|\mathbf{g}\|_2^2 + \frac{2\mathsf{A}^2}{\mathsf{T_s}} \sum_{m=1}^{\infty} \Bigl(\mathrm{Re}\bigl(\mathsf{K}_{CC}(m)\bigr) \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T_s}) \\
&\qquad\qquad + \mathrm{Re}\bigl(\mathsf{K}_{CC}(-m)\bigr) \mathsf{R}^*_{\mathbf{gg}}(-m\mathsf{T_s}) \Bigr) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T_s}} \mathsf{K}_{CC}(0)\, \|\mathbf{g}\|_2^2 + \frac{2\mathsf{A}^2}{\mathsf{T_s}} \sum_{m=1}^{\infty} \Bigl(\mathrm{Re}\bigl(\mathsf{K}_{CC}(m)\bigr) + \mathrm{Re}\bigl(\mathsf{K}_{CC}(-m)\bigr) \Bigr)\, \mathsf{R}_{\mathbf{gg}}(m\mathsf{T_s}) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T_s}} \mathsf{K}_{CC}(0)\, \|\mathbf{g}\|_2^2 + \frac{2\mathsf{A}^2}{\mathsf{T_s}} \sum_{m=1}^{\infty} \Bigl(\mathsf{K}_{CC}(m) + \mathsf{K}_{CC}(-m) \Bigr)\, \mathsf{R}_{\mathbf{gg}}(m\mathsf{T_s}) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T_s}} \mathsf{K}_{CC}(0)\, \|\mathbf{g}\|_2^2 + \frac{2\mathsf{A}^2}{\mathsf{T_s}} \sum_{m=1}^{\infty} \Bigl(\mathsf{K}_{CC}(m)\, \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T_s}) + \mathsf{K}_{CC}(-m)\, \mathsf{R}^*_{\mathbf{gg}}(-m\mathsf{T_s}) \Bigr) \\
&= \frac{2\mathsf{A}^2}{\mathsf{T_s}} \sum_{m=-\infty}^{\infty} \mathsf{K}_{CC}(m)\, \mathsf{R}^*_{\mathbf{gg}}(m\mathsf{T_s}),
\end{aligned}$$

where the second line follows because $\mathsf{K}_{CC}(0)$ is the variance of C_ℓ and is thus real; the third line because $\mathbf{g}$ is real, so its self-similarity is real and symmetric; the fourth line because the autocovariance function of a WSS complex SP is conjugate symmetric by (17.60); the fifth line again because the self-similarity of $\mathbf{g}$ is real and symmetric; and the final line by trivial algebra.

The intuitive explanation for this result is that, ignoring some mathematical technicalities, the two signals are "essentially" orthogonal and add up to the QAM signal. Since they are "essentially" orthogonal, the power in their sum should be the sum of their powers.

One would not expect a similar result for the operational PSD because the latter is related to the power in the result of filtering the QAM signal, and filtering is equivalent to replacing the pulse shape with a new pulse shape, which is in general not real. This destroys the orthogonality.

Moreover, filtering a QAM signal cannot be viewed as separately filtering each of the quadrature signals: there are also cross terms.

Problem 3 — *Sums of Independent Gaussians*

Recall that if Z is the sum of two independent random variables X_1 and X_2 with densities $f_{X_1}(\cdot)$ and $f_{X_2}(\cdot)$, then the density of Z is

$$f_Z(z) = (f_{X_1} \star f_{X_2})(z).$$

Compute now the density of Z when $X_1 \sim \mathcal{N}(0, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(0, \sigma_2^2)$:

$$\begin{aligned}
f_Z(z) &= (f_{X_1} \star f_{X_2})(z) \\
&= \int_{-\infty}^{\infty} f_{X_1}(\xi) f_{X_2}(z-\xi)\,\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_1^2}}\, e^{-\frac{\xi^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}}\, e^{-\frac{(z-\xi)^2}{2\sigma_2^2}}\,\mathrm{d}\xi \\
&= \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left(-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\xi^2 - \frac{2z\sigma_1^2}{\sigma_1^2+\sigma_2^2}\xi + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}z^2\right)\right)\mathrm{d}\xi \\
&= \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left(-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\left(\xi - \frac{z\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 + \frac{z^2\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2}\right)\right)\mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\, e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}} \\
&\qquad \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}} \exp\left(-\frac{\sigma_1^2+\sigma_2^2}{2\sigma_1^2\sigma_2^2}\left(\xi - \frac{z\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2\right)\mathrm{d}\xi \\
&= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}\, e^{-\frac{z^2}{2(\sigma_1^2+\sigma_2^2)}},
\end{aligned}$$

where in the last step we used the fact that the Gaussian density integrates to one. Therefore, $Z \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.

