



Exercise 2 of September 27, 2017

<http://www.isi.ee.ethz.ch/teaching/courses/it1.html>

Problem 1

Example of Joint Entropy

Let $P_{X,Y}(x, y)$ be given by

$P_{X,Y}(x, y)$	$y = 0$	$y = 1$
$x = 0$	1/3	1/3
$x = 1$	0	1/3

Find

- a) $H(X), H(Y)$,
- b) $H(X|Y), H(Y|X)$,
- c) $H(X, Y)$,
- d) $H(Y) - H(Y|X)$,
- e) $I(X; Y)$.

Problem 2

Zero Conditional Entropy

Show that $H(Y|X) = 0$ if and only if “ Y is a function of X ”, i.e., if and only if there exists a function $g(\cdot)$ such that $\Pr[Y = g(X)] = 1$.

Problem 3

Entropy of Functions of a Chance Variable

Let X be a discrete chance variable. Show that the entropy of a function of X is less than or equal to the entropy of X by justifying the following steps:

$$\begin{aligned} H(X, g(X)) &\stackrel{(a)}{=} H(X) + H(g(X)|X) \stackrel{(b)}{=} H(X); \\ H(X, g(X)) &\stackrel{(c)}{=} H(g(X)) + H(X|g(X)) \stackrel{(d)}{\geq} H(g(X)). \end{aligned}$$

Thus, $H(g(X)) \leq H(X)$. When does equality hold?

Problem 4**Entropy of a Sum**

Let X and Y be random variables that take values in $\{x_1, x_2, \dots, x_r\} \subset \mathbb{R}$ and in $\{y_1, y_2, \dots, y_s\} \subset \mathbb{R}$, respectively. Let $Z = X + Y$.

- Show that $H(Z|X) = H(Y|X)$. Argue that if X and Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of *independent* random variables adds uncertainty.
- Give an example (of necessarily dependent random variables) in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.
- Under what conditions does $H(Z) = H(X) + H(Y)$ hold?

Hint: You may find the conclusion of Problem 3 helpful.

Problem 5**Jensen's Inequality**

Let a_1, \dots, a_n be positive real numbers.

- Use Jensen's inequality to show that the arithmetic mean is greater than or equal to the geometric mean, i.e.,

$$\left(\prod_{k=1}^n a_k \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^n a_k.$$

Also find necessary and sufficient conditions for equality.

- Use Jensen's inequality to show that

$$\frac{1}{n} \sum_{k=1}^n a_k^\beta \geq \left(\frac{1}{n} \sum_{k=1}^n a_k \right)^\beta$$

for any $\beta \geq 1$. What happens if $0 < \beta < 1$?

- Suppose that you have written n exams with scores a_k that are numbers between 1 and 6. Would you rather have $\frac{1}{n} \sum_{k=1}^n a_k$ or $\sqrt{\frac{1}{n} \sum_{k=1}^n a_k^2}$ as your final score?