Information Theory I

Fall Semester 2016

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Model Answers to Exercise 13 of December 14, 2016

http://www.isi.ee.ethz.ch/teaching/courses/it1/

Problem 1 ***Source Coding for Bernoulli Sources***

The pairs $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent and identically distributed. From the distributed source coding theorem and its converse we know that the region of achievable rates is given by

$$\begin{aligned} \mathsf{R}_1 &\geq H(X|Y), \\ \mathsf{R}_2 &\geq H(Y|X), \\ \mathsf{R}_1 + \mathsf{R}_2 &\geq H(X,Y). \end{aligned}$$

For rate pairs in the interior of this region, $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ can be recovered with probability of error tending to zero as n tends to infinity. For rate pairs outside this region, the probability of error cannot tend to zero as n tends to infinity.
We denote by X, Y, and Z the random variables that describe a single outcome of the source. Note that although X and Z are independent, X and Y are not independent in general (e.g., for $p = \frac{1}{2}$ and $r = 0$, we have $X = Y$ and X is not deterministic). For the described source we obtain:

$$H(X,Y) \overset{\text{(i)}}{=} H(X,Z) \overset{\text{(ii)}}{=} H(X) + H(Z) = H_{\text{b}}(p) + H_{\text{b}}(r),$$

where (i) follows because the pair (X, Y) is an injective function of the pair (X, Z); (ii) follows because X and Z are independent; and $H_{\text{b}}(\cdot)$ denotes the binary entropy function. Furthermore,

$$H(Y|X) \overset{\text{(iii)}}{=} H(X,Y) - H(X) = H_{\text{b}}(p) + H_{\text{b}}(r) - H_{\text{b}}(p) = H_{\text{b}}(r),$$

where (iii) follows from the chain rule. Because Y is zero if and only if $X = Z = 0$ or $X = Z = 1$, we have

$$\Pr[Y = 0] = \Pr[X = 0, Z = 0] + \Pr[X = 1, Z = 1] = (1-p)(1-r) + pr,$$

and therefore

$$H(X|Y) = H(X,Y) - H(Y) = H_{\text{b}}(p) + H_{\text{b}}(r) - H_{\text{b}}\big((1-p)(1-r) + pr\big).$$

The region of achievable rates is thus given by:

$$\begin{aligned} \mathsf{R}_1 &\geq H_{\text{b}}(p) + H_{\text{b}}(r) - H_{\text{b}}\big((1-p)(1-r) + pr\big), \\ \mathsf{R}_2 &\geq H_{\text{b}}(r), \\ \mathsf{R}_1 + \mathsf{R}_2 &\geq H_{\text{b}}(p) + H_{\text{b}}(r). \end{aligned}$$

Problem 2 ***Slepian-Wolf***

a) We compute

$$\begin{aligned}
H(X,Y) &= -\sum_{x,y} p(x,y)\log p(x,y)\\
&= -3\alpha\log\alpha - 6\beta\log\beta\\
&= -3\alpha\log\alpha - (1-3\alpha)\log\left(\frac{1}{6}-\frac{\alpha}{2}\right);\\
H(X) &= -\sum_{x} p(x)\log p(x)\\
&= -3(\alpha+2\beta)\log(\alpha+2\beta)\\
&= \log 3;\\
H(Y|X) &= H(X,Y) - H(X)\\
&= -3\alpha\log\alpha - (1-3\alpha)\log\left(\frac{1}{6}-\frac{\alpha}{2}\right) - \log 3.
\end{aligned}$$

By symmetry, $H(X) = H(Y)$ and $H(X|Y) = H(Y|X)$. Hence, the rate region looks as shown in Figure 1.

