Model Answers to Exercise 11 of May 9, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1

(i) Let \( Y = (Y_1, Y_2)^T \). Then
\[
\begin{align*}
    f_{Y|H=0}(y) &= \frac{1}{2\pi\sigma_1\sigma_2} \exp \left( -\frac{(y_1 - A)^2}{2\sigma_1^2} - \frac{(y_2 - A)^2}{2\sigma_2^2} \right), \\
    f_{Y|H=1}(y) &= \frac{1}{2\pi\sigma_1\sigma_2} \exp \left( -\frac{(y_1 + A)^2}{2\sigma_1^2} - \frac{(y_2 + A)^2}{2\sigma_2^2} \right).
\end{align*}
\]

The likelihood ratio function is
\[
LR(y) = \frac{f_{Y|H=0}(y)}{f_{Y|H=1}(y)} = \exp \left( \frac{4Ay_1}{2\sigma_1^2} + \frac{4Ay_2}{2\sigma_2^2} \right),
\]
and the log likelihood-ratio function is
\[
\text{LLR}(y) = \frac{2Ay_1}{\sigma_1^2} + \frac{2Ay_2}{\sigma_2^2}.
\]

Since the prior is uniform, an optimal guessing rule is to guess “\( H = 0 \)” if \( \text{LLR}(y) \) is non-negative and to guess “\( H = 1 \)” if it is negative. Using the above expression for the log likelihood-ratio function and recalling the assumption that \( A \) is nonnegative leads to the rule
\[
\begin{align*}
    &\text{“} H = 0 \text{”} \quad \uparrow \\
    &\frac{y_1}{\sigma_1^2} + \frac{y_2}{\sigma_2^2} \quad \geq \quad 0, \quad \downarrow \\
    &\text{“} H = 1 \text{”}
\end{align*}
\]

(ii) From (1), an optimal decision rule (when \( \sigma_1 = 2\sigma_2 \)) is
\[
\begin{align*}
    &\text{“} H = 0 \text{”} \quad \uparrow \\
    &\frac{y_1}{4} + y_2 \quad \geq \quad 0, \quad \downarrow \\
    &\text{“} H = 1 \text{”}
\end{align*}
\]
which is illustrated in Figure 1.
(iii) From (1), a one dimensional sufficient statistic is

\[ T: (y_1, y_2) \mapsto \frac{y_1}{\sigma_1^2} + \frac{y_2}{\sigma_2^2}. \]

(iv) Under each hypothesis, \( T(Y_1, Y_2) \) is the sum of two independent Gaussian random variables and thus Gaussian. In particular,

\[
T(Y_1, Y_2) \sim \begin{cases} 
\mathcal{N}\left(\frac{A}{\sigma_1^2} + \frac{A}{\sigma_2^2}, \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) & \text{under } H = 0 \\
\mathcal{N}\left(-\frac{A}{\sigma_1^2} - \frac{A}{\sigma_2^2}, \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) & \text{under } H = 1.
\end{cases}
\]

Hence,

\[
\Pr(\text{error}) = \frac{1}{2} \Pr(\text{error} | H = 0) + \frac{1}{2} \Pr(\text{error} | H = 1) = \frac{1}{2} \Pr[T < 0 | H = 0] + \frac{1}{2} \Pr[T \geq 0 | H = 1] \\
= \frac{1}{2} Q\left(\frac{A}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right) + \frac{1}{2} Q\left(\frac{A}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right) \\
= Q\left(\frac{A}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}}\right).
\]

(v) Let \( \tilde{T} = Y_1 + Y_2 \). We have that

\[
\tilde{T} \sim \begin{cases} 
\mathcal{N}(2A, \sigma_1^2 + \sigma_2^2) & \text{under } H = 0 \\
\mathcal{N}(-2A, \sigma_1^2 + \sigma_2^2) & \text{under } H = 1.
\end{cases}
\]
Hence,

\[
\Pr(\text{error}) = \frac{1}{2} \Pr[\hat{T} < 0 \mid H = 0] + \frac{1}{2} \Pr[\hat{T} \geq 0 \mid H = 1]
\]

\[
= \frac{1}{2} Q \left( \frac{2A}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) + \frac{1}{2} Q \left( \frac{2A}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right)
\]

\[
= Q \left( \frac{2A}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right).
\]

This performance is indeed suboptimal. This can be seen by noting that \(Q(\cdot)\) is monotonic and by noting that

\[
\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \geq \frac{4}{\sigma_1^2 + \sigma_2^2},
\]

as can be verified by recalling that the harmonic mean cannot exceed the arithmetic mean:

\[
\frac{1}{\frac{1}{\alpha} + \frac{1}{\beta}} \leq \frac{\alpha + \beta}{2}, \quad \alpha, \beta > 0.
\]

**Problem 2**

*Ternary Gaussian Detection*

Because the prior is uniform, it follows from Proposition 21.6.1 (ii) that the MAP rule for this problem is

- guess “\(M = 2\)” if \(\langle Y, s \rangle_E \geq \frac{|s|^2}{2}\)
- guess “\(M = 3\)” if \(\langle Y, s \rangle_E \leq -\frac{|s|^2}{2}\)
- guess “\(M = 1\)” if \(|\langle Y, s \rangle_E| < \frac{|s|^2}{2}\).
Figure 3: The constellation points and their corresponding decision regions.

