Communication and Detection Theory

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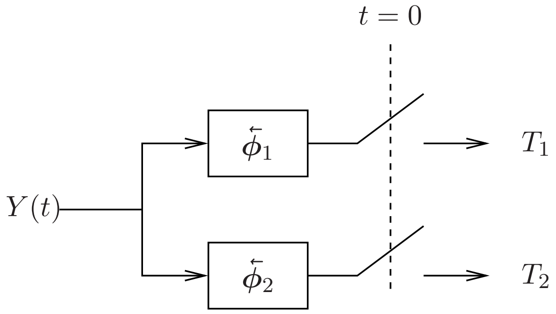
http://www.isi.ee.ethz.ch/teaching/courses/cdt

Problem 2 ***Signaling in White Gaussian Noise***

(i) The signals

$$\phi_1(t) = \frac{s_1(t)}{\|\mathbf{s}_1\|_2} = \frac{s_1(t)}{\mathsf{A}\sqrt{\mathsf{T}}}, \quad t \in \mathbb{R},$$
$$\phi_2(t) = \frac{s_2(t)}{\|\mathbf{s}_2\|_2} = \frac{s_2(t)}{\mathsf{A}\sqrt{\mathsf{T}}}, \quad t \in \mathbb{R}$$

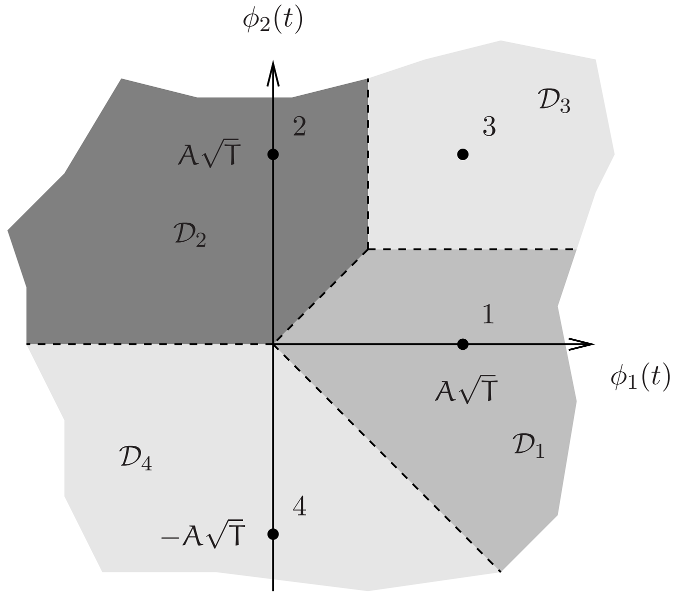
form an orthonormal basis for span $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4)$. Hence the pair (T_1, T_2), where $T_1 = \langle \mathbf{Y}, \boldsymbol{\phi}_1 \rangle$ and where $T_2 = \langle \mathbf{Y}, \boldsymbol{\phi}_2 \rangle$ forms a sufficient statistic for guessing M based on $\big(Y(t)\big)$. These inner products can be implemented using matched filters as shown in the figure below:



Conditional on $M = m$,

$$T_1 = \langle \mathbf{s}_m, \boldsymbol{\phi}_1 \rangle + Z_1,$$
$$T_2 = \langle \mathbf{s}_m, \boldsymbol{\phi}_2 \rangle + Z_2,$$

where Z_1 and Z_2 are IID $\mathcal{N}(0, \mathsf{N}_0/2)$. Thus the decision rule that minimizes the probability of error is the nearest neighbor decision rule: for a given $(T_1, T_2) = (t_1, t_2)$ we guess $m \in \{1, \ldots, 4\}$ whenever $(\langle \mathbf{s}_m, \boldsymbol{\phi}_1 \rangle, \langle \mathbf{s}_m, \boldsymbol{\phi}_2 \rangle)$ is closest to (t_1, t_2). The decision regions are indicated in the figure below.



(ii) Using the Union Bound and defining $\mathcal{B}_{m,m'} \subseteq \mathbb{R}^2$ as the set in which the a posteriori probability of $M = m'$ is greater or equal to that of $M = m$, we obtain

$$\begin{aligned}\Pr(\text{error} \,|\, M = 3) &\leq \sum_{m'=1,2,4} \int_{\mathcal{B}_{3,m'}} f_{\mathbf{T}|M=3}(\mathbf{t})\, \mathrm{d}\mathbf{t} \\ &= 2\mathcal{Q}\left(\frac{\mathsf{A}\sqrt{\mathsf{T}}}{2\sqrt{\mathsf{N}_0/2}}\right) + \mathcal{Q}\left(\frac{\mathsf{A}\sqrt{5\mathsf{T}}}{2\sqrt{\mathsf{N}_0/2}}\right).\end{aligned}$$

But, as seen in the figure, the last term is superfluous since $\mathcal{B}_{3,4} \subseteq (\mathcal{B}_{3,1} \cup \mathcal{B}_{3,2})$. Hence

$$\Pr\big(\text{error} \,\big|\, M = 3\big) \leq 2\mathcal{Q}\left(\frac{\mathsf{A}}{2}\sqrt{\frac{\mathsf{T}}{\mathsf{N}_0/2}}\right)$$

is a better upper bound.

