Model Answers to Exercise 13 of May 19, 2015

http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 1

**Constructing a SP from a RV**

(i) The process \( (X(t)) \) is not stationary. Indeed, the variance of \( X(t) \) is \( e^{-2|t|} \), which depends on \( t \). Thus, the distribution of \( X(t) \) depends on \( t \), and the process is not stationary.

(ii) This SP is Gaussian. Indeed, for every positive integer \( n \) and every choice of the epochs \( t_1, \ldots, t_n \in \mathbb{R} \), the random vector \( (X(t_1), \ldots, X(t_n))^T \) is Gaussian because it has the representation \( AW \), where \( W \) is a standard Gaussian and \( A \) is the \( n \times 1 \) matrix

\[
A = \begin{pmatrix}
e^{-|t_1|} \\
0 \\
\vdots \\
e^{-|t_n|}
\end{pmatrix}.
\]

Problem 2

**Delaying and Adding**

(i) The SP \( (Y(t)) \) is Gaussian. To show this, consider any positive integer \( n \) and any epochs \( t_1, \ldots, t_n \in \mathbb{R} \). To show that \( (Y(t)) \) is Gaussian, we need to show that the random \( n \)-vector \( (Y(t_1), \ldots, Y(t_n))^T \) is Gaussian. This follows because this vector is the result of applying a linear transformation to the random \( 2n \)-vector

\[
\begin{pmatrix}
X(t_1) \\
X(t_2 - t_D) \\
\vdots \\
X(t_n) \\
X(t_n - t_D)
\end{pmatrix},
\]

which is Gaussian because, by hypothesis, \( (X(t)) \) is a Gaussian SP. Indeed,

\[
\begin{pmatrix}
Y(t_1) \\
\vdots \\
Y(t_n)
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 1
\end{pmatrix} \begin{pmatrix}
X(t_1) \\
X(t_1 - t_D) \\
\vdots \\
X(t_n) \\
X(t_n - t_D)
\end{pmatrix}.
\]
ii) The mean of \( Y(t) \) is

\[
E[Y(t)] = E[X(t) + X(t - t_D)] \\
= E[X(t)] + E[X(t - t_D)] \\
= 2\mu_x, \quad t \in \mathbb{R}.
\]

As to the autocovariance function:

\[
E[(Y(t) - E[Y(t)])(Y(t + \tau) - E[Y(t)])] \\
= E[(X(t) + X(t - t_D) - 2\mu_x)(X(t + \tau) + X(t + \tau - t_D) - 2\mu_x)] \\
= E[(X(t) - \mu_x)(X(t + \tau) - \mu_x)] + E[(X(t - t_D) - \mu_x)(X(t + \tau - t_D) - \mu_x)] \\
+ E[\{(X(t) - \mu_x)(X(t + \tau - t_D) - \mu_x)\} + E[\{(X(t - t_D) - \mu_x)(X(t + \tau - t_D) - \mu_x)\}] \\
= K_{XX}(\tau) + K_{XX}(\tau + t_D) + K_{XX}(\tau - t_D) + K_{XX}(\tau) \\
= 2K_{XX}(\tau) + K_{XX}(\tau + t_D) + K_{XX}(\tau - t_D), \quad t, \tau \in \mathbb{R}.
\]

Thus, \( Y(t) \) is WSS.

(iii) Since \( Y(t) \) is WSS and Gaussian, it is stationary.

**Problem 3**

**Random Variables and Stochastic Processes**

(i) The random variable \( Z(0.2) \) can be written as

\[
Z(0.2) = aX + bY,
\]

where \( a = \cos(2\pi \cdot 0.2) \) and \( b = \sin(2\pi \cdot 0.2) \) are constants. Since a linear combination of independent Gaussians is Gaussian (Proposition 19.7.3), \( Z(0.2) \) is Gaussian.

