Problem 1 \quad \textbf{Hypothesis Testing}

(i) The two conditional probability densities can be computed using the rule relating the density of the transformed random variable to the density of the random variable (19.7):

\[ f_{Y|H=0}(y) = \frac{1}{2} e^{-|y-a|}, \quad y \in \mathbb{R}, \]
\[ f_{Y|H=1}(y) = \frac{1}{2} e^{-|y+a|}, \quad y \in \mathbb{R}. \]

The two densities are depicted in Figure 1.
The likelihood ratio is
\[
\text{LR}(y) = \frac{f_{Y|H=0}(y)}{f_{Y|H=1}(y)} = e^{y+a|-y-a|} = \begin{cases} 
  e^{-2a} & y \leq -a, \\
  e^{2y} & -a < y < a, \\
  e^{2a} & y \geq a.
\end{cases}
\]

Since the hypotheses are equally likely, we guess “\(H = 0\)” if \(\text{LR}(y) > 1\) and guess “\(H = 1\)” if \(\text{LR}(y) \leq 1\). Thus, an optimal decision rule is to guess “\(H = 0\)” if \(y > 0\), and to guess “\(H = 1\)” if \(y \leq 0\). This decision rule is depicted in Figure 1.

Let \(p^\ast\text{(error)}\) denote the probability of error of the optimal detector, then we have
\[
p^\ast\text{(error)} = p^\ast\text{(error}\mid H = 0) \Pr[H = 0] + p^\ast\text{(error}\mid H = 1) \Pr[H = 1]
\]
\[
= \frac{1}{4} \int_{-\infty}^{0} e^{-|y-a|} dy + \frac{1}{4} \int_{0}^{\infty} e^{-|y+a|} dy
\]
\[
= \frac{1}{4} e^{-(y-a)} \bigg|_{-\infty}^{0} - \frac{1}{4} e^{-(y+a)} \bigg|_{0}^{\infty}
\]
\[
= \frac{1}{2} e^{-a}.
\]

Applying the Bhattacharyya bound we get
\[
p^\ast\text{(error)} \leq \frac{1}{2} \int_{-\infty}^{\infty} \sqrt{\frac{1}{4} e^{-|y-a|} - |y+a|} dy
\]
\[
= \frac{1}{4} \left[ \left. \int_{-\infty}^{a} e^{y} dy + \int_{a}^{\infty} e^{-y} dy \right] + \left. \int_{-\infty}^{a} e^{-y} dy \right] + \left. \int_{a}^{\infty} e^{-y} dy \right]
\]
\[
= \frac{1}{4} (e^{-a} + 2ae^{-a} + e^{-a})
\]
\[
= \left( a + \frac{1}{2} \right) e^{-a}.
\]

As expected, for all values of \(a \geq 0\) the Bhattacharyya bound is indeed an upper bound on the exact probability of error computed in Part (iii).

**Problem 2**

**Binary Hypothesis Testing**

(i) Since \(f_{Y|H=1}(\cdot)\) integrates to one,
\[
1 = \int_{-\infty}^{\infty} f_{Y|H=1}(y) dy
\]
\[
= \int_{0}^{\infty} \beta e^{-\frac{y^2}{2}} dy
\]
\[
= \frac{1}{2} \int_{-\infty}^{\infty} \beta e^{-\frac{y^2}{2}} dy
\]
\[
= \frac{1}{2} \beta \sqrt{2\pi}
\]
\[
= \beta \sqrt{\frac{\pi}{2}}.
\]
Consequently, $\beta = \sqrt{\frac{2}{\pi}}$. 

(ii) An optimal rule is

\[
\text{guess } H = 0 \iff \left( \frac{\pi_0 f_{Y|H=0}(y)}{\pi_1 f_{Y|H=1}(y)} > 1 \right) 
\iff \left( \frac{e^{-y}}{e^{-y_2}} > \frac{\pi_1}{\pi_0} \sqrt{\frac{2}{\pi}} \right) 
\iff \left( \frac{y^2}{2} - y - \ln \left( \sqrt{\frac{2\pi_1}{\pi_0}} \right) > 0 \right) 
\iff \left( \frac{y^2}{2} - y - \gamma > 0 \right),
\]

where we have defined

\[\gamma = \ln \left( \sqrt{\frac{2\pi_1}{\pi_0}} \right).\]

The roots of the polynomial $\frac{y^2}{2} - y - \gamma$ are

\[y_{1,2} = 1 \pm \sqrt{1 + 2\gamma}.\]

Depending on the value of $\gamma$ (or the ratio $\pi_1/\pi_0$) we have to distinguish different cases:

