Exercise 6 of October 25, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it1.html

Problem 1  Strong Versus Weak Typicality

A sequence of chance variables $X_1, \ldots, X_{100}$ is drawn IID with $X_i$ taking on the values “True” and “False” equiprobably. Describe the set of strongly typical sequences and the set of weakly typical sequences for $\epsilon = 0.01$.

Problem 2  Random Box Size

Let $X_1, X_2, \ldots, X_n$ be IID uniform random variables over the unit interval $[0, 1]$. An $n$-dimensional rectangular box with side lengths $X_1, X_2, \ldots, X_n$ is constructed. Its volume is $V_n = \prod_{i=1}^n X_i$, and the edge length $L_n$ of an $n$-cube with the same volume is $L_n = V_n^{1/n}$.

a) Compute $\lim_{n \to \infty} (\mathbb{E}[V_n])^{1/n}$.

b) Show that $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \ln X_i = -1$ in probability.

c) Deduce from Part b) that $\lim_{n \to \infty} L_n = e^{-1}$ in probability.

Problem 3  From AEP to Kraft’s Inequality

Let $\ell_1, \ldots, \ell_d$ be the codeword lengths of a uniquely decodable fixed-to-variable binary code. Use Theorem 1 to show that

$$\sum_{i=1}^d 2^{-\ell_i} \leq 1.$$  (1)

Theorem 1 (Converse for the Source Coding Theorem). Consider an IID source with entropy $H(X)$ and a sequence of fixed-to-fixed codes which map $n$ source symbols to $n \rho_n$ bits. If

$$\lim_{n \to \infty} \rho_n < H(X),$$  (2)

then the probability of successful decoding for these codes tends to zero as $n$ tends to infinity.

Hint: Prove the claim by contradiction. Assume that there exists a uniquely decodable code $C$ whose codeword lengths do not satisfy (1). Construct a source for which the expected codeword length of $C$ is smaller than the entropy of the source. Use the extension of $C$ to devise a coding scheme that satisfies (2) and whose probability of successful decoding tends to one as $n$ tends to infinity, which leads to the desired contradiction.

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Problem 4  

Fano’s Inequality

Let $X$ take values in the set $\mathcal{X} = \{1, \ldots, m\}$ and let $p_i = \Pr[X = i]$ for $i = 1, \ldots, m$.

a) Suppose that you must guess $X$. Which guess would have the smallest probability of error? What is the probability of error $P^*_e$ associated with this guess?

b) Let $\hat{x}$ be a fixed guess, and let $P_e$ be the probability of error associated with this guess. Maximize $H(\hat{X})$ over all choices of $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m$ for which the guess $\hat{x}$ has a probability of error $\hat{P}_e = P_e$. Argue that this implies

$$H(X) \leq H_b(P_e) + P_e \log (m - 1).$$

c) Suppose that you must guess $X$ based on the observation of a correlated chance variable $Y$. Given some guessing rule, let $P_e$ be the probability of error and let $E$ be a chance variable which is zero if the guess is correct and one otherwise. Justify the following proof of Fano’s inequality:

$$H_b(P_e) + P_e \log (m - 1) \overset{(i)}{=} H(E) + \Pr[E = 1] \log (m - 1)$$

$$\quad \overset{(ii)}{\geq} H(E|Y) + \Pr[E = 1] \log (m - 1)$$

$$\quad \overset{(iii)}{=} \sum_{y \in \mathcal{Y}} P_Y(y) \left[ H(E|Y = y) + \Pr[E = 1|Y = y] \log (m - 1) \right]$$

$$\quad \overset{(iv)}{\geq} \sum_{y \in \mathcal{Y}} P_Y(y) H(X|Y = y)$$

$$\quad \overset{(v)}{=} H(X|Y).$$

Hint: Use the result from Part b).