Model Answers to Exercise 12 of December 2, 2015

Problem 1  
*On the Statistical Mean Value*

a) Let $p_n$ denote the following function of the sequence $x \in \mathcal{X}^n$

$$p_n(x) = \frac{1}{n} \sum_{k=1}^{n} I\{x_k = 1\}.$$  

Then, assuming that $n$ is an even number, we make the following derivation:

$$\Pr\left[\hat{P}_n = \frac{1}{2}\right] = \sum_{x: p_n(x) = 1/2} \Pr[X = x]$$

$$= \left|\left\{x: p_n(x) = \frac{1}{2}\right\}\right| \cdot \left(\frac{1}{2}\right)^n$$

$$= 2^{-n} \cdot \frac{n}{n!}$$

$$= 2^{-n} \cdot \frac{n!}{(n/2)!^2}$$

$$\approx 2^{-n} \cdot \frac{(\frac{n}{2})^n \sqrt{2\pi n}}{\left(\frac{(n/2)^{n/2} \sqrt{2\pi n/2}}{2}\right)^n}$$

$$= 2^{-n} \cdot \frac{(\frac{n}{2})^n \sqrt{2\pi n}}{(\frac{n}{2})^n \pi n}$$

$$= \sqrt{\frac{2}{\pi n}}$$

$$\to 0 \quad \text{as} \ n \to \infty,$$

where $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$. In (1) we have used Stirling’s approximation. Therefore

$$\lim_{n \to \infty} \Pr\left[\hat{P}_n = \frac{1}{2}\right] = 0.$$

b) The weak law of large numbers implies that

$$\hat{P}_n = \frac{1}{n} \sum_{k=1}^{n} I\{X_k = 1\} \quad \overset{n \to \infty}{\to} \quad \mathbb{E}[X] = \frac{1}{2} \quad \text{in probability.}$$

Therefore,

$$\lim_{n \to \infty} \Pr\left[0.4999 \leq \hat{P}_n \leq 0.5001\right] = 1.$$
Problem 2  

One Bit Quantization of a Single Gaussian Random Variable

We first prove that the optimum representation points are \( \pm a \) for some \( a > 0 \), and that it is optimal to map \( x \) to \( a \) if it is positive, and to map it to \( -a \) otherwise.

Having a one bit quantization means that we have two reconstruction points: one at \( \alpha \) and one at \( \beta \). Since the distortion measure is the quadratic error to the reconstruction point, the distortion is minimized by mapping \( x \) to the closest reconstruction point, i.e., for any constellation of reconstruction points \( \alpha, \beta \) the optimal corresponding regions are always of the form \((-\infty, (\alpha + \beta)/2] \) and \(((\alpha + \beta)/2, \infty) \).

We shall now compute the expected distortion as a function of the boundary point \( c = (\alpha + \beta)/2 \) and then show that the minimal distortion is achieved for \( c = 0 \). We assume \( c \geq 0 \), which incurs no loss in generality since the problem is symmetric with respect to the origin. The optimal reconstruction points for a given \( c \) are \( \alpha = E[X|X < c] \) and \( \beta = E[X|X \geq c] \).

This follows since the expected squared error of a random variable \( X \) to a fixed point is minimized if the fixed point is the expected value of \( X \). This can be seen by noting that for every \( \xi \in \mathbb{R} \)

\[
E \left[ (X - (E[X] + \xi))^2 \right] = E[X^2] - E[X]^2 + \xi^2
\]

is minimized by \( \xi = 0 \). Hence, the reconstruction points are

\[
E[X|X < c] = \int_{-\infty}^{c} x \frac{f_X(x)}{Pr[X < c]} \, dx
= \frac{1}{Pr[X < c]} \left( \int_{-\infty}^{0} x f_X(x) \, dx + \int_{0}^{c} x f_X(x) \, dx \right)
= \frac{1}{Pr[X < c]} \left( - \int_{0}^{\infty} x f_X(x) \, dx + \int_{0}^{c} x f_X(x) \, dx \right)
= \frac{1}{Pr[X < c]} \left( - \int_{\infty}^{c} x f_X(x) \, dx \right).
\]

and

\[
E[X|X \geq c] = \int_{c}^{\infty} x \frac{f_X(x)}{Pr[X \geq c]} \, dx
= \frac{1}{Pr[X \geq c]} \int_{c}^{\infty} x f_X(x) \, dx.
\]

We thus have

\[
\beta = -\alpha \cdot \frac{Pr[X < c]}{Pr[X \geq c]}.
\]

The expected distortion becomes

\[
E[d(X, \hat{X})] = \int_{-\infty}^{c} (x - \alpha)^2 f_X(x) \, dx + \int_{c}^{\infty} (x - \beta)^2 f_X(x) \, dx
= \int_{-\infty}^{\infty} x^2 f_X(x) \, dx - 2\alpha \int_{-\infty}^{c} x f_X(x) \, dx + \alpha^2 \int_{-\infty}^{c} f_X(x) \, dx
\]

\[
- 2\beta \int_{c}^{\infty} x f_X(x) \, dx + \beta^2 \int_{c}^{\infty} f_X(x) \, dx
= \sigma^2 - \alpha^2 Pr[X < c] - \beta^2 Pr[X \geq c]
\]

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\[= \sigma^2 - \alpha^2 \Pr[X < c] - \frac{\alpha^2 \Pr[X < c]^2}{\Pr[X \geq c]} \]
\[= \sigma^2 - \frac{1}{\Pr[X < c]} \left( \int_c^\infty x f_X(x) \, dx \right)^2 - \frac{1}{\Pr[X \geq c]} \left( \int_c^\infty x f_X(x) \, dx \right)^2 \]
\[= \sigma^2 - \left( \frac{1}{\Pr[X < c]} + \frac{1}{\Pr[X \geq c]} \right) \left( \int_c^\infty x f_X(x) \, dx \right)^2 . \]

A plot of this expression shows that its minimum is achieved for \( c = 0 \). Hence, the distortion is minimized by the symmetric repartition \((-\infty, 0], (0, \infty)\) with corresponding reconstruction points at \(-\beta\) and \(+\beta\). It remains to explicitly compute the value of \( \beta \) and the expected distortion. Both contain the expression
\[\int_0^\infty x f_X(x) \, dx = \int_0^\infty x \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \, dx = \frac{1}{\sqrt{2\pi \sigma^2}} \int_0^\infty x \, e^{-\frac{x^2}{2\sigma^2}} \, dx = \frac{1}{2\sqrt{2\pi \sigma^2}} \int_0^\infty e^{-\frac{x^2}{2\sigma^2}} \, du = \sqrt{\frac{1}{2\pi \sigma}}, \]
where the third equality follows from the substitution \( u = x^2 \). We obtain
\[\beta = \frac{1}{\Pr[X \geq 0]} \int_0^\infty x f_X(x) \, dx = 2 \sqrt{\frac{1}{2\pi \sigma}} = \sqrt{\frac{2}{\pi \sigma}} = -\alpha, \]
and
\[E[d(X, \hat{X})] = \sigma^2 - \left( \frac{1}{\Pr[X < 0]} + \frac{1}{\Pr[X \geq 0]} \right) \left( \int_0^\infty x f_X(x) \, dx \right)^2 \]
\[= \sigma^2 - \frac{4}{2\pi} \frac{1}{\sigma^2} = \frac{\pi}{2\pi} = \frac{\pi - 2}{\pi} \sigma^2 . \]