(ii) Yes, \( \{Z(t)\} \) is a Gaussian SP. To prove this, consider any positive integer \( n \) and arbitrary epochs \( t_1, \ldots, t_n \in \mathbb{R} \). We need to show that the random \( n \)-vector

\[
Z = (Z(t_1), \ldots, Z(t_n))^T,
\]

where \( Z(t_\nu) = X \cos(2\pi t_\nu) + Y \sin(2\pi t_\nu), \nu = 1, \ldots, n \) is a Gaussian vector. But this holds since \( Z = AW \), where \( A \) is the deterministic matrix

\[
A = \sigma \begin{pmatrix} \cos(2\pi t_1) & \sin(2\pi t_1) \\ \cos(2\pi t_2) & \sin(2\pi t_2) \\ \vdots & \vdots \\ \cos(2\pi t_n) & \sin(2\pi t_n) \end{pmatrix},
\]

and where \( W = (X/\sigma, Y/\sigma)^T \) is a standard Gaussian 2-vector.

(iii) Yes, \( \{Z(t)\} \) is stationary. To prove this it suffices to show that it is WSS, because it is Gaussian. We thus need to show that \( E[Z(t)] \) and \( E[X(t)X(t + \tau)] \) do not depend on \( t \). For the former we have

\[
E[Z(t)] = E[X] \cos(2\pi t) + E[Y] \sin(2\pi t) \\
= 0, \quad t \in \mathbb{R},
\]

and for the latter

\[
E[Z(t)Z(t + \tau)] = E[X^2] \cos(2\pi t) \cos(2\pi (t + \tau)) + E[Y^2] \sin(2\pi t) \sin(2\pi (t + \tau)) \\
= \sigma^2 \cos(2\pi \tau), \quad t, \tau \in \mathbb{R}.
\]
Problem 4

Classifying Stochastic Processes

The stochastic processes \((S(t)), \ (U(t)),\) and \((W(t))\) are Gaussian, whereas \((T(t))\) and \((V(t))\) are not.

All these processes are of finite variance. All have constant means. Indeed, \((S(t)), \ (U(t)),\) \((W(t)),\) and \((T(t))\) are centered and

\[
E[V(t)] = K_{XX}(\tau_4), \quad t \in \mathbb{R}.
\]

As to the autocovariances:

\[
E[S(t)S(t+\tau)] = E[(X(t+\tau) + Y(t+\tau + \tau_1))(X(t) + Y(t + \tau_1))]
= E[X(t+\tau)X(t)] + E[Y(t+\tau + \tau_1)Y(t + \tau_1)]
= K_{XX}(\tau) + K_{YY}(\tau), \quad t, \tau \in \mathbb{R},
\]

so \((S(t))\) is WSS.

For \((U(t))\) we compute

\[
E[U(t+\tau)U(t)] = E[(X(t+\tau) + X(t+\tau + \tau_3))(X(t) + X(t + \tau_3))]
= K_{XX}(\tau) + K_{XX}(\tau - \tau_3) + K_{XX}(\tau + \tau_3) + K_{XX}(\tau)
= 2K_{XX}(\tau) + K_{XX}(\tau - \tau_3) + K_{XX}(\tau + \tau_3), \quad t, \tau \in \mathbb{R},
\]

so \((U(t))\) is WSS.

For \((W(t))\) we compute

\[
E[W(t+\tau)W(t)] = E[(X(t+\tau) + X(-t - \tau))(X(t) + X(-t))]
= K_{XX}(\tau) + K_{XX}(2t + \tau) + K_{XX}(2t + \tau) + K_{XX}(-\tau)
= 2K_{XX}(\tau) + 2K_{XX}(2t + \tau), \quad t, \tau \in \mathbb{R}.
\]

Since this depends on \(t\) and not only on \(\tau\), the SP \((W(t))\) is not WSS.

For \((T(t))\) we compute

\[
E[T(t+\tau)T(t)] = E[X(t+\tau)Y(t + \tau + \tau_2)X(t)Y(t + \tau_2)]
= E[X(t+\tau)X(t)]E[Y(t + \tau + \tau_2)Y(t + \tau_2)]
= K_{XX}(\tau)K_{YY}(\tau), \quad t, \tau \in \mathbb{R},
\]

so \((T(t))\) is WSS.

