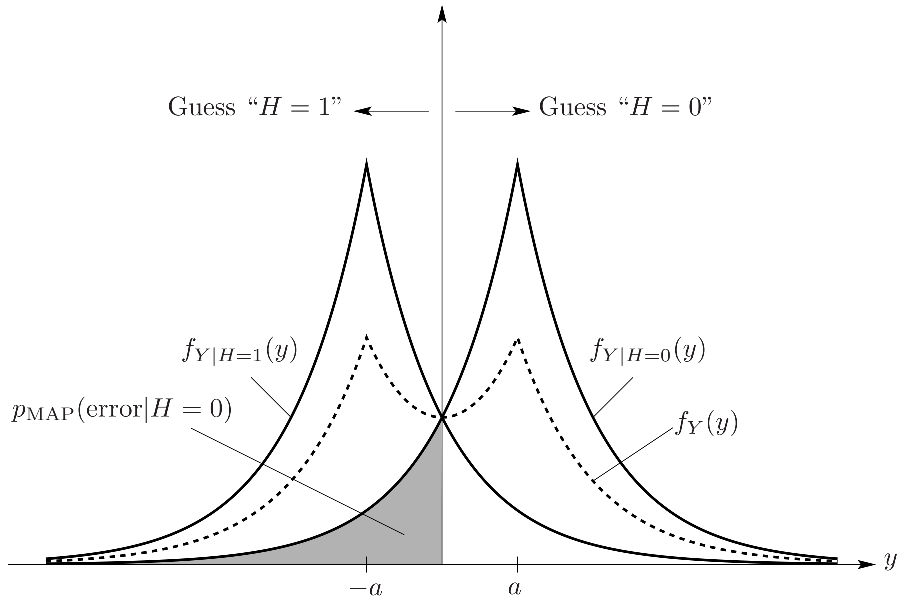

Figure 1: The conditional densities and the decision region.

Communication and Detection Theory

Signal and Information Processing Laboratory
Institut für Signal- und Informationsverarbeitung

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Problem 1 ***Hypothesis Testing***

(i) The two conditional probability densities can be computed using the rule relating the density of the transformed random variable to the density of the random variable (19.7):

$$f_{Y|H=0}(y) = \frac{1}{2}e^{-|y-a|}, \quad y \in \mathbb{R},$$
$$f_{Y|H=1}(y) = \frac{1}{2}e^{-|y+a|}, \quad y \in \mathbb{R}.$$

The two densities are depicted in Figure 1.

(ii) The likelihood ratio is

$$\begin{aligned}\text{LR}(y) &= \frac{f_{Y|H=0}(y)}{f_{Y|H=1}(y)}\\ &= e^{|y+a|-|y-a|}\\ &= \begin{cases} e^{-2a} & y \le -a,\\ e^{2y} & -a < y < a,\\ e^{2a} & y \ge a.\end{cases}\end{aligned}$$

Since the hypotheses are equally likely, we guess "$H = 0$" if $\text{LR}(y) > 1$ and guess "$H = 1$" if $\text{LR}(y) \le 1$. Thus, an optimal decision rule is to guess "$H = 0$" if $y > 0$, and to guess "$H = 1$" if $y \le 0$. This decision rule is depicted in Figure 1.

(iii) Let $p^*(\text{error})$ denote the optimal probability of error. Then,

$$\begin{aligned}p^*(\text{error}) &= p^*(\text{error}|H = 0)\Pr[H = 0] + p^*(\text{error}|H = 1)\Pr[H = 1]\\ &= \frac{1}{4}\int_{-\infty}^{0} e^{-|y-a|}\,\mathrm{d}y + \frac{1}{4}\int_{0}^{\infty} e^{-|y+a|}\,\mathrm{d}y\\ &= \frac{1}{4}e^{(y-a)}\Big|_{-\infty}^{0} - \frac{1}{4}e^{-(y+a)}\Big|_{0}^{\infty}\\ &= \frac{1}{2}e^{-a}.\end{aligned}$$

