Communication and Detection Theory

Signal and Information Processing Laboratory
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Model Answers to Exercise 11 of May 9, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1 — *A Multi-Antenna Receiver*

(i) Let $\mathbf{Y} = (Y_1, Y_2)^{\mathsf{T}}$. Then

$$f_{\mathbf{Y}|H=0}(\mathbf{y}) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{(y_1 - \mathrm{A})^2}{2\sigma_1^2} - \frac{(y_2 - \mathrm{A})^2}{2\sigma_2^2}\right),$$

$$f_{\mathbf{Y}|H=1}(\mathbf{y}) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left(-\frac{(y_1 + \mathrm{A})^2}{2\sigma_1^2} - \frac{(y_2 + \mathrm{A})^2}{2\sigma_2^2}\right).$$

The likelihood ratio function is

$$\mathrm{LR}(\mathbf{y}) = \frac{f_{\mathbf{Y}|H=0}(\mathbf{y})}{f_{\mathbf{Y}|H=1}(\mathbf{y})} = \exp\left(\frac{4\mathrm{A}y_1}{2\sigma_1^2} + \frac{4\mathrm{A}y_2}{2\sigma_2^2}\right),$$

and the log likelihood-ratio function is

$$\mathrm{LLR}(\mathbf{y}) = \frac{2\mathrm{A}y_1}{\sigma_1^2} + \frac{2\mathrm{A}y_2}{\sigma_2^2}.$$

Since the prior is uniform, an optimal guessing rule is to guess "$H = 0$" if $\mathrm{LLR}(\mathbf{y})$ is nonnegative and to guess "$H = 1$" if it is negative. Using the above expression for the log likelihood-ratio function and recalling the assumption that A is nonnegative leads to the rule

$$\frac{y_1}{\sigma_1^2} + \frac{y_2}{\sigma_2^2} \ \underset{\text{"}H=1\text{"} \leftarrow <}{\overset{\text{"}H=0\text{"} \leftarrow \geq}{}} \ 0. \tag{1}$$

(ii) From (1), an optimal decision rule (when $\sigma_1 = 2\sigma_2$) is

$$\frac{y_1}{4} + y_2 \ \underset{\text{"}H=1\text{"} \leftarrow <}{\overset{\text{"}H=0\text{"} \leftarrow \geq}{}} \ 0,$$

which is illustrated in Figure 1.

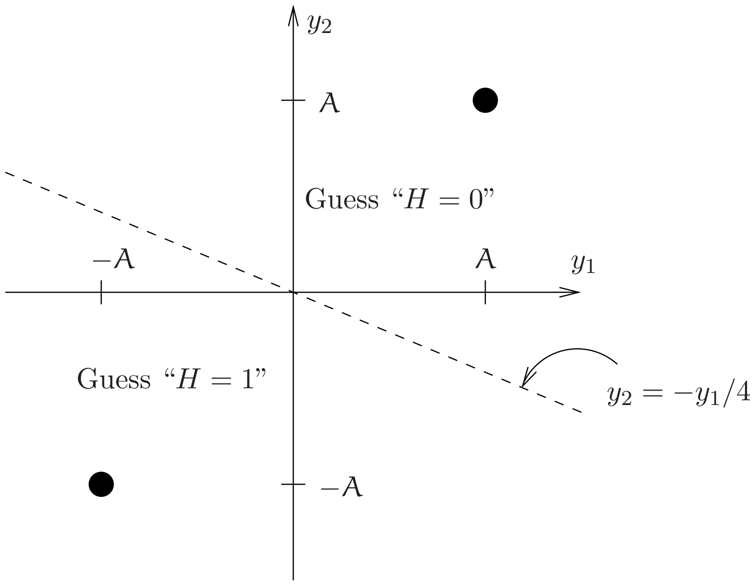

Figure 1: Decision regions.

