Problem 1  

**On the Statistical Mean Value**

Let $X_1, X_2, \ldots, X_n$ be IID random variables with $\Pr[X_k = 0] = \Pr[X_k = 1] = 1/2$, and let

$$\hat{P}_n = \frac{1}{n} \sum_{k=1}^{n} I\{X_k = 1\},$$

where $I\{\cdot\}$ is the indicator function:

$$I\{\text{statement}\} = \begin{cases} 
1 & \text{if statement is true}, \\
0 & \text{if statement is false}. 
\end{cases}$$

Hence, $\hat{P}_n$ is the ratio of the number of 1s in the sequence to the length of the sequence.

a) Compute $\lim_{n \to \infty} \Pr[\hat{P}_n = \frac{1}{2}]$.  

*Hint: You may find Stirling’s approximation (i.e., $n! \approx e^{-n} n^n \sqrt{2\pi n}$) useful.*

b) Compute $\lim_{n \to \infty} \Pr[0.4999 \leq \hat{P}_n \leq 0.5001]$.  

*Hint: This part requires no calculation.*

Problem 2  

**One Bit Quantization of a Single Gaussian Random Variable**

Let $X$ be a Gaussian random variable with zero mean and variance $\sigma^2$, i.e., $X \sim \mathcal{N}(0, \sigma^2)$, and let the distortion measure be squared error, i.e.,

$$d(x, \hat{x}) = (x - \hat{x})^2.$$

We do not allow block description. Show that the optimum reproduction points for 1 bit quantization are $\pm \sqrt{\frac{2}{\pi}} \sigma$, and that the expected distortion for 1 bit quantization is $\frac{\pi - 2}{\pi} \sigma^2$.

(You can assume that the optimum reproduction points are $\pm a$ for some $a > 0$, and that it is optimal to map $x$ to $a$ if it is positive, and to map it to $-a$ otherwise.)