Figure 4: On the left the shaded area is the region where the MAP guesses “m = 1”. On the right is the same region rotated by 45 degrees around the point (0, A). Also shown are the contour lines of the received vector’s density.

The conditional probabilities of error are now given by

\[
p(\text{error}|M = 1) = 2 \Phi\left(\frac{|s|}{2\sigma}\right),
\]
\[
p(\text{error}|M = 2) = \Phi\left(\frac{|s|}{2\sigma}\right),
\]
\[
p(\text{error}|M = 3) = \Phi\left(\frac{|s|}{2\sigma}\right).
\]

**Problem 3**

(i) Because the prior is uniform, it follows from Proposition 21.6.1 (ii) that the MAP rule for this problem coincides with the “nearest-neighbor” rule, and the decision regions are thus as depicted in Figure 3.

(ii) By symmetry, \(p(\text{error}|M = m)\) does not depend on \(m\). We shall thus compute it for \(M = 1\). We shall focus on \(p(\text{correct}|M = 1)\), which is \(1 - p(\text{error}|M = 1)\). This is the probability
that a two-dimensional Gaussian centered around \((0, A)\) will lie in the area where the MAP guesses \(m = 1\). Since the noise distribution is radially symmetric, this probability is equal to the probability that this two-dimensional Gaussian centered around \((0, A)\) will lie in the result of rotating this region by 45 degrees around \((0, A)\).

This latter probability can be expressed as \(\Pr[Z_1 \geq -A/\sqrt{2}, Z_2 \geq -A/\sqrt{2}]\). Thus

\[
p(\text{error}|M = m) = p(\text{error}|M = 1) = 1 - \left(1 - Q\left(\frac{A}{\sqrt{2}\sigma}\right)\right)^2.
\]

(iii) A brute force application of Proposition 21.5.3 yields the bound

\[
p(\text{error}|M = 1) \leq 2 Q\left(\frac{A}{\sqrt{2}\sigma}\right) + Q\left(\frac{A}{\sigma}\right),
\]

because \((a_1, b_1)\) has two neighbors at distance \(\sqrt{2}A\) and one neighbor at distance \(2A\).

This bound can be improved by restricting the sum in (21.51) to \(m' \in \{2, 4\}\) to yield

\[
p(\text{error}|M = 1) \leq 2 Q\left(\frac{A}{\sqrt{2}\sigma}\right).
\]

In this improved bound the region whose probability is counted twice is the region where the decoder declares \(m = 3\).

**Problem 4**  

**A 7-ary QAM problem**

![Figure 5: The decision regions for 7-ary QAM.](image)

(i) The decision regions are depicted in Figure 5.
(ii) A disk of radius \( \frac{A}{2} \) centered around the constellation point \((a_m, b_m)\) is fully contained in the region where the MAP guesses “\( M = m \)”. Consequently, conditional on \( M = m \), an error can occur only if the noise causes the received point \((Y^{(1)}, Y^{(2)})\) to lie outside this disk, i.e., only if the noise is of norm exceeding \( \frac{A}{2} \). The sum of the squares of the noise components has a mean-2\(\sigma^2\) exponential distribution (Note 19.8.1). Consequently, conditional on \( M = m \), the probability of error \( p(\text{error}|M = m) \) is upper-bounded by the probability that a mean-2\(\sigma^2\) exponential random variable exceeds \( \frac{A^2}{4} \). The density of a mean-2\(\sigma^2\) exponential random variable is

\[
\frac{1}{2\sigma^2} e^{-\frac{\xi^2}{2\sigma^2}} 1\{\xi > 0\}, \quad \xi \in \mathbb{R},
\]

so

\[
p(\text{error}|M = m) \leq \int_{\frac{A^2}{4}}^{\infty} \frac{1}{2\sigma^2} e^{-\frac{\xi^2}{2\sigma^2}} d\xi = e^{-\frac{A^2}{8\sigma^2}}.
\]

(iii) Proposition 21.5.3 would yield the bounds

\[
p(\text{error}|M = m) \leq 3Q\left(\frac{A}{2\sigma}\right) + 2Q\left(\frac{\sqrt{3}A}{2\sigma}\right) + Q\left(\frac{A}{\sigma}\right), \quad m = 1, 2, \ldots, 6,
\]

\[
p(\text{error}|M = 7) \leq 6Q\left(\frac{A}{2\sigma}\right),
\]

because for \( m \neq 0 \) the point \((a_m, b_m)\) has three neighbors at distance \( A \); two neighbors at distance \( \sqrt{3}A \); and one neighbor at distance \( 2A \), and because the point \((a_7, b_7)\) at the origin has six neighbors all of which are at distance \( A \).

For \( m \neq 7 \) we can indeed improve on the above bound by including fewer terms. This would yield

\[
p(\text{error}|M = m) \leq 3Q\left(\frac{A}{2\sigma}\right), \quad m = 1, 2, \ldots, 6.
\]

For example, when \( m = 1 \), it suffices to consider only \( m' = 2, 6, 7 \) in (21.36).

(iv) When \( A/\sigma \) is very large, the bounds of Proposition 21.5.3 are better. This follows from (19.18) and (19.19). When \( A/\sigma \) is very small, e.g., zero, then the bounds of Proposition 21.5.3 are inferior: at \( A/\sigma = 0 \) they yield bounds that are larger than one.