(iii) From (1), a one dimensional sufficient statistic is

$$T\colon (y_1, y_2) \mapsto \frac{y_1}{\sigma_1^2} + \frac{y_2}{\sigma_2^2}.$$

(iv) Under each hypothesis, $T(Y_1, Y_2)$ is the sum of two independent Gaussian random variables and thus Gaussian In particular,

$$T(Y_1, Y_2) \sim \begin{cases} \mathcal{N}\Big(\frac{A}{\sigma_1^2} + \frac{A}{\sigma_2^2}, \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big) & \text{under } H = 0 \\ \mathcal{N}\Big(-\frac{A}{\sigma_1^2} - \frac{A}{\sigma_2^2}, \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\Big) & \text{under } H = 1. \end{cases}$$

Hence,

$$\begin{aligned} \Pr(\text{error}) &= \frac{1}{2}\Pr(\text{error}\,|\,H=0) + \frac{1}{2}\Pr(\text{error}\,|\,H=1) \\ &= \frac{1}{2}\Pr[T < 0 | H = 0] + \frac{1}{2}\Pr[T \geq 0 | H = 1] \\ &= \frac{1}{2}\mathcal{Q}\left(\frac{A\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right) + \frac{1}{2}\mathcal{Q}\left(\frac{A\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right) \\ &= \mathcal{Q}\left(A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right). \end{aligned}$$

(v) Let $\tilde{T} = Y_1 + Y_2$. We have that

$$\tilde{T} \sim \begin{cases} \mathcal{N}\big(2A, \sigma_1^2 + \sigma_2^2\big) & \text{under } H = 0 \\ \mathcal{N}\big(-2A, \sigma_1^2 + \sigma_2^2\big) & \text{under } H = 1. \end{cases}$$

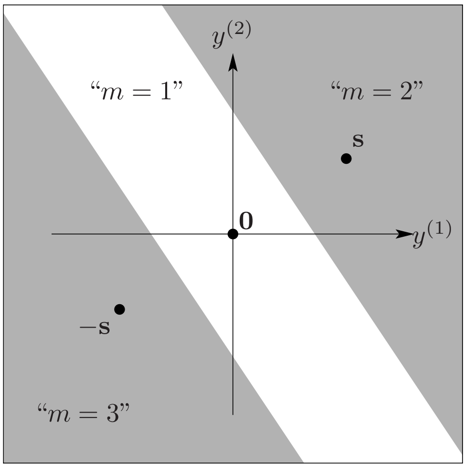

Figure 2: The decision regions of the MAP rule.

Hence,

$$\begin{aligned}\Pr(\text{error}) &= \frac{1}{2}\Pr\bigl[\tilde{T} < 0 \bigm| H = 0\bigr] + \frac{1}{2}\Pr\bigl[\tilde{T} \geq 0 \bigm| H = 1\bigr] \\ &= \frac{1}{2}\mathcal{Q}\left(\frac{2\mathrm{A}}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) + \frac{1}{2}\mathcal{Q}\left(\frac{2\mathrm{A}}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) \\ &= \mathcal{Q}\left(\frac{2\mathrm{A}}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right).\end{aligned}$$

This performance is indeed suboptimal. This can be seen by noting that $\mathcal{Q}(\cdot)$ is monotonic and by noting that

$$\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \geq \frac{4}{\sigma_1^2 + \sigma_2^2},$$

as can be verified by recalling that the harmonic mean cannot exceed the arithmetic mean:

$$\frac{1}{\frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)} \leq \frac{\alpha + \beta}{2}, \quad \alpha, \beta > 0.$$

Problem 2 — *Ternary Gaussian Detection*

Because the prior is uniform, it follows from Proposition 21.6.1 (ii) that the MAP rule for this problem is

$$\begin{aligned}&\text{guess “}M = 2\text{”if} \quad \langle \mathbf{Y}, \mathbf{s} \rangle_{\mathrm{E}} \geq \frac{\|\mathbf{s}\|^2}{2} \\ &\text{guess “}M = 3\text{”if} \quad \langle \mathbf{Y}, \mathbf{s} \rangle_{\mathrm{E}} \leq -\frac{\|\mathbf{s}\|^2}{2} \\ &\text{guess “}M = 1\text{”if} \quad |\langle \mathbf{Y}, \mathbf{s} \rangle_{\mathrm{E}}| < \frac{\|\mathbf{s}\|^2}{2}.\end{aligned}$$

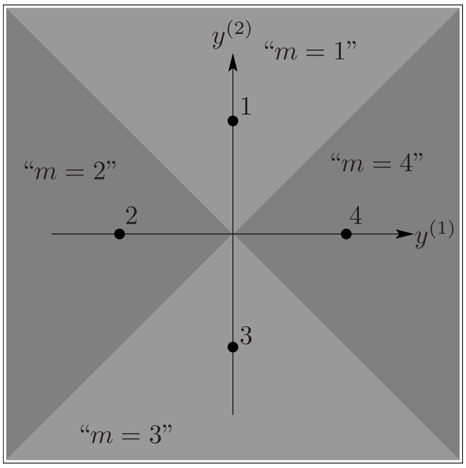

Figure 3: The constellation points and their corresponding decision regions.

