Information Theory I

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http://www.isi.ee.ethz.ch/teaching/courses/it1.html

Problem 1 — *Strong Versus Weak Typicality*

A sequence of chance variables $X_1, \ldots, X_{100}$ is drawn IID with X_i taking on the values "True" and "False" equiprobably. Describe the set of strongly typical sequences and the set of weakly typical sequences for $\epsilon = 0.01$.

Problem 2 — *Random Box Size*

Let $X_1, X_2, \ldots, X_n$ be IID uniform random variables over the unit interval $[0, 1]$. An n-dimensional rectangular box with side lengths $X_1, X_2, \ldots, X_n$ is constructed. Its volume is $V_n = \prod_{i=1}^n X_i$, and the edge length L_n of an n-cube with the same volume is $L_n = {V_n}^{1/n}$.

a) Compute $\lim_{n\to\infty} (\mathsf{E}[V_n])^{1/n}$.

b) Show that $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \ln X_i = -1$ in probability.

c) Deduce from Part b) that $\lim_{n\to\infty} L_n = e^{-1}$ in probability.

Problem 3 — *From AEP to Kraft's Inequality*

Let $\ell_1, \ldots, \ell_d$ be the codeword lengths of a uniquely decodable fixed-to-variable binary code. Use Theorem 1 to show that

$$\sum_{i=1}^d 2^{-\ell_i} \leq 1. \tag{1}$$

Theorem 1 (Converse for the Source Coding Theorem). *Consider an IID source with entropy $H(X)$ and a sequence of fixed-to-fixed codes which map n source symbols to $n\,\rho_n$ bits. If*

$$\lim_{n\to\infty} \rho_n < H(X), \tag{2}$$

then the probability of successful decoding for these codes tends to zero as n tends to infinity.

Hint: Prove the claim by contradiction. Assume that there exists a uniquely decodable code $\mathcal{C}$ whose codeword lengths do not satisfy (1). Construct a source for which the expected codeword length of $\mathcal{C}$ is smaller than the entropy of the source. Use the extension of $\mathcal{C}$ to devise a coding scheme that satisfies (2) and whose probability of successful decoding tends to one as n tends to infinity, which leads to the desired contradiction.

Problem 4 ***Fano's Inequality***

Let X take values in the set $\mathcal{X} = \{1, \ldots, m\}$ and let $p_i = \Pr[X = i]$ for $i = 1, \ldots, m$.

a) Suppose that you must guess X. Which guess would have the smallest probability of error? What is the probability of error P_{e}^* associated with this guess?

b) Let $\hat{x}$ be a fixed guess, and let P_{e} be the probability of error associated with this guess. Maximize $H(\tilde{X})$ over all choices of $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_m$ for which the guess $\hat{x}$ has a probability of error $\tilde{P}_{\text{e}} = P_{\text{e}}$. Argue that this implies

$$H(X) \leq H_{\text{b}}(P_{\text{e}}) + P_{\text{e}} \log(m-1).$$

c) Suppose that you must guess X based on the observation of a correlated chance variable Y. Given some guessing rule, let P_{e} be the probability of error and let E be a chance variable which is zero if the guess is correct and one otherwise. Justify the following proof of Fano's inequality:

$$\begin{aligned} H_{\text{b}}(P_{\text{e}}) + P_{\text{e}} \log(m-1) &\overset{\text{(i)}}{=} H(E) + \Pr[E=1] \log(m-1) \\ &\overset{\text{(ii)}}{\geq} H(E|Y) + \Pr[E=1] \log(m-1) \\ &\overset{\text{(iii)}}{=} \sum_{y \in \mathcal{Y}} P_Y(y) \Big[H(E|Y=y) + \Pr[E=1|Y=y] \log(m-1) \Big] \\ &\overset{\text{(iv)}}{\geq} \sum_{y \in \mathcal{Y}} P_Y(y) H(X|Y=y) \\ &\overset{\text{(v)}}{=} H(X|Y). \end{aligned}$$

Hint: Use the result from Part b).