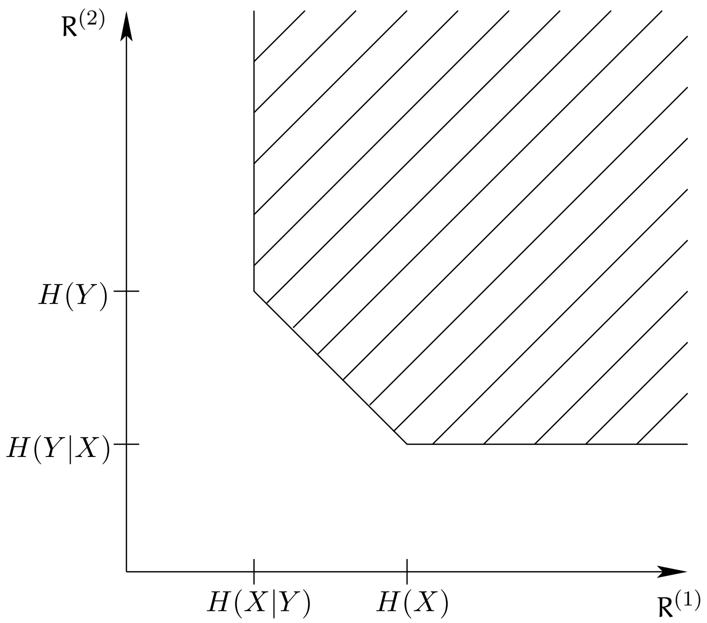

Figure 1: Rate region for general α.

b) This can be easily computed from $p(x,y)$:

$$\Pr[X=Y] = \sum_x p(x,x) = 3\alpha.$$

c) If $\alpha = \frac{1}{3}$, then we get

$$\begin{aligned}
H(X,Y) &= \log 3;\\
H(X) &= \log 3;\\
H(X|Y) &= H(Y|X) = 0.
\end{aligned}$$

This corresponds to the case where $X = Y$, which can also can be seen from the result in Part b). The rate region is shown in Figure 2.

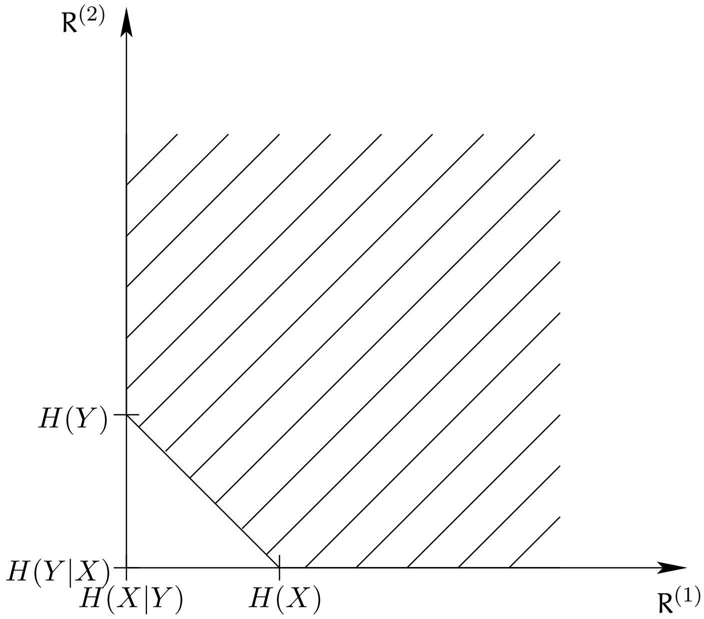

Figure 2: Rate region for $\alpha = \frac{1}{3}$.

d) If $\alpha = \frac{1}{9}$, then we get

$$\begin{aligned} H(X,Y) &= 2\log 3; \\ H(X) &= \log 3; \\ H(X|Y) &= \log 3. \end{aligned}$$

This corresponds to the case where $X \perp\!\!\!\perp Y$. The rate region is shown in Figure 3.