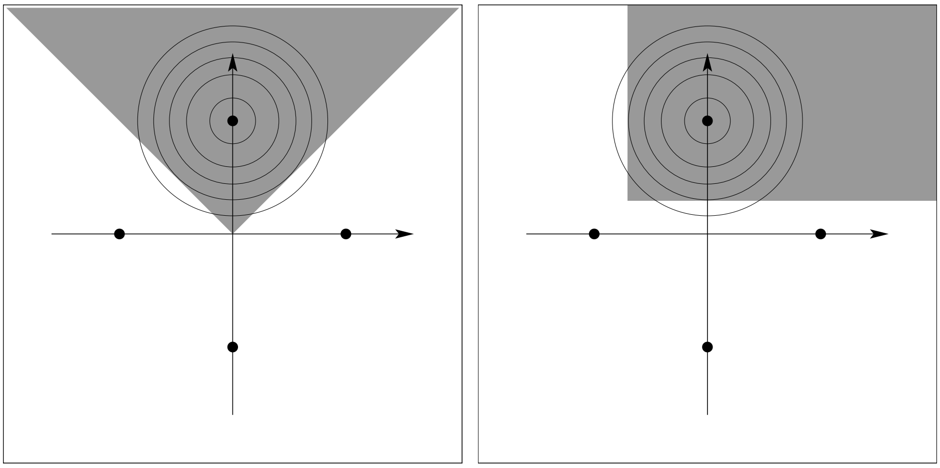

Figure 4: On the left the shaded area is the region where the MAP guesses "$m = 1$". On the right is the same region rotated by 45 degrees around the point $(0, A)$. Also shown are the contour lines of the received vector's density.

The conditional probabilities of error are now given by

$$p(\text{error}|M = 1) = 2\,\mathcal{Q}\Big(\frac{\|\mathbf{s}\|}{2\sigma}\Big),$$
$$p(\text{error}|M = 2) = \mathcal{Q}\Big(\frac{\|\mathbf{s}\|}{2\sigma}\Big),$$
$$p(\text{error}|M = 3) = \mathcal{Q}\Big(\frac{\|\mathbf{s}\|}{2\sigma}\Big).$$

Problem 3 ***4-PSK Detection***

(i) Because the prior is uniform, it follows from Proposition 21.6.1 (ii) that the MAP rule for this problem coincides with the "nearest-neighbor" rule, and the decision regions are thus as depicted in Figure 3.

(ii) By symmetry, $p(\text{error}|M = m)$ does not depend on m. We shall thus compute it for $M = 1$. We shall focus on $p(\text{correct}|M = 1)$, which is $1 - p(\text{error}|M = 1)$. This is the probability that a two-dimensional Gaussian centered around $(0, \mathsf{A})$ will lie in the area where the MAP guesses "$m = 1$". Since the noise distribution is radially symmetric, this probability is equal to the probability that this two-dimensional Gaussian centered around $(0, \mathsf{A})$ will lie in the result of rotating this region by 45 degrees around $(0, \mathsf{A})$.

This latter probability can be expressed as $\Pr[Z_1 \geq -\mathsf{A}/\sqrt{2},\ Z_2 \geq -\mathsf{A}/\sqrt{2}]$. Thus

$$p(\text{error}|M = m) = p(\text{error}|M = 1) = 1 - \left(1 - \mathcal{Q}\left(\frac{\mathsf{A}}{\sqrt{2}\sigma}\right)\right)^2.$$

(iii) A brute force application of Proposition 21.5.3 yields the bound

$$p(\text{error}|M = 1) \leq 2\,\mathcal{Q}\left(\frac{\mathsf{A}}{\sqrt{2}\sigma}\right) + \mathcal{Q}\left(\frac{\mathsf{A}}{\sigma}\right),$$

because (a_1, b_1) has two neighbors at distance $\sqrt{2}\mathsf{A}$ and one neighbor at distance $2\mathsf{A}$.

This bound can be improved by restricting the sum in (21.51) to $m' \in \{2, 4\}$ to yield

$$p(\text{error}|M = 1) \leq 2\,\mathcal{Q}\left(\frac{\mathsf{A}}{\sqrt{2}\sigma}\right).$$

In this improved bound the region whose probability is counted twice is the region where the decoder declares "$m = 3$".

