Problem 1  

“Double”-Gaussian Channel

a) Consider a discrete-time memoryless Gaussian channel with average-power constraint $E_s = 1$. Sketch the capacity of this channel as a function of the noise variance $N$.

b) You are given two parallel independent Gaussian channels with noise variances $N_1$ and $N_2$, respectively. You are allocated a total power of $E_s$ to use between these two channels. The sum of the noise variances in the two channels is $2N$, but you will be able to choose from the two schemes below as to how the noise is distributed ($\Delta$ and $N$ are fixed).

Scheme A: $N_1 = N_2 = N$,  
Scheme B: $N_1 = N + \Delta$, $N_2 = N - \Delta$, $0 < \Delta < N$.

i) If you must put half of your power allocation in each channel, which scheme would you prefer?

ii) If you are allowed to arbitrarily divide your power allocation between the two channels, which scheme would you prefer?

Problem 2  

Additive Noise Channel

Consider an average power limited memoryless additive noise channel where the output $Y_k$ is equal to the sum of the input signal $x_k$ and the noise sample $Z_k$:

$$Y_k = x_k + Z_k.$$  

The transmitter is average-power limited so that

$$\frac{1}{n} \sum_{k=1}^{n} x_k^2(w) \leq E_s.$$  

The IID noise process $\{Z_k\}$ is independent of the channel input ($\{Z_k\} \perp \perp \{X_k\}$) and equal to 0 with probability $1/10$ and is otherwise Gaussian with zero mean and variance $\sigma^2 > 0$:

$$Z_k = \begin{cases} 0 & \text{with probability } \frac{1}{10} \\ G_k & \text{with probability } \frac{9}{10} \end{cases}$$

where $\{G_k\}$ is IID $\sim \mathcal{N}(0, \sigma^2)$.
a) Find the capacity $C$ of this channel.

b) Describe explicitly a coding scheme that achieves this capacity.

**Problem 3  \hspace{2cm} \textit{Exponential Noise Channel}**

Consider the channel

\[ Y_k = X_k + Z_k, \]

where the inputs $\{X_k\}$ are nonnegative and where $\{Z_k\}$ are IID exponentially distributed with mean $\mu$. Assume that the inputs satisfy the mean constraint

\[ E[X_k] \leq \lambda, \quad k = 0, 1, \ldots \]

Show that the capacity of this channel is

\[ C = \log \left( 1 + \frac{\lambda}{\mu} \right). \]

**Problem 4  \hspace{2cm} \textit{DMC with a Cost Constraint}**

Consider a discrete memoryless channel with a finite input alphabet $\mathcal{X}$. For every input symbol $x \in \mathcal{X}$, define a cost $b(x) \in \mathbb{R}_0^+$. A cost constraint on a codebook for this channel is

\[ \frac{1}{n} \sum_{i=1}^{n} b(x_i(m)) \leq \beta, \quad m \in \{1, \ldots, 2^n R\}, \tag{1} \]

where $\beta$ is some positive constant. A rate $R$ is said to be achievable on this channel under cost constraint $\beta$ if there exists a sequence of $(n, R)$ codebooks satisfying (1) whose maximum error probability tends to zero as $n$ tends to infinity. The capacity under cost constraint $\beta$, denoted by $C(\beta)$, is defined as the supremum over all rates that are achievable under cost constraint $\beta$. Show that

\[ C(\beta) = \sup_{E[b(X)] \leq \beta} I(X; Y). \tag{2} \]

Further, show that (2) still holds if we replace (1) by the weaker condition

\[ \frac{1}{2^n R} \sum_{m=1}^{2^n R} \frac{1}{n} \sum_{i=1}^{n} b(x_i(m)) \leq \beta. \]