(iii) The exact error probability is

$$\begin{aligned}\Pr\big(\text{error} \,\big|\, M = 3\big) &= \Pr\Big[T_1 < \frac{1}{2}\mathsf{A}\sqrt{\mathsf{T}} \,\Big|\, M = 3\Big] + \Pr\Big[T_2 < \frac{1}{2}\mathsf{A}\sqrt{\mathsf{T}} \,\Big|\, M = 3\Big] \\ &\quad - \Pr\Big[T_1 < \frac{1}{2}\mathsf{A}\sqrt{\mathsf{T}} \text{ and } T_2 < \frac{1}{2}\mathsf{A}\sqrt{\mathsf{T}} \,\Big|\, M = 3\Big] \\ &= 2\mathcal{Q}\left(\frac{\mathsf{A}}{2}\sqrt{\frac{2\mathsf{T}}{\mathsf{N}_0}}\right) - \left(\mathcal{Q}\left(\frac{\mathsf{A}}{2}\sqrt{\frac{2\mathsf{T}}{\mathsf{N}_0}}\right)\right)^2.\end{aligned}$$

(iv) The average received signal energy is

$$\begin{aligned}\mathsf{E}\left[\int_{-\infty}^{\infty} S^2(t)\,\mathrm{d}t\right] &= \sum_{m=1}^{4} \Pr[M = m] \int_{-\infty}^{\infty} s_m^2(t)\,\mathrm{d}t \\ &= \frac{1}{4}\sum_{m=1}^{4} \|\mathbf{s}_m\|_2^2 \\ &= \frac{1}{4}\big(\mathsf{A}^2\mathsf{T} + \mathsf{A}^2\mathsf{T} + 2\mathsf{A}^2\mathsf{T} + \mathsf{A}^2\mathsf{T}\big) \\ &= \frac{5}{4}\mathsf{A}^2\mathsf{T}.\end{aligned}$$

Looking at the constellation diagram we note that the average signal

$$\mathsf{E}[S(t)] = \frac{1}{4}\sum_{m=1}^{4} s_m(t) = \begin{cases} \frac{3}{4}\mathsf{A} & t \in \left[0, \frac{\mathsf{T}}{2}\right], \\ \frac{1}{4}\mathsf{A} & t \in \left[\frac{\mathsf{T}}{2}, \mathsf{T}\right], \\ 0 & \text{otherwise}, \end{cases} \qquad t \in \mathbb{R}$$

does not have zero energy. Thus, to reduce the expected transmitted signal energy, we can subtract the average signal $t \mapsto \mathsf{E}[S(t)]$ from each $s_m(\cdot)$. This corresponds to shifting the "center" of the constellation diagram towards the origin and thereby does not change the distance between any two of the four signals. The transmitted energy becomes

$$\mathsf{E}\left[\int_{-\infty}^{\infty} S^2(t)\,\mathrm{d}t\right] - \int_{-\infty}^{\infty} \left(\mathsf{E}[S(t)]\right)^2 \mathrm{d}t = \frac{15}{16}\mathsf{A}^2\mathsf{T}.$$

Problem 3 — *Mismatched Decoding of Antipodal Signaling*

Conditional on $H = 0$ the detector errs if $\langle \mathbf{s}, \mathbf{s}' \rangle + \langle \mathbf{N}, \mathbf{s}' \rangle$ is negative. Since

$$\langle \mathbf{N}, \mathbf{s}' \rangle \sim \mathcal{N}\left(0, \frac{\mathsf{N}_0}{2}\left\|\mathbf{s}'\right\|_2^2\right),$$

we conclude that

$$\Pr(\text{error} \,|\, H = 0) = \mathcal{Q}\left(\frac{\langle \mathbf{s}, \mathbf{s}' \rangle}{\sqrt{\mathsf{N}_0/2}\,\left\|\mathbf{s}'\right\|_2}\right).$$

Similarly, conditional on $H = 1$, the decoder errs if $-\langle \mathbf{s}, \mathbf{s}' \rangle + \langle \mathbf{N}, \mathbf{s}' \rangle$ is positive, so

$$\Pr(\text{error} \,|\, H = 1) = \mathcal{Q}\left(\frac{\langle \mathbf{s}, \mathbf{s}' \rangle}{\sqrt{\mathsf{N}_0/2}\,\left\|\mathbf{s}'\right\|_2}\right),$$

and

$$\Pr(\text{error}) = \mathcal{Q}\left(\frac{\langle \mathbf{s}, \mathbf{s}' \rangle}{\sqrt{\mathsf{N}_0/2}\,\left\|\mathbf{s}'\right\|_2}\right).$$

