Information Theory I

Fall Semester 2017

Prof. Dr. A. Lapidoth

Signal and Information Processing Laboratory
Institut für Signal- und Informationsverarbeitung

Exercise 7 of November 1, 2017

http://www.isi.ee.ethz.ch/teaching/courses/it1.html

Problem 1 — *On the Achievable Rate*

Consider a discrete memoryless channel of law $W_1(y|x)$ over the finite input and output alphabets $\mathcal{X}$ and $\mathcal{Y}$. Let a second channel be defined over the input alphabet $\mathcal{X} \times \mathcal{X}$ and output alphabet $\mathcal{Y} \times \mathcal{Y}$ and be of law

$$W_2(y_1, y_2|x_1, x_2) = W_1(y_1|x_1)W_1(y_2|x_2).$$

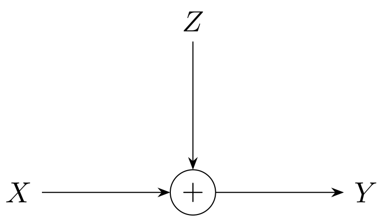
Based on the definition of achievable rate, show that

a) if R is achievable on the first channel, then $2\mathsf{R}$ is achievable on the second channel;

b) if R is achievable on the second channel, then $\mathsf{R}/2$ is achievable on the first channel.

Problem 2 — *An Additive Noise Channel*

Find the channel capacity of the following discrete memoryless channel:



where $\Pr[Z = 0] = \Pr[Z = a] = \frac{1}{2}$ for some fixed value $a \in \mathbb{R}$. The input X takes values in the binary alphabet $\mathcal{X} = \{0, 1\}$. Assume that Z is independent of X.
Hint: Observe that the channel capacity depends on the value of a, i.e., you need to introduce a case distinction!

Problem 3 — *Z-Channel*

The Z-channel has binary input and output alphabets and transition probabilities $W(\cdot|\cdot)$ given by the following matrix:

$$\mathsf{W} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad X, Y \in \{0, 1\}.$$

The channel is called *Z-channel* because it looks like the letter Z, see Figure 1. Find the capacity of the Z-channel and the maximizing input probability distribution.

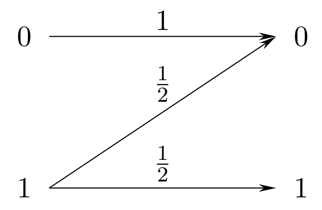


Figure 1: Z-Channel.

Problem 4 — *Independent Parallel Channels*

Consider two independent discrete memoryless channels $(\mathcal{X}_1, W_1(\cdot|\cdot), \mathcal{Y}_1)$ and $(\mathcal{X}_2, W_2(\cdot|\cdot), \mathcal{Y}_2)$ with capacities C_1 and C_2, respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, W_1(\cdot|\cdot) \times W_2(\cdot|\cdot), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which a tuple $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ is sent, and the output (Y_1, Y_2) is a tuple in $\mathcal{Y}_1 \times \mathcal{Y}_2$. Find the capacity of this channel.

Problem 5 — *Capacity of a Sum Channel*

a) Consider ν (in general different) discrete memoryless channels with capacities $\mathsf{C}_1, \mathsf{C}_2, \ldots, \mathsf{C}_\nu$. The *sum channel* associated therewith is the channel whose input and output alphabets are the unions of those of the original channels. (The input and output alphabets of the different channels are assumed disjoint.) That is, the sum channel has all ν channels available for use, but only one channel may be used at any given time. Show that the capacity of the sum channel is

$$\mathsf{C} = \log\left(\sum_{i=1}^{\nu} 2^{\mathsf{C}_i}\right)$$

(where C_i are measured in bits) and find s_i, the probability of using the i-th channel. Interpret C as the average of the capacities of the individual channels plus the information conveyed by the selection of a channel.

Hint: Introduce an auxiliary chance variable S with $\Pr[S = i] = s_i$ for all $i \in \{1, \ldots, \nu\}$.

b) Use the above result to find the capacity of the channel shown in Figure 2.

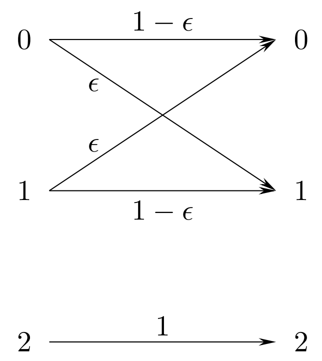


Figure 2: A sum channel.