Information Theory I

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http://www.isi.ee.ethz.ch/teaching/courses/it1/

Problem 1 — *Example of Joint Entropy*

Let $P_{X,Y}(x,y)$ be given by

$P_{X,Y}(x,y)$	$y=0$	$y=1$
$x=0$	$1/3$	$1/3$
$x=1$	0	$1/3$

Find

a) $H(X)$, $H(Y)$,

b) $H(X|Y)$, $H(Y|X)$,

c) $H(X,Y)$,

d) $H(Y) - H(Y|X)$,

e) $I(X;Y)$.

Problem 2 — *Zero Conditional Entropy*

Show that $H(Y|X) = 0$ if, and only if, Y is a function[1] of X.

Problem 3 — *Entropy of Functions of a Chance Variable*

Let X be a discrete chance variable. Show that the entropy of a function of X is less than or equal to the entropy of X by justifying the following steps:

$$H(X,g(X)) \overset{\text{(a)}}{=} H(X) + H(g(X)|X) \overset{\text{(b)}}{=} H(X);$$
$$H(X,g(X)) \overset{\text{(c)}}{=} H(g(X)) + H(X|g(X)) \overset{\text{(d)}}{\geq} H(g(X)).$$

Thus, $H(g(X)) \leq H(X)$. When does equality hold?

[1] More precisely, $\Pr[Y = g(X)] = 1$ for some function $g(\cdot)$.

Problem 4 ***Entropy of a Sum***

Let X and Y be random variables that take values in $\{x_1, x_2, \ldots, x_r\} \subset \mathbb{R}$ and in $\{y_1, y_2, \ldots, y_s\} \subset \mathbb{R}$, respectively. Let $Z = X + Y$.

a) Show that $H(Z|X) = H(Y|X)$. Argue that if X and Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of *independent* random variables adds uncertainty.

b) Give an example (of necessarily dependent random variables) in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

c) Under what conditions does $H(Z) = H(X) + H(Y)$ hold?

   Hint: You may find the conclusion of Problem 3 helpful.

Problem 5 ***Jensen's Inequality***

Let $a_1, \ldots, a_n$ be positive real numbers.

a) Use Jensen's inequality to show that the arithmetic mean is greater than or equal to the geometric mean, i.e.,

$$\left(\prod_{k=1}^{n} a_k\right)^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n} a_k.$$

   Also find necessary and sufficient conditions for equality.

b) Use Jensen's inequality to show that

$$\frac{1}{n}\sum_{k=1}^{n} a_k^{\beta} \geq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{\beta}$$

   for any $\beta \geq 1$. What happens if $0 < \beta < 1$?

c) Suppose that you have written n exams with scores a_k that are numbers between 1 and 6. Would you rather have $\frac{1}{n}\sum_{k=1}^{n} a_k$ or $\sqrt{\frac{1}{n}\sum_{k=1}^{n} a_k^2}$ as your final score?

Problem 6 ***Csiszár's Identity***

Show that for any pair of random vectors $(A_1, \ldots, A_n)$, $(B_1, \ldots, B_n)$

$$\sum_{i=1}^{n}\Big(I\big(A_{i+1}^{n}; B_i \big| B^{i-1}\big) - I\big(B^{i-1}; A_i \big| A_{i+1}^{n}\big)\Big) = 0,$$

where we use the shorthand notation $A^j = (A_1, \ldots, A_j)$, $A_j^k = (A_j, A_{j+1}, \ldots, A_k)$ for $j \leq k$, and B^0 and A_{n+1}^n are defined to be deterministic.