Problem 4 — *Imperfect Automatic Gain Control*

(i) It is optimal to base the decision on $T = \langle \mathbf{Y}, \mathbf{s}/\left\|\mathbf{s}\right\|_2 \rangle$. Thus, an optimal detector first passes the received signal $\big(Y(t),\ t \in \mathbb{R}\big)$ through a filter with impulse response $\vec{\mathbf{s}}/\left\|\mathbf{s}\right\|_2$ and samples the output at time $t = 0$. Conditional on $X = m$ the RV T is Gaussian with mean $mA\left\|s\right\|_2$ and variance $\mathsf{N}_0/2$. An optimal decision rule for guessing X based on T it thus given by

$$\phi_{\text{guess}}(t) = \begin{cases} +3 & \text{if } t \in [2\mathsf{A}\left\|\mathbf{s}\right\|_2, \infty), \\ +1 & \text{if } t \in [0, 2\mathsf{A}\left\|\mathbf{s}\right\|_2), \\ -1 & \text{if } t \in [-2\mathsf{A}\left\|\mathbf{s}\right\|_2, 0), \\ -3 & \text{if } t \in (-\infty, -2\mathsf{A}\left\|\mathbf{s}\right\|_2). \end{cases} \tag{1}$$

(ii) The probability of error can be computed as

$$\begin{aligned}
&\Pr(\text{error}) \\
&= \frac{1}{4}\bigl(\Pr(\text{error}\,|\,X=-3) + \Pr(\text{error}\,|\,X=-1) + \Pr(\text{error}\,|\,X=1) \\
&\quad + \Pr(\text{error}\,|\,X=3)\bigr) \\
&= \frac{1}{4}\left(\mathcal{Q}\biggl(\frac{\mathrm{A}\,\|\mathbf{s}\|_2}{\sqrt{\mathrm{N}_0/2}}\biggr) + 2\mathcal{Q}\biggl(\frac{\mathrm{A}\,\|\mathbf{s}\|_2}{\sqrt{\mathrm{N}_0/2}}\biggr) + 2\mathcal{Q}\biggl(\frac{\mathrm{A}\,\|\mathbf{s}\|_2}{\sqrt{\mathrm{N}_0/2}}\biggr) + \mathcal{Q}\biggl(\frac{\mathrm{A}\,\|\mathbf{s}\|_2}{\sqrt{\mathrm{N}_0/2}}\biggr)\right) \\
&= \frac{3}{2}\mathcal{Q}\biggl(\frac{\mathrm{A}\,\|\mathbf{s}\|_2}{\sqrt{\mathrm{N}_0/2}}\biggr).
\end{aligned}$$

(iii) Here, conditional on $X = m$, $T \sim \mathcal{N}(m(3/4)\mathrm{A}\,\|\mathbf{s}\|_2, \mathrm{N}_0/2)$. Using the decision rule given by (1), the probability of error is thus given by

$$\begin{aligned}
\Pr(\text{error}) &= \sum_{m\in\{-3,-1,+1,+3\}} \frac{1}{4}\Pr(\text{error}\,|\,X=m) \\
&= \frac{1}{4}\Biggl[\mathcal{Q}\biggl(\frac{\mathrm{A}\,\|\mathbf{s}\|_2/4}{\sqrt{\mathrm{N}_0/2}}\biggr) \\
&\quad + \mathcal{Q}\biggl(\frac{5\mathrm{A}\,\|\mathbf{s}\|_2/4}{\sqrt{\mathrm{N}_0/2}}\biggr) + \mathcal{Q}\biggl(\frac{3\mathrm{A}\,\|\mathbf{s}\|_2/4}{\sqrt{\mathrm{N}_0/2}}\biggr) \\
&\quad + \mathcal{Q}\biggl(\frac{5\mathrm{A}\,\|\mathbf{s}\|_2/4}{\sqrt{\mathrm{N}_0/2}}\biggr) + \mathcal{Q}\biggl(\frac{3\mathrm{A}\,\|\mathbf{s}\|_2/4}{\sqrt{\mathrm{N}_0/2}}\biggr) \\
&\quad + \mathcal{Q}\biggl(\frac{\mathrm{A}\,\|\mathbf{s}\|_2/4}{\sqrt{\mathrm{N}_0/2}}\biggr)\Biggr] \\
&= \frac{1}{2}\Biggl[\mathcal{Q}\biggl(\frac{\mathrm{A}\,\|\mathbf{s}\|_2/4}{\sqrt{\mathrm{N}_0/2}}\biggr) + \mathcal{Q}\biggl(\frac{3\mathrm{A}\,\|\mathbf{s}\|_2/4}{\sqrt{\mathrm{N}_0/2}}\biggr) + \mathcal{Q}\biggl(\frac{5\mathrm{A}\,\|\mathbf{s}\|_2/4}{\sqrt{\mathrm{N}_0/2}}\biggr)\Biggr].
\end{aligned}$$