Problem 1

Example of Joint Entropy

Let $P_{X,Y}(x,y)$ be given by

<table>
<thead>
<tr>
<th>$P_{X,Y}(x,y)$</th>
<th>$y = 0$</th>
<th>$y = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = 0$</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>$x = 1$</td>
<td>0</td>
<td>1/3</td>
</tr>
</tbody>
</table>

Find

a) $H(X)$, $H(Y)$,

b) $H(X|Y)$, $H(Y|X)$,

c) $H(X,Y)$,

d) $H(Y) - H(Y|X)$,

e) $I(X;Y)$.

Problem 2

Zero Conditional Entropy

Show that $H(Y|X) = 0$ if, and only if, $Y$ is a function\(^1\) of $X$.

Problem 3

Entropy of Functions of a Chance Variable

Let $X$ be a discrete chance variable. Show that the entropy of a function of $X$ is less than or equal to the entropy of $X$ by justifying the following steps:

\[
H(X, g(X)) \overset{(a)}{=} H(X) + H(g(X)|X) \overset{(b)}{=} H(X);
\]

\[
H(X, g(X)) \overset{(c)}{=} H(g(X)) + H(X|g(X)) \overset{(d)}{\geq} H(g(X)).
\]

Thus, $H(g(X)) \leq H(X)$. When does equality hold?

\(^1\)More precisely, $\Pr[Y = g(X)] = 1$ for some function $g(\cdot)$.
Problem 4

**Entropy of a Sum**

Let $X$ and $Y$ be random variables that take values in $\{x_1, x_2, \ldots, x_r\} \subset \mathbb{R}$ and in $\{y_1, y_2, \ldots, y_s\} \subset \mathbb{R}$, respectively. Let $Z = X + Y$.

a) Show that $H(Z|X) = H(Y|X)$. Argue that if $X$ and $Y$ are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of independent random variables adds uncertainty.

b) Give an example (of necessarily dependent random variables) in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

c) Under what conditions does $H(Z) = H(X) + H(Y)$ hold?

*Hint: You may find the conclusion of Problem 3 helpful.*

Problem 5

**Jensen's Inequality**

Let $a_1, \ldots, a_n$ be positive real numbers.

a) Use Jensen’s inequality to show that the arithmetic average is at least equal to the geometric average, i.e.,

$$\left( \prod_{k=1}^{n} a_k \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} a_k.$$ 

Also find necessary and sufficient conditions for equality.

b) Use Jensen’s inequality to show that

$$\frac{1}{n} \sum_{k=1}^{n} a_k^\beta \geq \left( \frac{1}{n} \sum_{k=1}^{n} a_k \right)^\beta$$

for any $\beta \geq 1$. What happens if $0 < \beta < 1$?

c) Suppose now that you have written $n$ exams with scores $a_k$ that are numbers between 1 and 6. Would you rather have as your final score $\frac{1}{n} \sum_{k=1}^{n} a_k$ or $\sqrt{\frac{1}{n} \sum_{k=1}^{n} a_k^2}$?

Problem 6

**Csiszár’s Identity**

Show that for any pair of random vectors $(A_1, \ldots, A_n), (B_1, \ldots, B_n)$

$$\sum_{i=1}^{n} \left( I(A_{i+1}; B_i|B^{i-1}) - I(B^{i-1}; A_i|A_{i+1}) \right) = 0,$$

where we use the shorthand notation $A^j = (A_1, \ldots, A_j)$, $A^k_j = (A_j, A_{j+1}, \ldots, A_k)$ for $j \leq k$, and $B^0$ and $A^n_{n+1}$ are defined to be deterministic.

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