- If $\gamma \geq 0$ (or $\pi_1/\pi_0 \geq \sqrt{\pi/2}$) the quadratic equation has exactly one positive real solution: $y_1 = 1 + \sqrt{1 + 2\gamma}$. Hence, we decide $H = 0$ only if $y > y_1$.
- If $-1/2 < \gamma < 0$ (or $\sqrt{\pi/2}e^{-1/2} < \pi_1/\pi_0 < \sqrt{\pi/2}$) the quadratic equation has two positive real solutions: $y_1 = 1 + \sqrt{1 + 2\gamma}$ and $y_2 = 1 - \sqrt{1 + 2\gamma}$. Hence, we decide $H = 0$ only if $0 < y < y_2$ or $y_1 < y$.
- If $\gamma = -1/2$ (or $\pi_1/\pi_0 = \sqrt{\pi/2}e^{-1/2}$) the quadratic equation has exactly one positive real solutions: $y_1 = 1$. Hence, we decide $H = 0$ only if $y \neq y_1$.
- Finally, if $\gamma < -1/2$ (or $\pi_1/\pi_0 < \sqrt{\pi/2}e^{-1/2}$) the quadratic solution does not have any real solution. Hence, we decide $H = 0$ for every $y$.

(iii) As above we have to distinguish the three different cases:

- $\gamma \geq 0$ (or $\pi_1/\pi_0 \geq \sqrt{\pi/2}$): The probability of error in this case is

\[
\text{Pr}(\text{error}|H = 0) = \int_{y_1}^{y_2} f_{Y|H=0}(y) \, dy = 1 - e^{-y_1},
\]

where $y_1 = 1 + \sqrt{1 + 2\gamma}$.
- $-1/2 < \gamma < 0$ (or $\sqrt{\pi/2}e^{-1/2} < \pi_1/\pi_0 < \sqrt{\pi/2}$): The probability of error in this case is

\[
\text{Pr}(\text{error}|H = 0) = \int_{y_1}^{y_2} f_{Y|H=0}(y) \, dy = e^{-y_2} - e^{-y_1},
\]

where $y_1$ as above and $y_2 = 1 - \sqrt{1 + 2\gamma}$.
- $\gamma \leq -1/2$ (or $\pi_1/\pi_0 \leq \sqrt{\pi/2}e^{-1/2}$): In this case we decide $H = 0$ except on a set of probability zero. Thus, the probability of error is

\[
\text{Pr}(\text{error}|H = 0) = 0.
\]
Problem 3  

**Hypothesis Testing with a Random Parameter**

(i) If we observe that $A = 2$, then the problem reduces to guessing $X$ based on $X + Z_2$, where $Z_2 \sim \mathcal{N}(0, 4\sigma^2)$. An optimal guessing rule is in this case to guess “$X = -1$” if $Y < 0$ and “$X = +1$” if $Y > 0$. If we observe that $A = 3$, then the problem reduces to guessing $X$ based on $X + Z_3$, where $Z_3 \sim \mathcal{N}(0, 9\sigma^2)$ with the same optimal rule. Thus, irrespective of the observed value of $A$, it is optimal to guess “$X = -1$” if $Y < 0$ and to guess “$X = +1$” if $Y > 0$.

(ii) The above rule remains optimal even if we do not observe $A$, because its implementation does not require knowledge of $A$. When $A$ is not observed we can thus do as well as if $A$ were observed.

Problem 4  

**Artifacts of Suboptimality**

(i) For Alice’s guessing rule

$$
\Pr(\text{error}) = \pi_0 \Pr[Y \leq 2 \mid H = 0] + \pi_1 \Pr[Y > 2 \mid H = 1]
= \frac{1}{2} \left(1 - \Phi\left(\frac{1}{\sigma}\right) + \Phi\left(\frac{3}{\sigma}\right)\right).
$$

(ii) As $\sigma \to \infty$ we get $\Pr(\text{error}) \to \frac{1}{2}$ and as $\sigma \to 0$ we get $\Pr(\text{error}) \to \frac{1}{2}$. But for $\sigma = 1$ we have $\Pr(\text{error}) < \frac{1}{2}$. Thus, $\Pr(\text{error})$ is not monotonically nondecreasing in $\sigma^2$.

(iii) The optimal guessing rule is to guess “$H = 0$” if $Y > 0$ and to guess “$H = 1$” if $Y < 1$. The associated minimal error probability is

$$
p^*(\text{error}) = \frac{1}{2} \Pr[Y \leq 0 \mid H = 0] + \frac{1}{2} \Pr[Y > 0 \mid H = 1]
= \Phi\left(\frac{1}{\sigma}\right).
$$

Since this is in general not equal to the probability of error of Alice’s rule, Alice’s rule does not minimize the probability of error.

(iv) For $\sigma = 0.1 \Rightarrow \Pr(\text{error}) \approx 0.5000$.

For $\sigma = 2 \Rightarrow \Pr(\text{error}) \approx 0.3791$.

Thus if $\sigma = 0.1$, then guessing $H$ based on $Y + Z$ where $Z \sim \mathcal{N}(0, 3.99), Z \perp Y$, will reduce the error probability by about 0.12.