Problem 1

(i) Let $Y = (Y_1, Y_2)^T$. Then

$$f_{Y|H=0}(y) = \frac{1}{2\pi\sigma_1\sigma_2} \exp \left(-\frac{(y_1 - A)^2}{2\sigma_1^2} - \frac{(y_2 - A)^2}{2\sigma_2^2}\right)$$

$$f_{Y|H=1}(y) = \frac{1}{2\pi\sigma_1\sigma_2} \exp \left(-\frac{(y_1 + A)^2}{2\sigma_1^2} - \frac{(y_2 + A)^2}{2\sigma_2^2}\right).$$

The likelihood ratio function is

$$LR(y) = \frac{f_{Y|H=0}(y)}{f_{Y|H=1}(y)} = \exp \left(\frac{4Ay_1}{2\sigma_1^2} + \frac{4Ay_2}{2\sigma_2^2}\right),$$

and the log likelihood-ratio function is

$$LLR(y) = \frac{2Ay_1}{\sigma_1^2} + \frac{2Ay_2}{\sigma_2^2}.$$ 

Since the prior is uniform, an optimal guessing rule is to guess “$H = 0$” if $LLR(y)$ is nonnegative and to guess “$H = 1$” if it is negative. Using the above expression for the log likelihood-ratio function and recalling the hypothesis that $A$ is nonnegative leads to the rule

$$\begin{align*}
\text{“}H = 0\text{”} & \uparrow \quad \geq \quad 0. \quad \text{(1)} \\
\frac{y_1}{\sigma_1^2} + \frac{y_2}{\sigma_2^2} & < \downarrow \\
\text{“}H = 1\text{”}
\end{align*}$$

(ii) From (1), an optimal decision rule (when $\sigma_1 = 2\sigma_2$) is

$$\begin{align*}
\text{“}H = 0\text{”} & \uparrow \\
\frac{y_1}{4} + y_2 & \geq \quad 0, \\
\text{“}H = 1\text{”} & \downarrow
\end{align*}$$

which is illustrated in Figure 1.
(iii) From (1), a one dimensional sufficient statistic is

\[ T: (y_1, y_2) \mapsto \frac{y_1}{\sigma_1^2} + \frac{y_2}{\sigma_2^2}. \]

(iv) Under each hypothesis, the sufficient statistic \( T \) is a sum of two independent Gaussian random variables. Thus \( T \) itself is a Gaussian random variable under each hypothesis. In particular,

\[
T \sim \begin{cases} 
\mathcal{N} \left( \frac{A}{\sigma_1^2} + \frac{A}{\sigma_2^2}, \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) & \text{under } H = 0 \\
\mathcal{N} \left( -\frac{A}{\sigma_1^2} - \frac{A}{\sigma_2^2}, \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) & \text{under } H = 1.
\end{cases}
\]

Hence,

\[
\Pr(\text{error}) = \frac{1}{2} \Pr(\text{error} | H = 0) + \frac{1}{2} \Pr(\text{error} | H = 1)
= \frac{1}{2} \Pr(T < 0 | H = 0) + \frac{1}{2} \Pr(T \geq 0 | H = 1)
= \frac{1}{2} \Phi \left( \frac{A \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} \right) + \frac{1}{2} \Phi \left( \frac{-A \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} \right)
= \Phi \left( \frac{A \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}} \right).
\]

(v) Let \( \tilde{T} = Y_1 + Y_2 \). We have that

\[
\tilde{T} \sim \begin{cases} 
\mathcal{N} \left( 2A, \sigma_1^2 + \sigma_2^2 \right) & \text{under } H = 0 \\
\mathcal{N} \left( -2A, \sigma_1^2 + \sigma_2^2 \right) & \text{under } H = 1.
\end{cases}
\]
Figure 2: The decision regions of the MAP rule.

Hence,

$\Pr(\text{error}) = \frac{1}{2} \Pr[\hat{T} < 0 \mid H = 0] + \frac{1}{2} \Pr[\hat{T} \geq 0 \mid H = 1]$

$= \frac{1}{2} Q\left(\frac{2A}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) + \frac{1}{2} Q\left(\frac{2A}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right)$

$= Q\left(\frac{2A}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right).$

This performance is indeed suboptimal. This can be seen by noting that $Q(\cdot)$ is monotonic and by noting that

$\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \geq \frac{4}{\sigma_1^2 + \sigma_2^2},$

as can be verified by recalling that the harmonic mean cannot exceed the arithmetic mean:

$\frac{1}{\frac{1}{\alpha} + \frac{1}{\beta}} \leq \frac{\alpha + \beta}{2}, \quad \alpha, \beta > 0.$

**Problem 2**

*Ternary Gaussian Detection*

Because the prior is uniform, it follows from Proposition 21.6.1 that the MAP rule for this problem is

- guess “$M = 2$” if $\langle Y, s \rangle_E \geq \frac{\|s\|^2}{2}$
- guess “$M = 3$” if $\langle Y, s \rangle_E \leq -\frac{\|s\|^2}{2}$
- guess “$M = 1$” if $|\langle Y, s \rangle_E| < \frac{\|s\|^2}{2}$. 

