Problem 1

*Optimal Code Lengths that Require One Bit above Entropy*

The source coding theorem asserts that an optimal code for a chance variable $X$ has expected length less than $H(X) + 1$. Show that this upper bound cannot be improved by constructing, for every $\epsilon > 0$, a chance variable for which the least expected length is larger than $H(X) + 1 - \epsilon$.

Problem 2

*Shannon Code*

Consider the following method for generating a code for a random variable $X$ which takes on $m$ values $\{1, 2, \ldots, m\}$ with positive probabilities $p_1, p_2, \ldots, p_m$. Assume that the probabilities are ordered so that $p_1 \geq p_2 \geq \cdots \geq p_m$. Define

$$F_i = \sum_{k=1}^{i-1} p_k,$$

the sum of the probabilities of all symbols less than $i$. Then the codeword for $i$ is the binary representation\(^1\) of the number $F_i \in [0, 1)$ rounded off to $l_i$ bits, where $l_i = \lceil \log \frac{1}{p_i} \rceil$.

a) Show that the code constructed by this process is prefix-free and the average length satisfies

$$H(X) \leq L < H(X) + 1.$$

b) Construct the code for the probability distribution $(0.5, 0.25, 0.125, 0.125)$.

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\(^1\)In cases where the binary representation is not unique we always take the terminating expansion. For example, we represent $1/2$ by $0.1$ instead of $0.0111\ldots$