



Exercise 5 of October 19, 2016

<http://www.isi.ee.ethz.ch/teaching/courses/it1/>

Problem 1

Huffman Coding

Consider the chance variable X that takes on the values (x_1, x_2, \dots, x_7) with respective probabilities $(0.49, 0.26, 0.12, 0.04, 0.04, 0.03, 0.02)$.

- Find a binary Huffman code for X .
- Find the expected code length for this encoding.
- Find a ternary Huffman code for X .

Problem 2

Bad Codes

Which of these codes cannot be Huffman codes for any probability assignment?

- $\{0, 10, 11\}$.
- $\{00, 01, 10, 110\}$.
- $\{01, 10\}$.

Problem 3

Optimal Codeword Lengths

Although the codeword lengths of an optimal variable length code are complicated functions of the message probabilities $\{p_1, p_2, \dots, p_m\}$, it can be said that less probable symbols are encoded into longer codewords. Suppose that the message probabilities are given in decreasing order $p_1 > p_2 \geq \dots \geq p_m$.

- Prove that for any binary Huffman code, if the most probable message symbol has probability $p_1 > \frac{2}{5}$, then that symbol must be assigned a codeword of length 1.
- Prove that for any binary Huffman code, if the most probable message symbol has probability $p_1 < \frac{1}{3}$, then that symbol must be assigned a codeword of length at least 2.

Problem 4**AEP**

Let X_1, X_2, \dots be drawn IID according to the probability mass function $p(x)$, $x \in \{1, 2, \dots, m\}$. Thus, $p(x_1, x_2, \dots, x_n) = \prod_{i=1}^n p(x_i)$. We know that

$$-\frac{1}{n} \log p(X_1, X_2, \dots, X_n) \rightarrow H(X)$$

in probability. Let $q(x_1, x_2, \dots, x_n) = \prod_{i=1}^n q(x_i)$, where q is another probability mass function on $\{1, 2, \dots, m\}$.

- a) Evaluate $\lim_{n \rightarrow \infty} \left(-\frac{1}{n} \log q(X_1, X_2, \dots, X_n)\right)$, where X_1, X_2, \dots are IID $\sim p(x)$, and express it as a function of relative entropies and entropies.
- b) Now evaluate the limit of the log likelihood ratio $\frac{1}{n} \log \frac{p(X_1, X_2, \dots, X_n)}{q(X_1, X_2, \dots, X_n)}$ when X_1, X_2, \dots are IID $\sim p(x)$.

Problem 5

Proof of Theorem 3.3.1 in Cover & Thomas: High-Probability Sets and the Typical Set

Let X_1, \dots, X_n be a random sequence chosen IID $\sim P_X$. Let $\mathcal{A}_\epsilon^{(n)}$ denote the set of weakly typical sequences for some fixed $\epsilon > 0$, and let $\mathcal{B}_\delta^{(n)}$ be an arbitrary set of length- n sequences such that

$$\Pr(\mathcal{B}_\delta^{(n)}) \geq 1 - \delta.$$

- a) Given any two sets \mathcal{A}, \mathcal{B} such that $\Pr(\mathcal{A}) \geq 1 - \epsilon_1$ and $\Pr(\mathcal{B}) \geq 1 - \epsilon_2$, show that

$$\Pr(\mathcal{A} \cap \mathcal{B}) \geq 1 - \epsilon_1 - \epsilon_2.$$

For n sufficiently large, show that this implies

$$\Pr(\mathcal{A}_\epsilon^{(n)} \cap \mathcal{B}_\delta^{(n)}) \geq 1 - \epsilon - \delta.$$

- b) For n sufficiently large, justify the following steps:

$$\begin{aligned} 1 - \epsilon - \delta &\stackrel{(i)}{\leq} \Pr(\mathcal{A}_\epsilon^{(n)} \cap \mathcal{B}_\delta^{(n)}) \\ &\stackrel{(ii)}{=} \sum_{\mathbf{x} \in \mathcal{A}_\epsilon^{(n)} \cap \mathcal{B}_\delta^{(n)}} P_{\mathbf{X}}(\mathbf{x}) \\ &\stackrel{(iii)}{\leq} \sum_{\mathbf{x} \in \mathcal{A}_\epsilon^{(n)} \cap \mathcal{B}_\delta^{(n)}} 2^{-n(H(X) - \epsilon)} \\ &\stackrel{(iv)}{=} |\mathcal{A}_\epsilon^{(n)} \cap \mathcal{B}_\delta^{(n)}| \cdot 2^{-n(H(X) - \epsilon)} \\ &\stackrel{(v)}{\leq} |\mathcal{B}_\delta^{(n)}| \cdot 2^{-n(H(X) - \epsilon)}. \end{aligned}$$

- c) Show that for any $\delta \in (0, 1)$, any $\delta' > 0$, and n sufficiently large,

$$\frac{1}{n} \log |\mathcal{B}_\delta^{(n)}| \geq H(X) - \delta'.$$