Exercise 2 of September 28, 2016

Problem 1

Let \( P_{X,Y}(x,y) \) be given by

\[
\begin{array}{c|cc}
  & y = 0 & y = 1 \\
\hline
  x = 0 & 1/3 & 1/3 \\
  x = 1 & 0 & 1/3 \\
\end{array}
\]

Find

a) \( H(X) \), \( H(Y) \),

b) \( H(X|Y) \), \( H(Y|X) \),

c) \( H(X,Y) \),

d) \( H(Y) - H(Y|X) \),

e) \( I(X;Y) \).

Problem 2

Zero Conditional Entropy

Show that \( H(Y|X) = 0 \) if, and only if, \( Y \) is a function\(^1\) of \( X \).

Problem 3

Entropy of Functions of a Chance Variable

Let \( X \) be a discrete chance variable. Show that the entropy of a function of \( X \) is less than or equal to the entropy of \( X \) by justifying the following steps:

\[
\begin{align*}
H(X, g(X)) & \overset{(a)}{=} H(X) + H(g(X)|X) \overset{(b)}{=} H(X); \\
H(X, g(X)) & \overset{(c)}{=} H(g(X)) + H(X|g(X)) \overset{(d)}{\geq} H(g(X)).
\end{align*}
\]

Thus, \( H(g(X)) \leq H(X) \). When does equality hold?

\(^1\)More precisely, \( \Pr[Y = g(X)] = 1 \) for some function \( g(\cdot) \).
Problem 4

**Entropy of a Sum**

Let $X$ and $Y$ be random variables that take values in $\{x_1, x_2, \ldots, x_r\} \subset \mathbb{R}$ and in $\{y_1, y_2, \ldots, y_s\} \subset \mathbb{R}$, respectively. Let $Z = X + Y$.

a) Show that $H(Z|X) = H(Y|X)$. Argue that if $X$ and $Y$ are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of independent random variables adds uncertainty.

b) Give an example (of necessarily dependent random variables) in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.

c) Under what conditions does $H(Z) = H(X) + H(Y)$ hold?

*Hint: You may find the conclusion of Problem 3 helpful.*

Problem 5

**Jensen’s Inequality**

Let $a_1, \ldots, a_n$ be positive real numbers.

a) Use Jensen’s inequality to show that the arithmetic mean is greater than or equal to the geometric mean, i.e.,

$$\left( \prod_{k=1}^{n} a_k \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} a_k.$$

Also find necessary and sufficient conditions for equality.

b) Use Jensen’s inequality to show that

$$\frac{1}{n} \sum_{k=1}^{n} a_k^{\beta} \geq \left( \frac{1}{n} \sum_{k=1}^{n} a_k \right)^{\beta}$$

for any $\beta \geq 1$. What happens if $0 < \beta < 1$?

c) Suppose that you have written $n$ exams with scores $a_k$ that are numbers between 1 and 6. Would you rather have $\frac{1}{n} \sum_{k=1}^{n} a_k$ or $\sqrt{\frac{1}{n} \sum_{k=1}^{n} a_k^2}$ as your final score?

Problem 6

**Csiszár’s Identity**

Show that for any pair of random vectors $(A_1, \ldots, A_n), (B_1, \ldots, B_n)$

$$\sum_{i=1}^{n} \left( I(A_{i+1}^n; B_i|B^{i-1}) - I(B^{i-1}; A_i|A_{i+1}^n) \right) = 0,$$

where we use the shorthand notation $A^j = (A_1, \ldots, A_j)$, $A^k_j = (A_j, A_{j+1}, \ldots, A_k)$ for $j \leq k$, and $B^0$ and $A_{n+1}^n$ are defined to be deterministic.

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