Finally,

\[
E[V(t+\tau)V(t)] = E[X(t+\tau)X(t+\tau + \tau_4)X(t)X(t + \tau_4)],
\]

and this does not depend on \(t\) (no need to compute!) because \((X(t))\) is stationary, so the joint distribution of

\[
X(t+\tau), X(t + \tau + \tau_4), X(t), X(t + \tau_4)
\]

does not depend on \(t\). Thus \((V(t))\) is WSS.

To summarize:

\((S(t)), \ (U(t)), \ (T(t)), \) and \((V(t))\) are WSS, and \((W(t))\) is not.
As to stationarity, \((S(t))\), and \((U(t))\) are Gaussian and WSS, so they are stationary. As to \((W(t))\), it is not WSS and it is of finite variance, so it cannot be stationary. It remains to determine whether \((T(t))\) and \((V(t))\) are stationary. They both are. Indeed, the joint law of

\[ T(t_1 + \tau), \ldots, T(t_n + \tau) \]

is determined by the joint law of

\[ X(t_1 + \tau), \ldots, X(t_n + \tau), Y(t_1 + \tau_2 + \tau), \ldots, Y(t_n + \tau_2 + \tau), \]

and this joint law does not depend on \(\tau\) because \((X(t))\) and \((Y(t))\) are stationary and independent. Similarly, \((V(t))\) is stationary because the joint law of

\[ V(t_1 + \tau), \ldots, V(t_n + \tau) \]

is determined by the joint law of

\[ X(t_1 + \tau), X(t_1 + \tau_4 + \tau), \ldots, X(t_n + \tau), X(t_n + \tau_4 + \tau), \]

and this joint law does not depend on \(\tau\) because \((X(t))\) is stationary. To summarize:

\((S(t)), (U(t)), (T(t)),\) and \((V(t))\) are stationary, and \((W(t))\) is not.

**Problem 5**

**A Linear Functional of a Gaussian SP**

Being a linear functional of a Gaussian stochastic process,

\[ Y \triangleq \int_0^2 X(t) \, dt \]

is a Gaussian random variable. Its distribution is thus determined by its mean and variance. Its mean is

\[
E[Y] = E\left[\int_0^2 X(t) \, dt\right] = \int_0^2 E[X(t)] \, dt = \int_0^2 2 \, dt = 4.
\]

Its variance is

\[
E[(Y - 4)^2] = E\left[\left(\int_0^2 X(t') \, dt' - 4\right)\left(\int_0^2 X(t) \, dt - 4\right)\right] \\
= E\left[\left(\int_0^2 (X(t') - 2) \, dt'\right)\left(\int_0^2 (X(t) - 2) \, dt\right)\right] \\
= \int_0^2 \int_0^2 E[(X(t') - 2)(X(t) - 2)] \, dt' \, dt \\
= \int_0^2 \int_0^2 e^{-(t' - t)} \, dt' \, dt \\
= \int_0^2 \int_{-t}^{2-t} e^{-|\tau|} \, d\tau \, dt \\
= \int_0^2 \left(\int_{-t}^{0} e^{\tau} \, d\tau + \int_{0}^{2-t} e^{-\tau} \, d\tau\right) \, dt \\
= \int_0^2 \left(1 - e^{-t} - e^{t-2} + 1\right) \, dt \\
= 2 + e^{-2} - 1 - 1 + e^{-2} + 2 \\
= 2 + 2e^{-2}.
\]
Thus \( Y \sim \mathcal{N}(4, 2 + 2e^{-2}) \), and

\[
\Pr[Y \geq 2] = 1 - \Phi\left( \frac{2}{\sqrt{2 + 2e^{-2}}} \right).
\]

\( \Phi \) denotes the cumulative distribution function of the standard normal distribution.