# Communication and Detection Theory

Signal and Information Processing Laboratory
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## Model Answers to Exercise 13 of May 23, 2017

http://www.isi.ee.ethz.ch/teaching/courses/cdt

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**Problem 1** ***Constructing a SP from a RV***

(i) The process $\big(X(t)\big)$ is not stationary. Indeed, the variance of $X(t)$ is $e^{-2|t|}$, which depends on $t$. Thus, the distribution of $X(t)$ depends on $t$, and the process is not stationary.

(ii) This SP is Gaussian. Indeed, for every positive integer $n$ and every choice of the epochs $t_1, \ldots, t_n$ the random vector $(X(t_1), \ldots, X(t_n))^{\mathsf{T}}$ is Gaussian because it has the representation $\mathsf{A}W$, where $W$ is a standard Gaussian and $\mathsf{A}$ is the $n \times 1$ matrix

$$\mathsf{A} = \begin{pmatrix} e^{-|t_1|} \\ \vdots \\ e^{-|t_n|} \end{pmatrix}.$$

**Problem 2** ***Delaying and Adding***

(i) The SP $\big(Y(t)\big)$ is Gaussian. To show this, consider any positive integer $n$ and any epochs $t_1, \ldots, t_n \in \mathbb{R}$. To show that $\big(Y(t)\big)$ is Gaussian, we need to show that the random $n$-vector $(Y(t_1), \ldots, Y(t_n))^{\mathsf{T}}$ is Gaussian. This follows because this vector is the result of applying a linear transformation to the random $2n$-vector

$$\Big(X(t_1), X(t_2 - t_{\mathrm{D}}), \ldots, X(t_n), X(t_n - t_{\mathrm{D}})\Big)^{\mathsf{T}},$$

which is Gassian because, by hypothesis, $\big(X(t)\big)$ is a Gaussian SP. Indeed,

$$\begin{pmatrix} Y(t_1) \\ \vdots \\ \vdots \\ . \\ Y(t_n) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} X(t_1) \\ X(t_1 - t_{\mathrm{D}}) \\ \vdots \\ \vdots \\ X(t_n) \\ X(t_n - t_{\mathrm{D}}) \end{pmatrix}.$$

(ii) The mean is $\big(Y(t)\big)$ is

$$\begin{aligned}\mathsf{E}\big[Y(t)\big] &= \mathsf{E}\big[X(t) + X(t - t_{\mathrm{D}})\big]\\ &= \mathsf{E}\big[X(t)\big] + \mathsf{E}\big[X(t - t_{\mathrm{D}})\big]\\ &= 2\mu_x, \quad t \in \mathbb{R}.\end{aligned}$$

As to the autocovariance function:

$$\begin{aligned}&\mathsf{E}\big[Y(t)Y(t+\tau)\big]\\ &\quad= \mathsf{E}\big[\big(X(t) + X(t - t_{\mathrm{D}})\big)\big(X(t+\tau) + X(t+\tau - t_{\mathrm{D}})\big)\big]\\ &\quad= \mathsf{E}\big[X(t)\,X(t+\tau)\big] + \mathsf{E}\big[X(t - t_{\mathrm{D}})\,X(t+\tau)\big] + \mathsf{E}\big[X(t)\,X(t+\tau - t_{\mathrm{D}})\big]\\ &\qquad+ \mathsf{E}\big[X(t - t_{\mathrm{D}})\,X(t+\tau - t_{\mathrm{D}})\big]\\ &\quad= \mathsf{K}_{XX}(\tau) + \mathsf{K}_{XX}(\tau + t_{\mathrm{D}}) + \mathsf{K}_{XX}(\tau - t_{\mathrm{D}}) + \mathsf{K}_{XX}(\tau)\\ &\quad= 2\,\mathsf{K}_{XX}(\tau) + \mathsf{K}_{XX}(\tau + t_{\mathrm{D}}) + \mathsf{K}_{XX}(\tau - t_{\mathrm{D}}), \quad t, \tau \in \mathbb{R}.\end{aligned}$$

Thus, $\big(Y(t)\big)$ is WSS.

(iii) Since $\big(Y(t)\big)$ is WSS and Gaussian, it is stationary.

## Problem 3 — *Random Variables and Stochastic Processes*

(i) The random variable $Z(0.2)$ can be written as

$$Z(0.2) = aX + bY,$$

where $a = \cos(2\pi \cdot 0.2)$ and $b = \sin(2\pi \cdot 0.2)$ are constants. Since a linear combination of independent Gaussians is Gaussian (Proposition 19.7.3), $Z(0.2)$ is Gaussian.