(iv) Applying the Bhattacharyya bound we get

$$\begin{aligned}p^*(\text{error}) &\le \frac{1}{2}\int_{-\infty}^{\infty}\sqrt{\frac{1}{4}e^{-|y-a|-|y+a|}}\,\mathrm{d}y\\ &= \frac{1}{4}\left(\int_{-\infty}^{-a} e^{y}\,\mathrm{d}y + \int_{-a}^{a} e^{-a}\,\mathrm{d}y + \int_{a}^{\infty} e^{-y}\,\mathrm{d}y\right)\\ &= \frac{1}{4}\left(e^{-a} + 2ae^{-a} + e^{-a}\right)\\ &= \left(\frac{a+1}{2}\right)e^{-a}.\end{aligned}$$

As expected, for all values of $a \ge 0$ the Bhattacharyya bound is indeed an upper bound on the exact probability error computed in Part (iii).

Problem 2 — *Binary Hypothesis Testing*

(i) Since $f_{Y|H=1}(\cdot)$ integrates to one,

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} f_{Y|H=1}(y)\,\mathrm{d}y\\ &= \int_{0}^{\infty}\beta e^{-\frac{y^2}{2}}\,\mathrm{d}y\\ &= \frac{1}{2}\int_{-\infty}^{\infty}\beta e^{-\frac{y^2}{2}}\,\mathrm{d}y\\ &= \frac{1}{2}\beta\sqrt{2\pi}\\ &= \beta\sqrt{\frac{\pi}{2}}.\end{aligned}$$

Consequently, $\beta = \sqrt{\frac{2}{\pi}}$.

(ii) An optimal rule is

$$\begin{aligned}
\text{guess "}H=0\text{"} &\iff \left(\frac{\pi_0 f_{Y|H=0}(y)}{\pi_1 f_{Y|H=1}(y)} > 1\right) \\
&\iff \left(\frac{e^{-y}}{e^{-\frac{y^2}{2}}} > \frac{\pi_1}{\pi_0}\sqrt{\frac{2}{\pi}}\right) \\
&\iff \left(\frac{y^2}{2} - y - \ln\left(\sqrt{\frac{2}{\pi}}\frac{\pi_1}{\pi_0}\right) > 0\right) \\
&\iff \left(\frac{y^2}{2} - y - \gamma > 0\right),
\end{aligned}$$

where we have defined

$$\gamma = \ln\left(\sqrt{\frac{2}{\pi}}\frac{\pi_1}{\pi_0}\right).$$

The roots of the polynomial $y^2/2 - y - \gamma$ are

$$y_{1,2} = 1 \pm \sqrt{1+2\gamma}.$$

Depending on the value of γ (or the ratio π_1/π_0) we have to distinguish different cases:

- If $\gamma > 0$ (or $\pi_1/\pi_0 > \sqrt{\pi/2}$) the quadratic equation has exactly one *positive* real solution: $y_1 = 1 + \sqrt{1+2\gamma}$. Hence, we decide $H = 0$ only if $y > y_1$.
- If $-1/2 < \gamma < 0$ (or $\sqrt{\pi/2}e^{-1/2} < \pi_1/\pi_0 < \sqrt{\pi/2}$) the quadratic equation has two *positive* real solutions: $y_1 = 1+\sqrt{1+2\gamma}$ and $y_2 = 1-\sqrt{1+2\gamma}$. Hence, we decide $H = 0$ only if $0 < y < y_2$ or $y_1 < y$.
- Finally, if $\gamma < -1/2$ (or $\pi_1/\pi_0 < \sqrt{\pi/2}e^{-1/2}$) the quadratic solution does not have any real solution. Hence, we decide $H = 0$ for every y.

