Problem 1

**Expectation of a Chance Variable**

In Information Theory one is frequently required to compute the expectation of \( g(X) \) where \( X \) is a chance variable taking values in \( \mathcal{X} \) and \( g: \mathcal{X} \to \mathbb{R} \cup \{-\infty, \infty\} \) is a function whose domain is \( \mathcal{X} \) and which takes values on the extended real line \( \mathbb{R} \cup \{-\infty, \infty\} \). In this case, to avoid terms of the form \( 0 \cdot \infty \), the expectation of \( g(X) \), rather than being defined as

\[
E[g(X)] = \sum_{x \in \mathcal{X}} P_X(x) g(x),
\]

as for ordinary real-valued functions, must be defined as

\[
E[g(X)] = \sum_{x \in \text{supp}(P_X)} P_X(x) g(x),
\]

where \( \text{supp}(P_X) \) denotes the support of \( P_X \) defined as

\[
\text{supp}(P_X) \triangleq \{ x \in \mathcal{X} : P_X(x) \neq 0 \}.
\]

The right-hand-side of (2) is taken to be \(+\infty\) if \( \Pr[g(X) = +\infty] > 0 \) and \( \Pr[g(X) = -\infty] = 0 \). It is understood to be \(-\infty\) if \( \Pr[g(X) = -\infty] > 0 \) and \( \Pr[g(X) = +\infty] = 0 \), and it is undefined if \( \Pr[g(X) = +\infty] > 0 \) and \( \Pr[g(X) = -\infty] > 0 \).

a) Let \( |\mathcal{X}| = L \) and \( |\text{supp}(P_X)| = L' \) where \( L' \leq L \). Here \( |\mathcal{A}| \) denotes the cardinality of the set \( \mathcal{A} \), i.e., the number of elements in \( \mathcal{A} \).

Find \( E\left[\frac{1}{P_X(X)}\right] \).

**Hint:** What is \( g(x) \)?

b) Suppose that \( X \) and \( X' \) are chance variables taking value in \( \mathcal{X} \). Write out the general expression for \( E[P_X(X)] \) and \( E[P_{X'}(X')] \).

c) Write out the general expression for \( E[-\log P_X(X)] \) and \( E[-\log P_{X'}(X')] \).

**Note:** We define \( a + \infty = \infty \) whenever \( a \neq -\infty \) and \( a \cdot \infty = \text{sgn}(a) \cdot \infty \) whenever \( a \neq 0 \).

Problem 2

**Statistical Independence**

Let \( X_1, X_2, \ldots, X_n \) be a sequence of binary, independent and identically distributed (IID) random variables. Assume \( n > 1 \), and assume

\[
\Pr[X_i = 1] = \Pr[X_i = 0] = \frac{1}{2}, \quad i = 1, \ldots, n.
\]

Let \( Z \) be a parity check on \( X_1, \ldots, X_n \), i.e., \( Z = X_1 \oplus X_2 \oplus \cdots \oplus X_n \) where \( 0 \oplus 0 = 1 \oplus 1 = 0 \) and \( 0 \oplus 1 = 1 \oplus 0 = 1 \).
a) Is $Z$ statistically independent of $X_1$?

b) Are $Z, X_1, \ldots, X_{n-1}$ statistically independent?

c) Are $Z, X_1, \ldots, X_n$ statistically independent?

d) Is $Z$ statistically independent of $X_1$ if $\Pr[X_i = 1] = p \neq \frac{1}{2}$ for all $i$? You may take $n = 2$ here.

**Problem 3**

**On the Expectation of a Discrete Random Variable**

Let the random variable $T$ take on only positive integer values. Show that

$$
E[T] = \sum_{v=1}^{\infty} \Pr[T \geq v].
$$

**Problem 4**

**Markov’s Inequality and Chebyshev’s Inequality**

a) *(Markov Inequality)* For any nonnegative random variable $X$ and any $\delta > 0$, show that

$$
\Pr[X \geq \delta] \leq \frac{E[X]}{\delta}.
$$

Exhibit a random variable $X$ and some $\delta$ that achieves this inequality with equality.

*Hint: In the definition of expectation split the sum/integration into two parts according to whether $x \geq \delta$ or $x < \delta$.*

b) *(Chebyshev’s Inequality)* Let $Y$ be a random variable with mean $\mu$ and variance $\sigma^2$. By letting $X = (Y - \mu)^2$, show that for any $\epsilon > 0$,

$$
\Pr[|Y - \mu| \geq \epsilon] \leq \frac{\sigma^2}{\epsilon^2}.
$$

*Hint: Use a).*

c) *(The Weak Law of Large Numbers)* Let $Z_1, Z_2, \ldots, Z_n$ be a sequence of IID random variables with mean $\mu$ and variance $\sigma^2$. Let $\bar{Z}_n = \frac{1}{n} \sum_{k=1}^{n} Z_k$ be the sample mean. Show that

$$
\Pr[|\bar{Z}_n - \mu| \geq \epsilon] \leq \frac{\sigma^2}{n\epsilon^2}.
$$

Thus, $\Pr[|\bar{Z}_n - \mu| > \epsilon] \to 0$ as $n \to \infty$. This is known as the weak law of large numbers.

*Hint: Use b).*

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