(ii) We show that $\big(Z(t)\big)$ is a Gaussian SP as follows. Consider any positive integer $n$ and arbitrary epochs $t_1, \ldots, t_n \in \mathbb{R}$. We need to show that the random-$n$ vector

$$\mathbf{Z} \triangleq \big(Z(t_1), \ldots, Z(t_n)\big)^{\mathsf{T}},$$

where $Z(t_\nu) = X\cos(2\pi t_\nu) + Y\sin(2\pi t_\nu)$, $\nu = 1, \ldots, n$ is a Gaussian vector. To show this we express it as $\mathbf{Z} = \mathsf{A}\mathbf{W}$, where $\mathsf{A}$ is the deterministic matrix

$$\mathsf{A} = \sigma \begin{pmatrix} \cos 2\pi t_1 & \sin 2\pi t_1\\ \cos 2\pi t_2 & \sin 2\pi t_2\\ \vdots & \vdots\\ \cos 2\pi t_n & \sin 2\pi t_n \end{pmatrix}$$

and where $\mathbf{W} = (X/\sigma, Y/\sigma)^{\mathsf{T}}$ is a standard Gaussian 2-vector.

(iii) Yes, $\big(Z(t)\big)$ is stationary. To prove this it suffices to show that it is WSS, because it is Gaussian. We thus need to show that $\mathsf{E}[Z(t)]$ and $\mathsf{E}[Z(t)\,Z(t+\tau)]$ do not depend on $t$. For the former we have

$$\begin{aligned}\mathsf{E}\big[Z(t)\big] &= \mathsf{E}\big[X\big]\cos 2\pi t + \mathsf{E}\big[Y\big]\sin 2\pi t\\ &= 0, \quad t \in \mathbb{R},\end{aligned}$$

and for the latter

$$\begin{aligned}\mathsf{E}\big[Z(t)Z(t+\tau)\big] &= \mathsf{E}\big[X^2\big]\cos(2\pi t)\cos(2\pi(t+\tau)) + \mathsf{E}\big[Y^2\big]\sin(2\pi t)\sin(2\pi(t+\tau))\\ &= \sigma^2\cos(2\pi\tau), \quad t, \tau \in \mathbb{R}.\end{aligned}$$

The stochastic processes $\bigl(S(t)\bigr)$, $\bigl(U(t)\bigr)$, and $\bigl(W(t)\bigr)$ are Gaussian, whereas $\bigl(T(t)\bigr)$ and $\bigl(V(t)\bigr)$ are not.

All these processes are of finite variance. All have constant means. Indeed, $\bigl(S(t)\bigr)$, $\bigl(U(t)\bigr)$, $\bigl(W(t)\bigr)$, and $\bigl(T(t)\bigr)$ are centered and

$$\mathsf{E}\bigl[V(t)\bigr] = \mathsf{K}_{XX}(\tau_4), \quad t \in \mathbb{R}.$$

As to the autocovariances:

$$\begin{aligned}\mathsf{E}\bigl[S(t)\,S(t+\tau)\bigr] &= \mathsf{E}\Bigl[\bigl(X(t+\tau)+Y(t+\tau+\tau_1)\bigr)\bigl(X(t)+Y(t+\tau_1)\bigr)\Bigr] \\ &= \mathsf{E}\bigl[X(t+\tau)\,X(t)\bigr] + \mathsf{E}\bigl[Y(t+\tau+\tau_1)\,Y(t+\tau_1)\bigr] \\ &= \mathsf{K}_{XX}(\tau) + \mathsf{K}_{YY}(\tau), \quad t,\tau \in \mathbb{R},\end{aligned}$$

so $\bigl(S(t)\bigr)$ is WSS.
For $\bigl(U(t)\bigr)$ we compute

$$\begin{aligned}\mathsf{E}\bigl[U(t+\tau)\,U(t)\bigr] &= \mathsf{E}\Bigl[\bigl(X(t+\tau)+X(t+\tau+\tau_3)\bigr)\bigl(X(t)+X(t+\tau_3)\bigr)\Bigr] \\ &= \mathsf{K}_{XX}(\tau) + \mathsf{K}_{XX}(\tau-\tau_3) + \mathsf{K}_{XX}(\tau+\tau_3) + \mathsf{K}_{XX}(\tau) \\ &= 2\,\mathsf{K}_{XX}(\tau) + \mathsf{K}_{XX}(\tau-\tau_3) + \mathsf{K}_{XX}(\tau+\tau_3), \quad t,\tau \in \mathbb{R},\end{aligned}$$

so $\bigl(U(t)\bigr)$ is WSS.
For $\bigl(W(t)\bigr)$ we compute

$$\begin{aligned}\mathsf{E}\bigl[W(t+\tau)\,W(t)\bigr] &= \mathsf{E}\Bigl[\bigl(X(t+\tau)+X(-t-\tau)\bigr)\bigl(X(t)+X(-t)\bigr)\Bigr] \\ &= \mathsf{K}_{XX}(\tau) + \mathsf{K}_{XX}(2t+\tau) + \mathsf{K}_{XX}(2t+\tau) + \mathsf{K}_{XX}(-\tau) \\ &= 2\,\mathsf{K}_{XX}(\tau) + 2\,\mathsf{K}_{XX}(2t+\tau), \quad t,\tau \in \mathbb{R}.\end{aligned}$$