(iii) As above we have to distinguish the three different cases:

- $\gamma > 0$ (or $\pi_1/\pi_0 > \sqrt{\pi/2}$): The probability of error in this case is

$$\Pr(\text{error}|H=0) = \int_0^{y_1} f_{Y|H=0}(y)\,\mathrm{d}y = 1 - e^{-y_1},$$

  where $y_1 = 1 + \sqrt{1+2\gamma}$.
- $-1/2 < \gamma < 0$ (or $\sqrt{\pi/2}e^{-1/2} < \pi_1/\pi_0 < \sqrt{\pi/2}$): The probability of error in this case is

$$\Pr(\text{error}|H=0) = \int_{y_1}^{y_2} f_{Y|H=0}(y)\,\mathrm{d}y = e^{-y_2} - e^{-y_1},$$

  where y_1 as above and $y_2 = 1 - \sqrt{1+2\gamma}$.
- $\gamma < -1/2$ (or $\pi_1/\pi_0 < \sqrt{\pi/2}e^{-1/2}$): In this case we always decide $H = 0$. Thus, the probability of error is

$$\Pr(\text{error}|H=0) = 0.$$

Problem 3 ***Hypothesis Testing with a Random Parameter***

(i) If we observe that $A = 2$, then the problem reduces to guessing X based on $X + Z_2$, where $Z_2 \sim \mathcal{N}(0, 4\sigma^2)$. An optimal guessing rule is in this case to guess "$X = -1$" if $Y < 0$ and "$X = +1$" if $Y > 0$. If we observe that $A = 3$, then the problem reduces to guessing X based on $X + Z_3$, where $Z_3 \sim \mathcal{N}(0, 9\sigma^2)$ with the same optimal rule. Thus, irrespective of the observed value of A, it is optimal to guess "$X = -1$" if $Y < 0$ and to guess "$X = +1$" if $Y > 0$.

(ii) The above rule remains optimal even if we do not observe A, because its implementation does not require knowledge of A. When A is not observed we can thus do as well as if A were observed.

Problem 4 ***Artifacts of Suboptimality***

(i) For Alice's guessing rule

$$\begin{aligned}\Pr(\text{error}) &= \pi_0 \Pr\bigl[Y \le 2 \,\big|\, H = 0\bigr] + \pi_1 \Pr\bigl[Y > 2 \,\big|\, H = 1\bigr] \\ &= \frac{1}{2}\left(1 - \mathcal{Q}\left(\frac{1}{\sigma}\right) + \mathcal{Q}\left(\frac{3}{\sigma}\right)\right).\end{aligned}$$

(ii) As $\sigma \to \infty$ we get $\Pr(\text{error}) \to \frac{1}{2}$ and as $\sigma \to 0$ we get $\Pr(\text{error}) \to \frac{1}{2}$. But for $\sigma = 1$ we have $\Pr(\text{error}) < \frac{1}{2}$. Thus, $\Pr(\text{error})$ is not monotonically nondecreasing in σ^2.

(iii) An optimal guessing rule is to guess "$H = 0$" if $Y > 0$ and to guess "$H = 1$" if $Y \le 0$. The associated minimal probability of decoding error is

$$\begin{aligned}p^*(\text{error}) &= \frac{1}{2}\Pr\bigl[Y \le 0 \,\big|\, H = 0\bigr] + \frac{1}{2}\Pr\bigl[Y > 0 \,\big|\, H = 1\bigr] \\ &= \mathcal{Q}\left(\frac{1}{\sigma}\right).\end{aligned}$$

Since this is in general not equal to the probability of error of Alice's rule, Alice's rule does not mimimize the probability of error.

(iv) For $\sigma = 0.1 \Rightarrow \Pr(\text{error}) \approx 0.5000$.
For $\sigma = 2 \Rightarrow \Pr(\text{error}) \approx 0.2583$.
Thus if $\sigma = 0.1$, then guessing H based on $Y + Z$ where $Z \sim \mathcal{N}(0, 1.9)\,, Z \perp\!\!\!\perp Y$, will reduce the error probability by almost 0.25.

