Information Theory I

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Prof. Dr. A. Lapidoth

Signal and Information Processing Laboratory
Institut für Signal- und Informationsverarbeitung

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http://www.isi.ee.ethz.ch/teaching/courses/it1/

Problem 1 *Properties of* $\mathsf{R}(\mathsf{D})$

Consider a discrete source $X \in \mathcal{X}$ with distribution $Q(\cdot)$ and a distortion measure $d(x, \hat{x})$. Let $\mathsf{R}(\mathsf{D})$ be the rate distortion function for this source and distortion measure. Let $d'(x, \hat{x}) = d(x, \hat{x}) - w(x)$ be a new distortion measure (i.e., $w(\cdot)$ is such that $d'(x, \hat{x}) \geq 0$ for all $x, \hat{x}$), and let $\mathsf{R}'(\mathsf{D})$ be the corresponding rate distortion function. Show that

$$\mathsf{R}'(\mathsf{D}) = \mathsf{R}(\mathsf{D} + \bar{w}),$$

where $\bar{w} = \sum_x Q(x)w(x)$. Use this to show that there is no essential loss of generality in assuming that

$$\min_{\hat{x}} d(x, \hat{x}) = 0,$$

i.e., that for each $x \in \mathcal{X}$ there is (at least) one symbol $\hat{x}$ that reproduces the source with zero distortion. (This result is due to Pinkston.)

Problem 2 *Erasure Distortion*

Consider $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$. Let $\hat{X}$ take value in $\hat{\mathcal{X}} = \{0, 1, ?\}$, and let the distortion measure be given by

$$d(x, \hat{x}) \triangleq \begin{cases} 0 & \text{if } x = \hat{x}, \\ 1 & \text{if } \hat{x} = ?, x \in \{0, 1\}, \\ \infty & \text{if } \hat{x} = 0, x = 1 \text{ or } \hat{x} = 1, x = 0, \end{cases}$$

so the distortion is 1 for an "erasure" and ∞ for an "error".

a) Calculate the rate distortion function for this source and plot the rate distortion region.

b) Can you suggest a simple scheme to achieve any value of the rate distortion function for this source?

Problem 3 *Rate Distortion Function with Infinite Distortion*

Find the rate distortion function

$$\mathsf{R}(\mathsf{D}) = \min_{q(\hat{x}|x):\ \mathsf{E}[d(X,\hat{X})]\le \mathsf{D}} I(X;\hat{X})$$

for $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ and distortion

$$d(x,\hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x}, \\ 1 & \text{if } x = 1,\ \hat{x} = 0, \\ \infty & \text{if } x = 0,\ \hat{x} = 1. \end{cases}$$

Problem 4 *Rate Distortion for Uniform Source with Hamming Distortion*

Consider a source X that is uniformly distributed over $\{1, 2, \ldots, m\}$ and the Hamming distortion

$$d(x,\hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x}, \\ 1 & \text{if } x \neq \hat{x}. \end{cases}$$

Find the rate distortion function.