Since this depends on $t$ and not only on $\tau$, the SP $\bigl(W(t)\bigr)$ is not WSS.
For $\bigl(T(t)\bigr)$ we compute

$$\begin{aligned}\mathsf{E}\bigl[T(t+\tau)\,T(t)\bigr] &= \mathsf{E}\bigl[X(t+\tau)\,Y(t+\tau+\tau_2)\,X(t)\,Y(t+\tau_2)\bigr] \\ &= \mathsf{E}\bigl[X(t+\tau)\,X(t)\bigr]\;\mathsf{E}\bigl[Y(t+\tau+\tau_2)\,Y(t+\tau_2)\bigr] \\ &= \mathsf{K}_{XX}(\tau)\;\mathsf{K}_{YY}(\tau), \quad t,\tau \in \mathbb{R},\end{aligned}$$

so $\bigl(T(t)\bigr)$ is WSS.
Finally,

$$\mathsf{E}\bigl[V(t+\tau)\,V(t)\bigr] = \mathsf{E}\bigl[X(t+\tau)\,X(t+\tau+\tau_4)\,X(t)\,X(t+\tau_4)\bigr],$$

and this does not depend on $t$ (no need to compute!) because $\bigl(X(t)\bigr)$ is stationary, so the joint distribution of

$$X(t+\tau), X(t+\tau+\tau_4), X(t), X(t+\tau_4)$$

does not depend on $t$. Thus $\bigl(V(t)\bigr)$ is WSS.
To summarize:

$\bigl(S(t)\bigr)$, $\bigl(U(t)\bigr)$, $\bigl(T(t)\bigr)$, and $\bigl(V(t)\bigr)$ are WSS, and $\bigl(W(t)\bigr)$ is not.

As to stationarity, $\big(S(t)\big)$, and $\big(U(t)\big)$ are Gaussian and WSS, so they are stationary. As to $\big(W(t)\big)$, it is not WSS and it is of finite variance, so it cannot be stationary. It remains to determine whether $\big(T(t)\big)$ and $\big(V(t)\big)$ are stationary. They both are. Indeed, the joint law of

$$T(t_1+\tau), \ldots, T(t_n+\tau)$$

is determined by the joint law of

$$X(t_1+\tau), \ldots, X(t_n+\tau), Y(t_1+\tau_2+\tau), \ldots, Y(t_n+\tau_2+\tau),$$

and this joint law does not depend on $\tau$ because $\big(X(t)\big)$ and $\big(Y(t)\big)$ are stationary and independent. Similarly, $\big(V(t)\big)$ is stationary because the joint law of

$$V(t_1+\tau), \ldots, V(t_n+\tau)$$

is determined by the joint law of

$$X(t_1+\tau), X(t_1+\tau_4+\tau), \ldots, X(t_n+\tau), X(t_n+\tau_4+\tau),$$

and this joint law does not depend on $\tau$ because $\big(X(t)\big)$ is stationary. To summarize:

$$\big(S(t)\big),\ \big(U(t)\big),\ \big(T(t)\big), \text{ and } \big(V(t)\big) \text{ are stationary, and } \big(W(t)\big) \text{ is not.}$$

## Problem 5 — ***A Linear Functional of a Gaussian SP***

Being a linear functional of a Gaussian stochastic process, $Y$ is a Gaussian random variable. Its distribution is thus determined by its mean and variance. Its mean is

$$\mathsf{E}[Y] = \mathsf{E}\left[\int_0^2 X(t)\,\mathrm{d}t\right] = \int_0^2 \mathsf{E}\big[X(t)\big]\,\mathrm{d}t = \int_0^2 2\,\mathrm{d}t = 4.$$

Its variance is

$$\begin{aligned}
\mathsf{E}\big[(Y-4)^2\big] &= \mathsf{E}\left[\left(\int_0^2 X(t')\,\mathrm{d}t' - 4\right)\left(\int_0^2 X(t)\,\mathrm{d}t - 4\right)\right] \\
&= \mathsf{E}\left[\left(\int_0^2 \big(X(t')-2\big)\,\mathrm{d}t'\right)\left(\int_0^2 \big(X(t)-2\big)\,\mathrm{d}t\right)\right] \\
&= \int_0^2\int_0^2 \mathsf{E}\big[\big(X(t')-2\big)\big(X(t)-2\big)\big]\,\mathrm{d}t'\,\mathrm{d}t \\
&= \int_0^2\int_0^2 e^{-|t'-t|}\,\mathrm{d}t'\,\mathrm{d}t \\
&= \int_0^2\int_{-t}^{2-t} e^{-|\tau|}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_0^2 \left(\int_{-t}^{0} e^{\tau}\,\mathrm{d}\tau + \int_0^{2-t} e^{-\tau}\,\mathrm{d}\tau\right)\mathrm{d}t \\
&= \int_0^2 \big(1 - e^{-t} - e^{t-2} + 1\big)\,\mathrm{d}t \\
&= 2 + e^{-2} - 1 - 1 + e^{-2} + 2 \\
&= 2 + 2e^{-2}.
\end{aligned}$$

Thus $Y \sim \mathcal{N}\big(4, 2+2e^{-2}\big)$, and

$$\Pr\big[Y \geq 2\big] = 1 - \mathcal{Q}\left(\frac{2}{\sqrt{2+2e^{-2}}}\right).$$