Information Theory I

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http://www.isi.ee.ethz.ch/teaching/courses/it1/

Problem 1 — *Strong Versus Weak Typicality*

A sequence of chance variables $X_1, \ldots, X_{100}$ is drawn IID with X_i taking on the values "True" and "False" equiprobably. Describe the set of strongly typical sequences and the set of weakly typical sequences for $\epsilon = 0.01$.

Problem 2 — *Random Box Size*

Let $X_1, X_2, \ldots, X_n$ be IID uniform random variables over the unit interval $[0,1]$. An n-dimensional rectangular box with side lengths $X_1, X_2, \ldots, X_n$ is constructed. Its volume is $V_n = \prod_{i=1}^n X_i$, and the edge length L_n of an n-cube with the same volume is $L_n = {V_n}^{1/n}$.

a) Compute $\lim_{n\to\infty} (\mathsf{E}[V_n])^{1/n}$.

b) Show that $\lim_{n\to\infty} \Pr\left[\left|\frac{1}{n}\sum_{i=1}^n \ln X_i - (-1)\right| \geq \epsilon_1\right] = 0$ for all $\epsilon_1 > 0$.

c) Deduce from Part b) that $\lim_{n\to\infty} \Pr\left[\left|L_n - e^{-1}\right| \geq \epsilon_2\right] = 0$ for all $\epsilon_2 > 0$.

Problem 3 — *From AEP to Kraft's Inequality*

Let $\ell_1, \ldots, \ell_d$ be the codeword lengths of a uniquely decodable one-to-variable code. Use Theorem 1 to show that

$$\sum_{i=1}^d 2^{-\ell_i} \leq 1. \tag{1}$$

Theorem 1 (Converse for the Source Coding Theorem). *Consider a source with entropy $H(X)$ satisfying the AEP and a sequence of codes which map n source symbols to $n\rho_n$ bits. If*

$$\limsup_{n\to\infty} \rho_n < H(X), \tag{2}$$

the probability of successful decoding for these codes tends to zero as n tends to infinity.

Hint: Prove the claim by contradiction. Assume that there exists a uniquely decodable code $\mathscr{C}$ whose codeword lengths do not satisfy (1). Construct a source for which the expected codeword length of $\mathscr{C}$ is smaller than the entropy of the source. Use the extension of $\mathscr{C}$ to devise a coding scheme that satisfies (2) and whose probability of successful decoding tends to one as n tends to infinity, which leads to the desired contradiction.

Problem 4 ***On the Achievable Rate***

Consider a discrete memoryless channel of law $W_1(y|x)$ over the finite input and output alphabets $\mathcal{X}$ and $\mathcal{Y}$. Let a second channel be defined over the input alphabet $\mathcal{X} \times \mathcal{X}$ and output alphabet $\mathcal{Y} \times \mathcal{Y}$ and be of law

$$W_2(y_1, y_2|x_1, x_2) = W_1(y_1|x_1)W_1(y_2|x_2).$$

Based on the definition of achievable rate, show that

a) if R is achievable on the first channel, then $2\mathsf{R}$ is achievable on the second channel;

b) if R is achievable on the second channel, then $\mathsf{R}/2$ is achievable on the first channel.