Exercise 12 of December 7, 2016

http://www.isi.ee.ethz.ch/teaching/courses/it1/

Problem 1 ***Rate Distortion with Two Distortion Functions***

a) Consider a source sequence X^n drawn IID according to the distribution $P_X(\cdot)$. For $x \in \mathcal{X}$ and $\hat{x} \in \hat{\mathcal{X}}$, consider two distortion functions $d_1(x, \hat{x})$ and $d_2(x, \hat{x})$. We define a triple $(\mathsf{R}, \mathsf{D}_1, \mathsf{D}_2)$ to be achievable if for every $\epsilon > 0$ there exists some blocklength n_0 such that for every $n > n_0$ there exist an encoder $f^{(n)} \colon \mathcal{X}^n \to \{1, \ldots, 2^{n\mathsf{R}}\}$ and a reconstructor $g^{(n)} \colon \{1, \ldots, 2^{n\mathsf{R}}\} \to \hat{\mathcal{X}}^n$ satisfying $\mathsf{E}[d_1(X^n, \hat{X}^n)] \leq \mathsf{D}_1 + \epsilon$ and $\mathsf{E}[d_2(X^n, \hat{X}^n)] \leq \mathsf{D}_2 + \epsilon$. Show that $(\mathsf{R}, \mathsf{D}_1, \mathsf{D}_2)$ is achievable if, and only if,

$$\mathsf{R} \geq \min_{\substack{P_{\hat{X}|X}(\cdot|\cdot): \\ \mathsf{E}[d_1(X,\hat{X})] \leq \mathsf{D}_1, \\ \mathsf{E}[d_2(X,\hat{X})] \leq \mathsf{D}_2}} I\big(X; \hat{X}\big). \tag{1}$$

b) Now consider the case with one encoder

$$f^{(n)} \colon \mathcal{X}^n \to \{1, \ldots, 2^{n\mathsf{R}}\}$$

and two reconstructors

$$g_1^{(n)} \colon \{1, \ldots, 2^{n\mathsf{R}}\} \to \hat{\mathcal{X}}_1^n,$$
$$g_2^{(n)} \colon \{1, \ldots, 2^{n\mathsf{R}}\} \to \hat{\mathcal{X}}_2^n$$

with corresponding distortion functions $d_1(x, \hat{x}_1)$ and $d_2(x, \hat{x}_2)$, $x \in \mathcal{X}$, $\hat{x}_1 \in \hat{\mathcal{X}}_1$, $\hat{x}_2 \in \hat{\mathcal{X}}_2$. In this case we say that $(\mathsf{R}, \mathsf{D}_1, \mathsf{D}_2)$ is achievable if for every $\epsilon > 0$ there exists some blocklength n_0 such that for all $n > n_0$ there exist $f^{(n)}, g_1^{(n)}, g_2^{(n)}$ such that $\mathsf{E}[d_1(X^n, \hat{X}_1^n)] \leq \mathsf{D}_1 + \epsilon$ and $\mathsf{E}[d_2(X^n, \hat{X}_2^n)] \leq \mathsf{D}_2 + \epsilon$. Show that $(\mathsf{R}, \mathsf{D}_1, \mathsf{D}_2)$ is achievable if, and only if,

$$\mathsf{R} \geq \min_{\substack{P_{\hat{X}_1,\hat{X}_2|X}(\cdot,\cdot|\cdot): \\ \mathsf{E}[d_1(X,\hat{X}_1)] \leq \mathsf{D}_1, \\ \mathsf{E}[d_2(X,\hat{X}_2)] \leq \mathsf{D}_2}} I\big(X; \hat{X}_1, \hat{X}_2\big). \tag{2}$$

Hint: Use Part a).

c) Show that if $\hat{\mathcal{X}}_1 = \hat{\mathcal{X}}_2$ in Part b), then the RHS of (2) is less than or equal to the RHS of (1).

Problem 2 ***Source–Channel Separation with Feedback***

Consider an IID source $\{U_k\}$ taking value in the finite set $\mathcal{U}$, a bounded distortion function $d(u, \hat{u})$, $u \in \mathcal{U}$, $\hat{u} \in \hat{\mathcal{U}}$, and a discrete memoryless channel $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. A combined source–channel coding scheme in the presence of feedback is described as follows: the encoder is a sequence of mappings $f_i \colon (U^k, Y^{i-1}) \mapsto X_i$, and the decoder is a mapping $g \colon Y^n \mapsto \hat{U}^k$. Show that this combined source–channel coding scheme with feedback can achieve expected distortion D only if

$$\mathsf{R}(\mathsf{D}) \leq \frac{n}{k}\mathsf{C},$$

where $\mathsf{R}(\mathsf{D})$ is the rate-distortion function of the source

$$\mathsf{R}(\mathsf{D}) = \min_{P_{\hat{U}|U} \colon \mathsf{E}[d(U,\hat{U})] \leq \mathsf{D}} I(U; \hat{U})$$

and where C is the capacity of the channel

$$\mathsf{C} = \max_{P_X} I(P_X, W).$$

Problem 3 ***Source-Channel Separation and Average Bit-Error Probability***

Consider a combined source-channel coding scheme comprising a DMC of capacity C, a source that emits IID Ber(1/2) bits $U_1, \ldots, U_k$, an encoder that maps the k bits to n channel input symbols $X_1, \ldots, X_n$, and a decoder that maps the resulting n channel output symbols $Y_1, \ldots, Y_n$ to an estimate of the source bits $\hat{U}_1, \ldots, \hat{U}_k$. Suppose that the rate (in bits per channel use) exceeds the capacity of the channel, i.e., $k/n > \mathsf{C}$. Show that

$$\frac{1}{k}\sum_{i=1}^{k} \Pr\bigl[U_i \neq \hat{U}_i\bigr] \geq H_{\mathrm{b}}^{-1}\Bigl(1 - \frac{n}{k}\mathsf{C}\Bigr) > 0,$$

where $H_{\mathrm{b}}^{-1}(\cdot)$ denotes the inverse of the binary entropy function restricted to $[0, 1/2]$. In other words, if $k/n > \mathsf{C}$, then the average bit-error probability cannot be made arbitrarily small.
Hint: Use Problem 2 with the distortion function $d(u, \hat{u}) = I\{u \neq \hat{u}\}$ *and use the fact that for a binary source with Hamming distortion,* $\mathsf{R}(\mathsf{D}) = 1 - H_{\mathrm{b}}(\mathsf{D})$ *for* $\mathsf{D} \in [0, 1/2]$.