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The conditional probabilities of error are now given by

\[ p(\text{error}|M = 1) = 2 Q\left(\frac{||s||}{2\sigma}\right), \]
\[ p(\text{error}|M = 2) = Q\left(\frac{||s||}{2\sigma}\right), \]
\[ p(\text{error}|M = 3) = Q\left(\frac{||s||}{2\sigma}\right). \]

**Problem 3**

(i) Because the prior is uniform, it follows from Proposition 21.6.1 (ii) that the MAP rule for this problem coincides with the “nearest-neighbor” rule, and the decision regions are thus as depicted in Figure 3.
(ii) By symmetry, \( p(\text{error}|M = m) \) does not depend on \( m \). We shall thus compute it for \( M = 1 \). We shall focus on \( p(\text{correct}|M = 1) \), which is \( 1 - p(\text{error}|M = 1) \). This is the probability that a two-dimensional Gaussian centered around \((0, A)\) will lie in the area where the MAP guesses “
m = 1”. Since the noise distribution is radially symmetric, this probability is equal to the probability that this two-dimensional Gaussian centered around \((0, A)\) will lie in the result of rotating this region by 45 degrees around \((0, A)\).

This latter probability can be expressed as \( \Pr[Z_1 \geq -A/\sqrt{2}, Z_2 \geq -A/\sqrt{2}] \). Thus

\[
p(\text{error}|M = m) = p(\text{error}|M = 1) = 1 - \left(1 - \Phi\left(\frac{A}{\sqrt{2}\sigma}\right)\right)^2.
\]

(iii) A brute force application of Proposition 21.5.3 yields the bound

\[
p(\text{error}|M = 1) \leq 2 \Phi\left(\frac{A}{\sqrt{2}\sigma}\right) + \Phi\left(\frac{A}{\sigma}\right),
\]

because \((a_1, b_1)\) has two neighbors at distance \( \sqrt{2}A \) and one neighbor at distance \( 2A \).

This bound can be improved by restricting the sum in (21.50) to \( m' \in \{2, 4\} \) to yield

\[
p(\text{error}|M = 1) \leq 2 \Phi\left(\frac{A}{\sqrt{2}\sigma}\right).
\]

In this improved bound the region whose probability is counted twice is the region where the decoder declares “\( M = 3 \)”.

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Problem 4

**A 7-ary QAM problem**

(i) The decision regions are depicted in Figure 5.

(ii) A disk of radius \( A/2 \) centered around the constellation point \((a_m, b_m)\) is fully contained in the region where the MAP guesses “\( M = m' \)”. Consequently, conditional on \( M = m \), an error can occur only if the noise causes the received point \((Y^{(1)}, Y^{(2)})\) to lie outside this disk, i.e., only if the noise is of norm exceeding \( A/2 \). The sum of the squares of the noise components has a mean-2\(\sigma^2\) exponential distribution (Note 19.8.1). Consequently, conditional on \( M = m \), the probability of error \( p(\text{error}|M = m) \) is upper-bounded by the probability that a mean-2\(\sigma^2\) exponential random variable exceeds \( A^2/4 \). The density of a mean-2\(\sigma^2\) exponential random variable is

\[
\frac{1}{2\sigma^2} e^{-\frac{\xi}{2\sigma^2}} \mathbb{1}\{\xi > 0\}, \quad \xi \in \mathbb{R},
\]

so

\[
p(\text{error}|M = m) \leq \int_{A^2/4}^{\infty} \frac{1}{2\sigma^2} e^{-\frac{\xi}{2\sigma^2}} d\xi = e^{-\frac{A^2}{8\sigma^2}}.
\]

(iii) Proposition 21.5.3 would yield the bounds

\[
p(\text{error}|M = m) \leq 3 \Phi\left(\frac{A}{2\sigma}\right) + 2 \Phi\left(\frac{\sqrt{3}A}{2\sigma}\right) + \Phi\left(\frac{A}{\sigma}\right), \quad m = 1, 2, \ldots, 6,
\]
Figure 5: The decision regions for 7-ary QAM.

\[ p(\text{error}|M = 7) \leq 6 Q\left(\frac{A}{2\sigma}\right), \]

because for \( m \neq 7 \) the point \((a_m, b_m)\) has three neighbors at distance \( A \); two neighbors at distance \( \sqrt{3}A \); and one neighbor at distance \( 2A \), and because the point \((a_7, b_7)\) at the origin has six neighbors all of which are at distance \( A \).

For \( m \neq 7 \) we can indeed improve on the above bound by including fewer terms. This would yield

\[ p(\text{error}|M = m) \leq 3 Q\left(\frac{A}{2\sigma}\right), \quad m = 1, 2, \ldots, 6. \]

For example, when \( m = 1 \), it suffices to consider only \( m' = 2, 6, 7 \) in (21.35).

(iv) When \( A/\sigma \) is very large, the bounds of Proposition 21.5.3 are better. This follows from (19.18) and (19.19). When \( A/\sigma \) is very small, e.g., zero, then the bounds of Proposition 21.5.3 are inferior: at \( A/\sigma = 0 \) Proposition 21.5.3 yields bounds that are larger than one.