Problem 4 — *A 7-ary QAM problem*

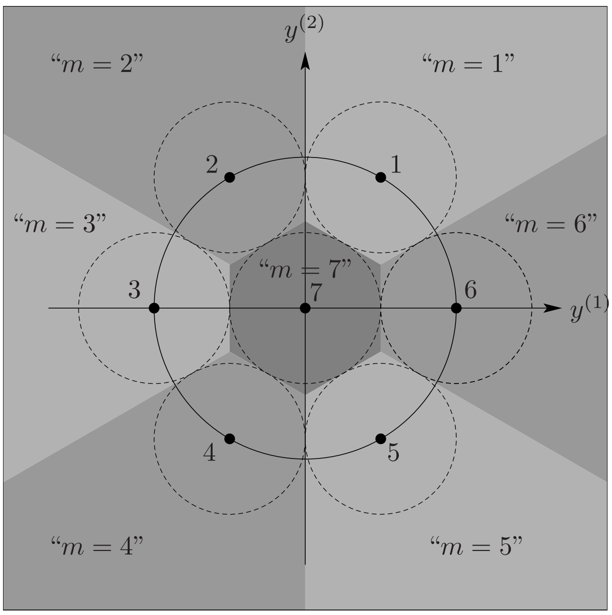

Figure 5: The decision regions for 7-ary QAM.

(i) The decision regions are depicted in Figure 5.

(ii) A disk of radius $\mathsf{A}/2$ centered around the constellation point (a_m, b_m) is fully contained in the region where the MAP guesses "$M = m$". Consequently, conditional on $M = m$, an error can occur only if the noise causes the received point $(Y^{(1)}, Y^{(2)})$ to lie outside this disk, i.e., only if the noise is of norm exceeding $\mathsf{A}/2$. The sum of the squares of the noise components has a mean-$2\sigma^2$ exponential distribution (Note 19.8.1). Consequently, conditional on $M = m$, the probability of error $p(\text{error}|M = m)$ is upper-bounded by the probability that a mean-$2\sigma^2$ exponential random variable exceeds $\mathsf{A}^2/4$. The density of a mean-$2\sigma^2$ exponential random variable is

$$\frac{1}{2\sigma^2}\, e^{-\frac{\xi}{2\sigma^2}}\, \mathrm{I}\{\xi > 0\}, \quad \xi \in \mathbb{R},$$

so

$$\begin{aligned} p(\text{error}|M = m) &\leq \int_{\mathsf{A}^2/4}^{\infty} \frac{1}{2\sigma^2}\, e^{-\frac{\xi}{2\sigma^2}}\, \mathrm{d}\xi \\ &= e^{-\frac{\mathsf{A}^2}{8\sigma^2}}. \end{aligned}$$

(iii) Proposition 21.5.3 would yield the bounds

$$\begin{aligned} p(\text{error}|M = m) &\leq 3\,\mathcal{Q}\Big(\frac{\mathsf{A}}{2\sigma}\Big) + 2\,\mathcal{Q}\Big(\frac{\sqrt{3}\mathsf{A}}{2\sigma}\Big) + \mathcal{Q}\Big(\frac{\mathsf{A}}{\sigma}\Big), \quad m = 1, 2, \ldots, 6, \\ p(\text{error}|M = 7) &\leq 6\,\mathcal{Q}\Big(\frac{\mathsf{A}}{2\sigma}\Big), \end{aligned}$$

because for $m \neq 0$ the point (a_m, b_m) has three neighbors at distance A; two neighbors at distance $\sqrt{3}\mathsf{A}$; and one neighbor at distance $2\mathsf{A}$, and because the point (a_7, b_7) at the origin has six neighbors all of which are at distance A.

For $m \neq 7$ we can indeed improve on the above bound by including fewer terms. This would would yield

$$p(\text{error}|M = m) \leq 3\,\mathcal{Q}\Big(\frac{\mathsf{A}}{2\sigma}\Big), \quad m = 1, 2, \ldots, 6.$$

For example, when $m = 1$, it suffices to consider only $m' = 2, 6, 7$ in (21.36).

(iv) When A/σ is very large, the bounds of Proposition 21.5.3 are better. This follows from (19.18) and (19.19). When A/σ is very small, e.g., zero, then the bounds of Proposition 21.5.3 are inferior: at $\mathsf{A}/\sigma = 0$ they yield bounds that are larger than one.