Problem 1  \( \text{Parallel Channels and Waterfilling} \)

Consider a pair of parallel Gaussian channels, i.e.,
\[
\begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix} =
\begin{pmatrix}
X_1 \\
X_2
\end{pmatrix} +
\begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix},
\]
where
\[
\begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix} \sim \mathcal{N}
\left(0,
\begin{pmatrix}
\sigma_1^2 & 0 \\
0 & \sigma_2^2
\end{pmatrix}
\right)
\]
and where we have a power constraint \( \mathbb{E}[X_1^2 + X_2^2] \leq P \). Assume \( \sigma_1^2 > \sigma_2^2 \). At what power does the channel stop behaving like a single channel with noise variance \( \sigma_2^2 \), and begin behaving like a pair of channels?

Problem 2  \( \text{Multipath Gaussian Channel} \)

Consider a channel with input power constraint \( P \) where the signal takes two different paths and the two noisy signals are added together at the receive antenna. More explicitly, the output is given by
\[
Y = Y_1 + Y_2,
\]
where
\[
Y_1 = X + Z_1, \\
Y_2 = X + Z_2.
\]

a) Find the capacity of this channel if \( Z_1 \) and \( Z_2 \) are jointly Gaussian with covariance matrix
\[
K_Z =
\begin{pmatrix}
\sigma_1^2 & \rho \sigma_1 \sigma_2 \\
\rho \sigma_1 \sigma_2 & \sigma_2^2
\end{pmatrix}
\]

b) What is the capacity for \( \rho = 0 \), \( \rho = 1 \), and \( \rho = -1 \)?

Problem 3  \( \text{Bandlimited Gaussian Channel} \)

Consider the bandlimited Gaussian channel with noise power spectral density \( N_0/2 \) and power \( P \). The capacity of this channel is
\[
W \log \left( 1 + \frac{P}{N_0 W} \right) \text{ bits per second},
\]
where \( W \) is the bandwidth of the channel. What would you rather have, twice the bandwidth or twice the power?