Problem 4 — *Computing Probabilities*

The solution hinges on (19.8), i.e., that for $\sigma \neq 0$

$$\left(X \sim \mathcal{N}(\mu, \sigma^2)\right) \implies \left(\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)\right).$$

(i) By simple transformation

$$\begin{aligned}
\Pr[X \leq 2] &= 1 - \Pr[X > 2] \\
&= 1 - \mathcal{Q}\left(\frac{2-1}{\sqrt{3}}\right) \\
&= 1 - \mathcal{Q}\left(\frac{1}{\sqrt{3}}\right).
\end{aligned}$$

(ii) Since X and Y are independent, so are $2X$ and $3Y$. And since scaling Gaussians produces Gaussians, $2X \sim \mathcal{N}(2, 12)$ and $3Y \sim \mathcal{N}(-6, 36)$. (Note that if X is of variance σ^2 then aX is of variance $a^2\sigma^2$.) Since the sum of independent Gaussians is Gaussian, $2X + 3Y \sim \mathcal{N}(-4, 48)$ and

$$\begin{aligned}
\Pr[2X + 3Y > -2] &= \mathcal{Q}\left(\frac{-2-(-4)}{\sqrt{48}}\right) \\
&= \mathcal{Q}\left(\frac{1}{2\sqrt{3}}\right).
\end{aligned}$$

Problem 5 — *Bounds on the Q-Function*

a) Using a change of variable $\eta = z - \xi$, we can express $\mathcal{Q}(\xi)$ as

$$\begin{aligned}\mathcal{Q}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\xi}^{\infty} e^{-z^2/2} \, \mathrm{d}z \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(\eta+\xi)^2/2} \, \mathrm{d}\eta \\ &= \frac{1}{\sqrt{2\pi}} \, e^{-\xi^2/2} \int_{0}^{\infty} e^{-\eta^2/2-\xi\eta} \, \mathrm{d}\eta. \end{aligned} \tag{1}$$

b) Since $e^{-x} \leq 1$ for $x \geq 0$, we can substitute $x = \eta^2/2$ to establish that $e^{-\eta^2/2} \leq 1$. Next we show that $e^{-x} \geq 1 - x$ for all $x \geq 0$. Note that the slope of $1 - x$ is constant and is equal to -1. The slope of e^{-x} is $-e^{-x}$ which is greater than -1 for $x \geq 0$. Since e^{-x} and $1 - x$ are both zero when $x = 0$, it follows that $1 - x$ always decreases faster than e^{-x} and thus $1 - x \leq e^{-x}$ for all $x \geq 0$. We can substitute $x = \eta^2/2$ to establish that $1 - \eta^2/2 \leq e^{-\eta^2/2}$. In summary, we have established that

$$1 - \eta^2/2 \leq e^{-\eta^2/2} \leq 1 \qquad \eta \in \mathbb{R}. \tag{2}$$

c) We use the right inequality in (2) to upper bound $e^{-\eta^2/2}$ in (1):

$$\begin{aligned}\mathcal{Q}(\xi) &= \frac{1}{\sqrt{2\pi}} \, e^{-\xi^2/2} \int_{0}^{\infty} e^{-\eta^2/2} e^{-\xi\eta} \, \mathrm{d}\eta \\ &\leq \frac{1}{\sqrt{2\pi}} \, e^{-\xi^2/2} \int_{0}^{\infty} 1 \cdot e^{-\xi\eta} \, \mathrm{d}\eta \\ &= \frac{1}{\sqrt{2\pi}} \, e^{-\xi^2/2} \cdot \left. \frac{e^{-\xi\eta}}{-\xi} \right|_{\eta=0}^{\infty} \\ &= \frac{1}{\sqrt{2\pi}\xi} \, e^{-\xi^2/2}. \end{aligned} \tag{3}$$

In the same fashion, we use the left inequality in (2) to lower bound $e^{-\eta^2/2}$ in (1) as follows

$$\begin{aligned}\mathcal{Q}(\xi) &= \frac{1}{\sqrt{2\pi}} \, e^{-\xi^2/2} \int_{0}^{\infty} e^{-\eta^2/2} e^{-\xi\eta} \, \mathrm{d}\eta \\ &\geq \frac{1}{\sqrt{2\pi}} \, e^{-\xi^2/2} \int_{0}^{\infty} (1 - \eta^2/2) \cdot e^{-\xi\eta} \, \mathrm{d}\eta \\ &= \frac{1}{\sqrt{2\pi}} \, e^{-\xi^2/2} \cdot \left(\frac{1}{\xi} - \frac{1}{\xi^3} \right). \end{aligned}$$

Combining the two bounds we get the desired result:

$$\left(1 - \frac{1}{\xi^2}\right) \frac{1}{\sqrt{2\pi}\xi} \, e^{-\xi^2/2} \leq \mathcal{Q}(\xi) \leq \frac{1}{\sqrt{2\pi}\xi} \, e^{-\xi^2/2}.$$