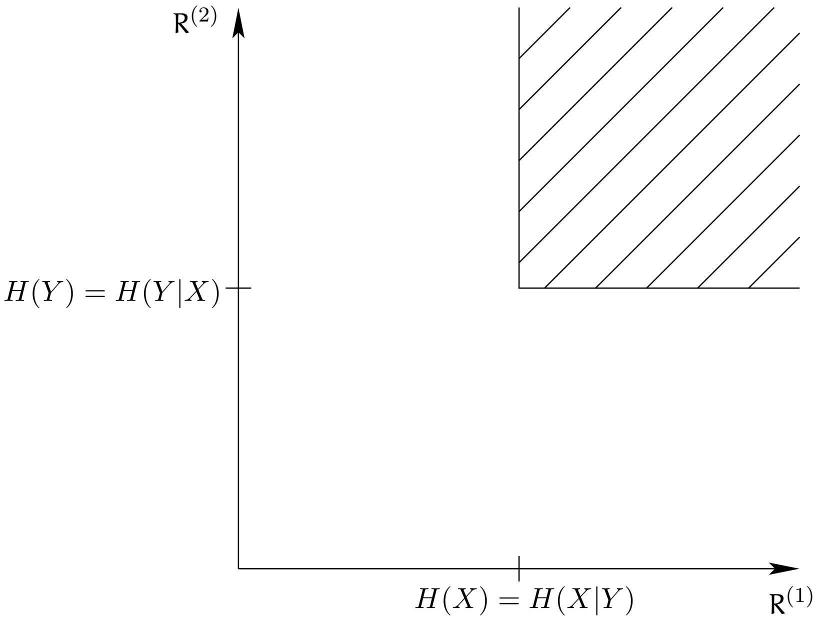

Figure 3: Rate region for $\alpha = \frac{1}{9}$.

Problem 3 *Slepian-Wolf*

We need to compute some entropies:

$$
\begin{aligned}
H(U_1) &= -\left(\alpha + (m-1)\cdot\frac{\beta}{m-1}\right)\log\left(\alpha + (m-1)\cdot\frac{\beta}{m-1}\right) - (m-1)\cdot\frac{\gamma}{m-1}\log\frac{\gamma}{m-1} \\
&= -(\alpha+\beta)\log(\alpha+\beta) - \gamma\log\frac{\gamma}{m-1} \\
&= -(1-\gamma)\log(1-\gamma) - \gamma\log\gamma + \gamma\log(m-1) \\
&= H_{\mathrm{b}}(\gamma) + \gamma\log(m-1), \\
H(U_2) &= -\left(\alpha + (m-1)\cdot\frac{\gamma}{m-1}\right)\log\left(\alpha + (m-1)\cdot\frac{\gamma}{m-1}\right) - (m-1)\cdot\frac{\beta}{m-1}\log\frac{\beta}{m-1} \\
&= -(\alpha+\gamma)\log(\alpha+\gamma) - \beta\log\frac{\beta}{m-1} \\
&= -(1-\beta)\log(1-\beta) - \beta\log\beta + \beta\log(m-1) \\
&= H_{\mathrm{b}}(\beta) + \beta\log(m-1), \\
H(U_1,U_2) &= -\alpha\log\alpha - (m-1)\cdot\frac{\beta}{m-1}\log\frac{\beta}{m-1} - (m-1)\cdot\frac{\gamma}{m-1}\log\frac{\gamma}{m-1} \\
&= -(1-\beta-\gamma)\log(1-\beta-\gamma) - \beta\log\frac{\beta}{m-1} - \gamma\log\frac{\gamma}{m-1} \\
&= -(1-\beta-\gamma)\log(1-\beta-\gamma) - \beta\log\beta - \gamma\log\gamma + (\beta+\gamma)\log(m-1), \\
H(U_1|U_2) &= H(U_1,U_2) - H(U_2) \\
&= -(1-\beta-\gamma)\log(1-\beta-\gamma) - \beta\log\beta - \gamma\log\gamma + (\beta+\gamma)\log(m-1) \\
&\quad - H_{\mathrm{b}}(\beta) - \beta\log(m-1) \\
&= -(1-\beta-\gamma)\log(1-\beta-\gamma) + (1-\beta)\log(1-\beta) - \gamma\log\gamma + \gamma\log(m-1), \\
H(U_2|U_1) &= H(U_1,U_2) - H(U_1) \\
&= -(1-\beta-\gamma)\log(1-\beta-\gamma) - \beta\log\beta - \gamma\log\gamma + (\beta+\gamma)\log(m-1) \\
&\quad - H_{\mathrm{b}}(\gamma) - \gamma\log(m-1) \\
&= -(1-\beta-\gamma)\log(1-\beta-\gamma) + (1-\gamma)\log(1-\gamma) - \beta\log\beta + \beta\log(m-1).
\end{aligned}
$$

Here β and γ must satisfy $\beta,\gamma \geq 0$ and $\beta+\gamma \leq 1$.
The rate region is now given by

$$
\begin{aligned}
\mathsf{R}^{(1)} &\geq H(U_1|U_2), \\
\mathsf{R}^{(2)} &\geq H(U_2|U_1), \\
\mathsf{R}^{(1)} + \mathsf{R}^{(2)} &\geq H(U_1,U_2).
\end{aligned}
$$