Problem 1

**Huffman Coding**

Consider the chance variable $X$ that takes on the values $(x_1, x_2, \ldots, x_7)$ with respective probabilities $(0.49, 0.26, 0.12, 0.04, 0.04, 0.03, 0.02)$.

a) Find a binary Huffman code for $X$.

b) Find the expected code length for this encoding.

c) Find a ternary Huffman code for $X$.

Problem 2

**Bad Codes**

Which of these codes cannot be Huffman codes for any probability assignment?

a) \{0, 10, 11\}.

b) \{00, 01, 10, 110\}.

c) \{01, 10\}.

Problem 3

**Optimal Codeword Lengths**

Although the codeword lengths of an optimal variable length code are complicated functions of the message probabilities \(\{p_1, p_2, \ldots, p_m\}\), it can be said that less probable symbols are encoded into longer codewords. Suppose that the message probabilities are given in decreasing order \(p_1 > p_2 \geq \cdots \geq p_m\).

a) Prove that for any binary Huffman code, if the most probable message symbol has probability \(p_1 > \frac{1}{2}\), then that symbol must be assigned a codeword of length 1.

b) Prove that for any binary Huffman code, if the most probable message symbol has probability \(p_1 < \frac{1}{4}\), then that symbol must be assigned a codeword of length at least 2.
Problem 4

Let \( X_1, X_2, \ldots \) be drawn IID according to the probability mass function \( p(x), x \in \{1, 2, \ldots, m\} \). Thus, \( p(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n p(x_i) \). We know that
\[
-\frac{1}{n} \log p(X_1, X_2, \ldots, X_n) \to H(X)
\]
in probability. Let \( q(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n q(x_i) \), where \( q \) is another probability mass function on \( \{1, 2, \ldots, m\} \).

(a) Evaluate \( \lim_{n \to \infty} \left( -\frac{1}{n} \log q(X_1, X_2, \ldots, X_n) \right) \), where \( X_1, X_2, \ldots \) are IID \( \sim p(x) \), and express it as a function of relative entropies and entropies.

(b) Now evaluate the limit of the log likelihood ratio \( \frac{1}{n} \log \frac{p(X_1, X_2, \ldots, X_n)}{q(X_1, X_2, \ldots, X_n)} \) when \( X_1, X_2, \ldots \) are IID \( \sim p(x) \).

Problem 5

\textit{Proof of Theorem 3.3.1 in Cover & Thomas: High-Probability Sets and the Typical Set}

Let \( X_1, \ldots, X_n \) be a random sequence chosen IID \( \sim P_X(\cdot) \). Fix some \( \epsilon < \frac{1}{2} \), let \( A^{(n)}_\epsilon \) denote the set of weakly typical sequences, and let \( B^{(n)}_\delta \) be an arbitrary set of length-\( n \) sequences such that
\[
\Pr(B^{(n)}_\delta) > 1 - \delta.
\]

(a) For some \( 0 < \epsilon_1, \epsilon_2 < \frac{1}{2} \) and given any two sets \( A, B \) such that \( \Pr(A) > 1 - \epsilon_1 \) and \( \Pr(B) > 1 - \epsilon_2 \) show that
\[
\Pr(A \cap B) > 1 - \epsilon_1 - \epsilon_2.
\]
Show that from this then follows that
\[
\Pr(A^{(n)}_\epsilon \cap B^{(n)}_\delta) > 1 - \epsilon - \delta.
\]

(b) Justify each step in the chain of equalities/inequalities below:
\[
1 - \epsilon - \delta < \Pr(A^{(n)}_\epsilon \cap B^{(n)}_\delta)
= \sum_{x \in A^{(n)}_\epsilon \cap B^{(n)}_\delta} P_X(x)
\leq \sum_{x \in A^{(n)}_\epsilon \cap B^{(n)}_\delta} 2^{-n(H(X) - \epsilon)}
= |A^{(n)}_\epsilon \cap B^{(n)}_\delta| \cdot 2^{-n(H(X) - \epsilon)}
\leq |B^{(n)}_\delta| \cdot 2^{-n(H(X) - \epsilon)}.
\]

(c) Complete the proof of Theorem 3.3.1 in Cover & Thomas (p. 63), i.e., show that for \( 0 < \delta < \frac{1}{2} \) and \( \delta' > 0 \)
\[
\frac{1}{n} \log |B^{(n)}_\delta| > H(X) - \delta'
\]
for \( n \) sufficiently large.